

CONCERNING SINGULAR TRANSFORMATIONS  $B_k$  OF SURFACES  
APPLICABLE TO QUADRICS\*

BY

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INTRODUCTION

The theory of transformations  $B_k$  of deforms by flexure of quadrics† may be regarded as a later development of the concepts introduced by Sophus Lie in his researches on the transformations of surfaces of constant curvature. The attempt to make a possible extension of the same principles to researches of another class of applicable surfaces leads us naturally to propose a problem of arranging plane elements, or *facettes*, in space—which for the sake of clearness we will explain in detail.‡ A *facette* consists of a plane and a point of it called the *center*. We think of a surface  $S$  as the totality of its  $\infty^2$  facettes  $f$ , the planes of the facettes being tangent to  $S$  at their respective centers. We consider associated with each facette  $f$  a *simple infinity* of facettes  $f'$ , in accordance with any continuous law whatever. We imagine also that in every deformation of  $S$ , each facette  $f$  and the  $\infty^1$  associated facettes  $f'$  are carried along as *an invariable system*. In each configuration of  $S$  the associated facettes  $f'$  form a triply infinite system, and in general they cannot be arranged into a series of  $\infty^1$  surfaces  $S'$ , each consisting of  $\infty^2$  facettes  $f'$ . The problem in view consists precisely in determining all the cases for which the above circumstance is true *in all deformations of  $S$* . Of the general problem thus stated it is easy to indicate an infinity of particular solutions amongst which are immediately evident those in which each of the  $\infty^1$  surfaces  $S'$  remains constituted always of the same  $\infty^2$  facettes  $f'$ . But, in view of the eventual applications to problems of deformation, it is opportune to limit the problem much more, and to suppose that every facette  $f$  and each of its associated facettes  $f'$  has the center of one in the plane of the other. Thus the surface  $S$  and each of its transforms  $S'$  are always the focal surfaces of the rectilinear congruence formed by the joins of corresponding points.

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† Cf. the author's *Lezioni di geometria differenziale*, vol. 3 (Pisa-Spörri, 1909).

‡ Cf. *Lezioni*, vol. 3, § 39; also my communication to the Fourth International Congress of Mathematicians held in Rome (1908) as reported in *Atti del Congresso*, vol. 2, p. 273.

Upon the problem thus limited we have had the interesting researches of A. V. Bäcklund,\* which, even if they have not been useful, in essentially new cases, for the theory of applicability, they have nevertheless led to a better understanding of the reasons for the success of the method of transformations in the case of surfaces applicable to quadrics. Our aim in the present paper is an analogous one, in that by attacking a particular case of the problem, geometrically further circumscribed and definite, we arrive at complete results. The particular case which we wish to treat is that which corresponds to the special transformations  $B_k$ , indicated in my book as those *singular* transformations, which arise when, in place of quadrics homofocal to the given quadric, we use its *focal conics*. In this case, the  $\infty^1$  facettes  $f'$  associated with a facette  $f$  of the quadric are distributed as follows: 1°. their centers are distributed on the intersection of the plane of  $f$  and the fixed plane of the focal conic; 2°. the planes of  $f'$  envelope the cone which from the center of  $f$  projects the focal conic. It is upon the first of these properties that we fix our attention, and suppose that, given a surface  $S_0$  and a fixed plane  $\pi$ , with each facette  $f$  of  $S_0$  there are associated  $\infty^1$  facettes  $f'$ , whose centers lie on the intersection of the plane of  $f$  (i. e., the tangent plane of  $S_0$ ) and of  $\pi$ ; as for the planes of  $f'$  we subject them to the single condition of passing through the center of  $V$  of  $f$ , so that they envelope a cone with vertex  $V$ , as to whose form no hypothesis is made. Under these conditions we propose to solve the following problem:

**PROBLEM A.** *To find what must be assumed concerning  $S_0$  and its relation with the fixed plane  $\pi$ , so that, as  $S_0$  undergoes any deformation by flexure, and each of its facettes  $f$  moves carrying with it the invariable system of  $\infty^1$  facettes  $f'$ , it will always be true that the  $\infty^3$  facettes  $f'$  will be assembled in a series of  $\infty^1$  surfaces  $S'$ .*

We shall demonstrate that when the case where  $S_0$  is developable (which is of no interest) is excluded, all the solutions of problem (A) are obtained if one takes any quadric whatever which has the fixed plane  $\pi$  for a plane of symmetry (principal), so that one is necessarily led back to deformations of quadrics. One sees, therefore, if the quadric is general, that the distribution of the facettes is precisely the one above mentioned which presents itself in the theory of singular transformations  $B_k$ . However, when the quadric is one of revolution, *having the fixed plane for a meridian plane*, the cone enveloped by the planes of the  $\infty^1$  facettes  $f'$  associated with a facette  $f$  breaks up into two pencils of planes. The transformed surfaces  $S'$  are in this case

\* *Ueber eine Transformation von Luigi Bianchi*, *Annali di Matematica*, ser. 3, vol. 23 (1914); also, *Sätze aus Bianchi's Theorie der auf die Flächen zweiter Ordnung abwickelbaren Flächen*, *u.s.w.*, *Kongl. Svenska Vetenskapsakademiens Handlingar*, vol. 55, no. 22 (1916).

also applicable to one another and to the same quadric, which nevertheless is different from the given quadric, and in this respect the special case differs from the general one. The transformations which we have obtained in the special case are not any less singular transformations  $B_k$ ; all of them are coördinated by the same geometrical principle, contained in problem (A).

In conclusion, the researches which we set forth in this memoir, like those already recorded by Bäcklund, have not extended the domain of transformations to a new class of applicable surfaces, but they serve to characterize all the better the rather unusual circumstance presented by the deforms of quadrics.

1. FUNDAMENTAL EQUATIONS OF PROBLEM (A)

Let  $S_0$  be a given surface, and  $x_0, y_0, z_0$  the coördinates of a generic point  $P_0$  on  $S_0$ . Assuming for curvilinear coördinates  $u, v$  on  $S_0$  the two coördinates  $x_0, y_0$ , we write the parametric equations in the form

$$x_0 = u, \quad y_0 = v, \quad z_0 = z_0(u, v).$$

The first and second derivatives of  $z_0$  with respect to  $u$  and  $v$  will be indicated by  $p, q, r, s, t$  in accordance with the notation of Monge. The linear element of  $S_0$  is

$$(1) \quad ds_0^2 = (1 + p^2) du^2 + 2pq dudv + (1 + q^2) dv^2$$

$$(E = 1 + p^2, \quad F = pq, \quad G = 1 + q^2, \quad EG - F^2 = 1 + p^2 + q^2),$$

and the values of the Christoffel symbols for this differential form (1) are

$$(2) \quad \begin{aligned} \left\{ \begin{matrix} 11 \\ 1 \end{matrix} \right\} &= \frac{pr}{1 + p^2 + q^2}, & \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\} &= \frac{ps}{1 + p^2 + q^2}, & \left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\} &= \frac{pt}{1 + p^2 + q^2}, \\ \left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\} &= \frac{qr}{1 + p^2 + q^2}, & \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\} &= \frac{qs}{1 + p^2 + q^2}, & \left\{ \begin{matrix} 22 \\ 2 \end{matrix} \right\} &= \frac{qt}{1 + p^2 + q^2}. \end{aligned}$$

Let  $S$  be any surface applicable to  $S_0$ , with which then it has as first fundamental form (1), and  $S$  will be intrinsically defined by its second fundamental form  $Ddu^2 + 2D'dudv + D''dv^2$ . In place of these functions  $D, D', D''$  we make use of

$$\Delta = \frac{D}{\sqrt{EG - F^2}}, \quad \Delta' = \frac{D'}{\sqrt{EG - F^2}}, \quad \Delta'' = \frac{D''}{\sqrt{EG - F^2}},$$

and we remark that these functions are required to satisfy only the *equation of Gauss*,

$$(3) \quad \Delta\Delta'' - \Delta'^2 = \frac{rt - s^2}{(1 + p^2 + q^2)^2},$$

and the *equations of Codazzi*, which because of (2) are reducible to

$$(4) \quad \begin{aligned} \frac{\partial \Delta}{\partial v} - \frac{\partial \Delta'}{\partial u} &= \frac{q}{1 + p^2 + q^2} (2s\Delta' - t\Delta - r\Delta''), \\ \frac{\partial \Delta''}{\partial u} - \frac{\partial \Delta'}{\partial v} &= \frac{p}{1 + p^2 + q^2} (2s\Delta' - t\Delta - r\Delta''). \end{aligned}$$

We retain the customary notation  $x, y, z$  for the coördinates of the moving point  $P(u, v)$  on  $S$ , and  $X, Y, Z$  for the direction-cosines of the normal to  $S$  at  $P$ .

In conformity with the conditions of problem (A), we make correspond to each facette

$$f \equiv (x, y, z; X, Y, Z)$$

of  $S$  a simple infinity of facettes

$$f' \equiv (x', y', z'; X', Y', Z'),$$

with centers distributed on a line in the tangent plane to  $S$  and with their planes passing through the lines joining the centers  $(x, y, z), (x', y', z')$  of the facettes. Accordingly we write the formulas

$$(5) \quad \begin{aligned} x' &= x + l \frac{\partial x}{\partial u} + m \frac{\partial x}{\partial v}, \\ X' &= \alpha \frac{\partial x}{\partial u} + \beta \frac{\partial x}{\partial v} + \gamma X, \end{aligned}$$

with analogous expressions for the  $y$ 's and  $z$ 's, where  $l, m; \alpha, \beta, \gamma$  are functions of  $u, v$  and of a parameter  $\lambda$ , which must remain unchanged in the deformation of  $S$ , since the  $\infty^1$  facettes  $f'$  are rigidly attached to  $f$ .

When  $S$  takes the initial form  $S_0$ , we have

$$x = x_0 = u, \quad y = y_0 = v, \quad z = z_0,$$

and consequently

$$x'_0 = u + l, \quad y'_0 = v + m, \quad z'_0 = z_0 + lp + mq.$$

Hence if we assume the fixed plane  $\pi$  of the problem to be  $x = 0$ , we must have  $x'_0 = 0$ , thence  $l = -u$ , but  $m$  will remain variable. If we take as parameter  $m = \lambda$ , then the first equation of (5) becomes

$$(6) \quad x' = x - u \frac{\partial x}{\partial u} + \lambda \frac{\partial x}{\partial v}.$$

We must make the further requirement that the planes of  $f$  and  $f'$  shall intersect in the join of their centers, that is

$$X'(x' - x) + Y'(y' - y) + Z'(z' - z) = 0,$$

which by means of the second equations of (5) and (6) is reducible to

$$(E\alpha + F\beta)u = (F\alpha + G\beta)\lambda.$$

If we alter  $\alpha, \beta, \gamma$  in (5) by a factor of proportionality (to be signified hereafter by the use of the sign  $\equiv$  in place of  $=$ ), we can take

$$(7) \quad \begin{aligned} E\alpha + F\beta &= (1 + p^2 + q^2)\lambda, \\ F\alpha + G\beta &= (1 + p^2 + q^2)u. \end{aligned}$$

From these follow

$$(7^*) \quad \begin{aligned} \alpha &= (1 + q^2)\lambda - pq\lambda, \\ \beta &= -pq\lambda + (1 + p^2)u, \end{aligned}$$

and then we have

$$(8) \quad X' \equiv \alpha \frac{\partial x}{\partial u} + \beta \frac{\partial x}{\partial v} + \gamma X,$$

where  $\alpha$  and  $\beta$  are functions of  $u, v, \lambda$  as given by (7\*), and  $\gamma$  is an indeterminate function of  $u, v, \lambda$  subject to the condition that it does not vary as  $S$  is deformed.

The formulas (6) and (8) define in space  $\infty^3$  facettes  $f'$ , and we must now seek the condition to be satisfied so that, in any deformation whatever of  $S$ , these facettes  $f'$  can be arranged into  $\infty^1$  surfaces  $S'$ . It is necessary and sufficient that, for every configuration  $S$ , one can determine  $\lambda$  as a function of  $u, v$  and an arbitrary constant so that the following equations are satisfied:

$$(9) \quad \begin{aligned} X' \frac{\partial x'}{\partial u} + Y' \frac{\partial y'}{\partial u} + Z' \frac{\partial z'}{\partial u} &= 0, \\ X' \frac{\partial x'}{\partial v} + Y' \frac{\partial y'}{\partial v} + Z' \frac{\partial z'}{\partial v} &= 0. \end{aligned}$$

## 2. THE DIFFERENTIAL EQUATIONS FOR $\lambda = \lambda(u, v)$

The derivatives of (6) with respect to  $u$  and  $v$  lead to the expressions (cf. *Lezioni*, vol. III, p. 7)

$$(10) \quad \begin{aligned} \frac{\partial x'}{\partial u} &= L_0 \frac{\partial x}{\partial u} + M_0 \frac{\partial x}{\partial v} + \frac{\partial \lambda}{\partial u} \frac{\partial x}{\partial v} + \sqrt{1 + p^2 + q^2} (\Delta' \lambda - \Delta u) X, \\ \frac{\partial x'}{\partial v} &= P_0 \frac{\partial x}{\partial u} + Q_0 \frac{\partial x}{\partial v} + \frac{\partial \lambda}{\partial v} \frac{\partial x}{\partial v} + \sqrt{1 + p^2 + q^2} (\Delta'' \lambda - \Delta' u) X, \end{aligned}$$

where we have put

$$\begin{aligned} L_0 &= \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\} \lambda - \left\{ \begin{matrix} 11 \\ 1 \end{matrix} \right\} u, & M_0 &= \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\} \lambda - \left\{ \begin{matrix} 11 \\ 2 \end{matrix} \right\} u, \\ P_0 &= \left\{ \begin{matrix} 22 \\ 1 \end{matrix} \right\} \lambda - \left\{ \begin{matrix} 12 \\ 1 \end{matrix} \right\} u, & Q_0 &= \left\{ \begin{matrix} 22 \\ 2 \end{matrix} \right\} \lambda - \left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\} u + 1, \end{aligned}$$

which by (2) become

$$(11) \quad \begin{aligned} L_0 &= \frac{p(\lambda s - ur)}{1 + p^2 + q^2}, & M_0 &= \frac{q(\lambda s - ur)}{1 + p^2 + q^2}, \\ P_0 &= \frac{p(\lambda t - us)}{1 + p^2 + q^2}, & Q_0 &= \frac{q(\lambda t - us)}{1 + p^2 + q^2} + 1. \end{aligned}$$

In consequence of (8) and (10) the conditions (9) are reducible to

$$\begin{aligned} (F\alpha + G\beta) \frac{\partial \lambda}{\partial u} + (E\alpha + F\beta) L_0 + (F\alpha + G\beta) M_0 \\ + \gamma \sqrt{1 + p^2 + q^2} (\Delta' \lambda - \Delta u) = 0, \\ (F\alpha + G\beta) \frac{\partial \lambda}{\partial v} + (E\alpha + F\beta) P_0 + (F\alpha + G\beta) Q_0 \\ + \gamma \sqrt{1 + p^2 + q^2} (\Delta'' \lambda - \Delta' u) = 0. \end{aligned}$$

Making use of (7) and (11), and introducing in place of  $\gamma = \gamma(u, v, \lambda)$  the function

$$(12) \quad \Theta = \Theta(u, v, \lambda) = \frac{\gamma}{u \sqrt{1 + p^2 + q^2}},$$

we give the differential equations for  $\lambda$  the definitive form

$$(A) \quad \begin{aligned} \frac{\partial \lambda}{\partial u} &= \frac{(qu + p\lambda)(ur - \lambda s)}{u(1 + p^2 + q^2)} + \Theta(\Delta u - \Delta' \lambda), \\ \frac{\partial \lambda}{\partial v} + 1 &= \frac{(qu + p\lambda)(us - \lambda t)}{u(1 + p^2 + q^2)} + \Theta(\Delta' u - \Delta'' \lambda). \end{aligned}$$

We have now to examine how the unknown function  $z_0 = z_0(u, v)$ , which defines the initial configuration  $S_0$  of our surface, and  $\Theta = \Theta(u, v, \lambda)$ , which fixes the ordering of the facettes, must be taken in order that *the equations (A) shall admit a solution  $\lambda$  involving an arbitrary constant, that is shall form a completely integrable system.* We remark that it must hold for any configuration of  $S$ , that is for any choice of  $\Delta, \Delta', \Delta''$ , provided only that the latter satisfy equations (3) and (4) of Gauss and Codazzi.

### 3. THE CONDITIONS OF INTEGRABILITY

If we indicate by  $\Omega$  the expression obtained by taking the derivatives of the first and second equations of (A) with respect to  $v$  and  $u$  respectively and subtracting the results, we must have  $\Omega = 0$ , when we take account of (A). Now using the symbols  $\partial/\partial u, \partial/\partial v, \partial/\partial \lambda$  for the partial derivatives with respect to  $u, v, \lambda$ , regarded as independent, we find

$$\begin{aligned} \Omega = & \frac{\partial}{\partial u} \left[ \frac{(qu + p\lambda)(\lambda t - us)}{u(1 + p^2 + q^2)} \right] + \frac{\partial}{\partial v} \left[ \frac{(qu + p\lambda)(ur - \lambda s)}{u(1 + p^2 + q^2)} \right] \\ & + \frac{2pt\lambda + u(qt - ps)}{u(1 + p^2 + q^2)} \frac{\partial \lambda}{\partial u} + \frac{-2ps\lambda + u(pr - qs)}{u(1 + p^2 + q^2)} \frac{\partial \lambda}{\partial v} \\ & + \frac{\partial \Theta}{\partial u} (\Delta'' \lambda - \Delta' u) + \frac{\partial \Theta}{\partial v} (\Delta u - \Delta' \lambda) + \frac{\partial \Theta}{\partial \lambda} \left\{ (\Delta'' \lambda - \Delta' u) \frac{\partial \lambda}{\partial u} \right. \\ & + (\Delta u - \Delta' \lambda) \frac{\partial \lambda}{\partial v} \left. \right\} + \Theta u \left( \frac{\partial \Delta}{\partial v} - \frac{\partial \Delta'}{\partial u} \right) + \Theta \lambda \left( \frac{\partial \Delta''}{\partial u} - \frac{\partial \Delta'}{\partial v} \right) \\ & + \Theta \Delta'' \frac{\partial \lambda}{\partial u} - \Theta \Delta' \left( \frac{\partial \lambda}{\partial v} + 1 \right). \end{aligned}$$

If we substitute for the differences  $(\partial \Delta / \partial v) - (\partial \Delta' / \partial u)$ ,  $(\partial \Delta'' / \partial u) - (\partial \Delta' / \partial v)$  their values as given by the equations (4) of Codazzi, and for  $\partial \lambda / \partial u$ ,  $\partial \lambda / \partial v$  those which follow from (A), we have

$$\begin{aligned} \Omega = & \frac{\partial}{\partial u} \left[ \frac{(qu + p\lambda)(\lambda t - us)}{u(1 + p^2 + q^2)} \right] + \frac{\partial}{\partial v} \left[ \frac{(qu + p\lambda)(ur - \lambda s)}{u(1 + p^2 + q^2)} \right] \\ & + \frac{2pt\lambda + u(qt - ps)}{u(1 + p^2 + q^2)} \left\{ \frac{(qu + p\lambda)(ur - \lambda s)}{u(1 + p^2 + q^2)} + \Theta (\Delta u - \Delta' \lambda) \right\} \\ & + \frac{-2ps\lambda + u(pr - qs)}{u(1 + p^2 + q^2)} \left\{ -1 + \frac{(qu + p\lambda)(us - \lambda t)}{u(1 + p^2 + q^2)} \right. \\ & + \Theta (\Delta' u - \Delta'' \lambda) \left. \right\} + \frac{\partial \Theta}{\partial u} (\Delta'' \lambda - \Delta' u) + \frac{\partial \Theta}{\partial v} (\Delta u - \Delta' \lambda) \\ & + \frac{\partial \Theta}{\partial \lambda} \left\{ (\Delta u - \Delta' \lambda) \left( -1 + \frac{(qu + p\lambda)(us - \lambda t)}{u(1 + p^2 + q^2)} \right) \right. \\ & + \frac{(qu + p\lambda)(ur - \lambda s)}{u(1 + p^2 + q^2)} (\Delta'' \lambda - \Delta' u) \left. \right\} \\ & + \frac{\Theta (qu + p\lambda)}{1 + p^2 + q^2} \left( 2s\Delta' - t\Delta - r\Delta'' + \frac{ur - \lambda s}{u} \Delta'' + \frac{\lambda t - us}{u} \Delta' \right) \\ & + \Theta^2 [(\Delta u - \Delta' \lambda) \Delta'' - (\Delta' u - \Delta'' \lambda) \Delta']. \end{aligned} \tag{13}$$

Because of the equation (4) of Gauss the coefficient of  $\Theta^2$  in equation (13) is reducible to  $u(rt - s^2)/(1 + p^2 + q^2)^2$ , and  $\Omega$  is seen to be linear in  $\Delta$ ,  $\Delta'$ ,  $\Delta''$ , so that we may write

$$(13^*) \quad \Omega = a\Delta + b\Delta' + c\Delta'' + d,$$

where  $a, b, c, d$  are functions of  $u, v, \lambda$ . Thus on our hypothesis that  $\Omega = 0$

whatever be  $\Delta, \Delta', \Delta''$ , provided only that they satisfy the Gauss and Codazzi equations, it follows from known considerations (cf. *Lezioni*, vol. 2, p. 254) that the functions  $a, b, c, d$  must vanish separately, and that whatever be  $\lambda$ . We turn to the discussion of these conditions.

4. THE FUNCTION  $\Theta^2$  AS A POLYNOMIAL OF THE SECOND DEGREE IN  $\lambda$

We commence with the fourth coefficient  $d$ , which from the above observations is found to be

$$d = \Theta^2 \frac{u(rt - s^2)}{(1 + p^2 + q^2)^2} + \frac{\partial}{\partial u} \left[ \frac{(qu + p\lambda)(\lambda t - us)}{u(1 + p^2 + q^2)} \right] \\ + \frac{\partial}{\partial v} \left[ \frac{(qu + p\lambda)(ur - \lambda s)}{u(1 + p^2 + q^2)} \right] + \frac{2ps\lambda + u(qs - pr)}{u(1 + p^2 + q^2)} \\ + \frac{qu + p\lambda}{u^2(1 + p^2 + q^2)^2} \{ (ur - \lambda s)[2pt\lambda + u(qt - ps)] \\ + (us - \lambda t)[-2ps\lambda + u(pr - qs)] \}.$$

In the last term the coefficient within the brackets  $\{ \}$  of

$$\frac{qu + p\lambda}{u^2(1 + p^2 + q^2)^2}$$

is reducible to an expression linear in  $\lambda$ , namely

$$u(qu + p\lambda)(rt - s^2),$$

and, if the differentiation indicated is carried out, it is found that the third derivatives of  $z_0$  cancel one another and the resulting expression for  $d$  is

$$d = \Theta^2 u \frac{rt - s^2}{(1 + p^2 + q^2)^2} + \left\{ \frac{(rt - s^2)(1 - p^2 + q^2)}{u(1 + p^2 + q^2)^2} - \frac{pt}{u^2(1 + p^2 + q^2)} \right\} \lambda^2 \\ - \frac{4pq(rt - s^2)}{(1 + p^2 + q^2)^2} \lambda + \frac{u(rt - s^2)(1 + p^2 - q^2)}{(1 + p^2 + q^2)^2} + \frac{2ps}{u(1 + p^2 + q^2)} \lambda \\ - \frac{pr}{1 + p^2 + q^2} + \frac{rt - s^2}{u(1 + p^2 + q^2)^2} (p^2 \lambda^2 + 2pqu\lambda + q^2 u^2).$$

On the assumption that  $S_0$  is not a developable surface we have  $rt - s^2 \neq 0$ , so that when  $d$  is equated to zero, we find that  $\Theta^2$  is a quadratic in  $\lambda$  of the form

$$(14) \quad \Theta^2 = A\lambda^2 + 2B\lambda + C,$$

where the functions  $A, B, C$  have the values



$$\begin{aligned}
 (15) \quad A &= \frac{pt(1+p^2+q^2)}{u^3(rt-s^2)} - \frac{1+q^2}{u^2}, \\
 B &= -\frac{ps(1+p^2+q^2)}{u^2(rt-s^2)} + \frac{pq}{u}, \\
 C &= \frac{pr(1+p^2+q^2)}{u(rt-s^2)} - (1+p^2).
 \end{aligned}$$

Because of the form (14) of  $\Theta^2$  one has at once a consequence with respect to the type of cone enveloped by the planes of the  $\infty^1$  facettes  $f'$  associated with a facette  $f$ . The formula (8), which gives the direction-cosines of the planes of  $f'$  is reducible by (7\*), (12), and (14) to

$$\begin{aligned}
 (16) \quad X' \equiv & \{(1+q^2)\lambda - pqu\} \frac{\partial x}{\partial u} + \{-pq\lambda + (1+p^2)u\} \frac{\partial x}{\partial v} \\
 & + u\sqrt{1+p^2+q^2}\sqrt{A\lambda^2 + 2B\lambda + C} \cdot X.
 \end{aligned}$$

Since, for fixed values of  $u$  and  $v$ , these expressions are linear in  $\lambda$  and the radical  $\sqrt{A\lambda^2 + 2B\lambda + C}$ , we have the result:

*The cone enveloped by the planes of the  $\infty^1$  facettes  $f'$ , associated with a facette  $f$ , must be a quadric cone (with vertex at the center of  $f$ ).*

##### 5. THE DIFFERENTIAL EQUATIONS FOR $A, B, C$

We calculate now the other three coefficients  $a, b, c$  in (13\*) which also must be equated to zero. It is found that

$$\begin{aligned}
 a &= u \frac{\partial \Theta}{\partial v} + \left\{ \frac{(qu + p\lambda)(us - \lambda t)}{1 + p^2 + q^2} - u \right\} \frac{\partial \Theta}{\partial \lambda} + \frac{p(\lambda t - us)}{1 + p^2 + q^2} \Theta, \\
 b &= -u \frac{\partial \Theta}{\partial u} - \lambda \frac{\partial \Theta}{\partial v} + \left\{ \frac{(qu + p\lambda)(\lambda t - us)}{u(1 + p^2 + q^2)} \lambda + \lambda \right. \\
 &\quad + \left. \frac{(qu + p\lambda)(\lambda s - ur)}{1 + p^2 + q^2} \right\} \frac{\partial \Theta}{\partial \lambda} + \frac{2(qu + p\lambda)}{1 + p^2 + q^2} \Theta \\
 &\quad + \frac{(qu + p\lambda)(\lambda t - us)}{u(1 + p^2 + q^2)} \Theta + \frac{-2p\lambda s + u(pr - qs)}{1 + p^2 + q^2} \Theta \\
 &\quad - \lambda \frac{2pt\lambda + u(qt - ps)}{u(1 + p^2 + q^2)} \Theta, \\
 c &= \lambda \left\{ \frac{\partial \Theta}{\partial u} + \frac{(qu + p\lambda)(ur - \lambda s)}{u(1 + p^2 + q^2)} \frac{\partial \Theta}{\partial \lambda} + \frac{p(\lambda s - ur)}{u(1 + p^2 + q^2)} \Theta \right\}.
 \end{aligned}$$

It is readily seen that  $b$  is a linear combination of  $a$  and  $c$ , so that there remains for consideration only  $a = 0, c = 0$ . If these be multiplied by  $\Theta$ ,

we have on account of (14)

$$\begin{aligned}\Theta \frac{\partial \Theta}{\partial \lambda} &= A\lambda + B, \\ \Theta \frac{\partial \Theta}{\partial u} &= \frac{1}{2} \frac{\partial A}{\partial u} \lambda^2 + \frac{\partial B}{\partial u} \lambda + \frac{1}{2} \frac{\partial C}{\partial u}, \\ \Theta \frac{\partial \Theta}{\partial v} &= \frac{1}{2} \frac{\partial A}{\partial v} \lambda^2 + \frac{\partial B}{\partial v} \lambda + \frac{1}{2} \frac{\partial C}{\partial v};\end{aligned}$$

so that these two equations are equivalent to

$$\begin{aligned}u \left( \frac{1}{2} \frac{\partial A}{\partial v} \lambda^2 + \frac{\partial B}{\partial v} \lambda + \frac{1}{2} \frac{\partial C}{\partial v} \right) + (A\lambda + B) \left( \frac{(qu + p\lambda)(us - \lambda t)}{1 + p^2 + q^2} - u \right) \\ + \frac{p(\lambda t - us)}{1 + p^2 + q^2} (A\lambda^2 + 2B\lambda + C) = 0, \\ \frac{1}{2} \frac{\partial A}{\partial u} \lambda^2 + \frac{\partial B}{\partial u} \lambda + \frac{1}{2} \frac{\partial C}{\partial u} + (A\lambda + B) \frac{(qu + p\lambda)(ur - \lambda s)}{1 + p^2 + q^2} \\ + \frac{p(\lambda s - ur)}{u(1 + p^2 + q^2)} (A\lambda^2 + 2B\lambda + C) = 0.\end{aligned}$$

The first members of these equations (since the terms in  $\lambda^3$  cancel one another) are quadratics in  $\lambda$ , which from our hypothesis must *vanish identically*. It results that *all* the first derivatives of  $A$ ,  $B$ , and  $C$  are expressible linearly in terms of  $A$ ,  $B$ ,  $C$  as follows:

$$\begin{aligned}\frac{\partial A}{\partial u} &= \frac{2qs}{1 + p^2 + q^2} A - \frac{2ps}{u(1 + p^2 + q^2)} B, \\ \frac{\partial B}{\partial u} &= \frac{-qur}{1 + p^2 + q^2} A + \frac{pr + qs}{1 + p^2 + q^2} B - \frac{ps}{u(1 + p^2 + q^2)} C, \\ \frac{\partial C}{\partial u} &= \frac{-2qur}{1 + p^2 + q^2} B + \frac{2pr}{1 + p^2 + q^2} C, \\ (17) \quad \frac{\partial A}{\partial v} &= \frac{2qt}{1 + p^2 + q^2} A - \frac{2pt}{u(1 + p^2 + q^2)} B, \\ \frac{\partial B}{\partial v} &= \left( 1 - \frac{qus}{1 + p^2 + q^2} \right) A + \frac{ps + qt}{1 + p^2 + q^2} B - \frac{pt}{u(1 + p^2 + q^2)} C, \\ \frac{\partial C}{\partial v} &= 2 \left( 1 - \frac{qus}{1 + p^2 + q^2} \right) B + \frac{2ps}{1 + p^2 + q^2} C.\end{aligned}$$

Now, by means of (15), the functions  $A$ ,  $B$ ,  $C$  are expressible in terms of a single unknown function  $z_0$  and its first and second derivatives  $p$ ,  $q$ ,  $r$ ,  $s$ ,  $t$ , so that the system (17) is a system of the *third order in partial derivatives* of  $z_0$ , which must be studied and integrated. But before doing this it will be useful to obtain from these equations a consequence with regard to the discriminant  $\nabla = B^2 - AC$  of the quadratic  $\Theta^2$  in  $\lambda$ . With the aid of (17) we find

$$\frac{\partial \nabla}{\partial u} = \frac{2(pr + qs)}{1 + p^2 + q^2} \nabla = \nabla \cdot \frac{\partial}{\partial u} \log(1 + p^2 + q^2),$$

$$\frac{\partial \nabla}{\partial v} = \frac{2(ps + qt)}{1 + p^2 + q^2} \nabla = \nabla \cdot \frac{\partial}{\partial v} \log(1 + p^2 + q^2),$$

whence on integration

$$(18) \quad \nabla = B^2 - AC = k(1 + p^2 + q^2),$$

where  $k$  is a constant. In the geometrical problem there is an essential difference between the cases  $k \neq 0$ , and  $k = 0$ . In the first case ( $k \neq 0$ ), as  $\Theta^2$  is not a perfect square, the planes of the  $\infty^1$  facettes  $f'$ , associated with a facette  $f$ , envelope a quadric cone. In the second case ( $k = 0$ ), the polynomial  $\Theta^2$  is a perfect square and the cone decomposes into two pencils of planes.

#### 6. APPLICATION OF THE TRANSFORMATION OF LEGENDRE

For the integration of the system (17), where  $A$ ,  $B$ ,  $C$  have the values (15), it is opportune to effect a change of the independent variables and of the unknown function given by the *transformation of Legendre*, as suggested by the form of the system (17). Thus we take  $p$  and  $q$  for independent variables (which is possible as  $S_0$  is not developable), and for the unknown function

$$Z_0 = pu + qv - z_0.$$

The notation of Monge, namely  $P$ ,  $Q$ ;  $R$ ,  $S$ ,  $T$ , referring to the first and second derivatives of  $Z_0$  with respect to  $p$  and  $q$ , will be used as above, and we have the known formulas

$$P = u, \quad Q = v, \quad R = \frac{t}{rt - s^2}, \quad S = \frac{-s}{rt - s^2}, \quad T = \frac{r}{rt - s^2}.$$

Now

$$Rr + Ss = 1, \quad Sr + Ts = 0,$$

$$Rs + St = 0, \quad Ss + Tt = 1,$$

so that the formulas (15) become

$$\begin{aligned}
 (19) \quad A &= \frac{p(1+p^2+q^2)}{P^3} R - \frac{1+q^2}{P^2}, \\
 B &= \frac{p(1+p^2+q^2)}{P^2} S + \frac{pq}{P}, \\
 C &= \frac{p(1+p^2+q^2)}{P} T - (1+p^2).
 \end{aligned}$$

By transforming equations (17) in accordance with the formulas of differentiation

$$\frac{\partial}{\partial p} = R \frac{\partial}{\partial u} + S \frac{\partial}{\partial v}, \quad \frac{\partial}{\partial q} = S \frac{\partial}{\partial u} + T \frac{\partial}{\partial v},$$

we have the result

$$\begin{aligned}
 \frac{\partial A}{\partial p} &= 0, \\
 \frac{\partial A}{\partial q} &= \frac{2q}{1+p^2+q^2} A - \frac{2p}{P(1+p^2+q^2)} B, \\
 \frac{\partial B}{\partial p} &= \left( S - \frac{qP}{1+p^2+q^2} \right) A + \frac{p}{1+p^2+q^2} B, \\
 \frac{\partial B}{\partial q} &= TA + \frac{q}{1+p^2+q^2} B - \frac{p}{P(1+p^2+q^2)} C, \\
 \frac{\partial C}{\partial p} &= 2 \left( S - \frac{qP}{1+p^2+q^2} \right) B + \frac{2p}{1+p^2+q^2} C, \\
 \frac{\partial C}{\partial q} &= 2TB.
 \end{aligned}$$

Finally, if the expressions from (19) are substituted in these equations, we find that the six first derivatives of  $R$ ,  $S$ ,  $T$ , necessarily satisfying the conditions

$$\frac{\partial R}{\partial q} = \frac{\partial S}{\partial p}, \quad \frac{\partial S}{\partial q} = \frac{\partial T}{\partial p},$$

are given by the following equations:

$$\begin{aligned}
 (B) \quad \frac{\partial R}{\partial p} &= \frac{3}{P} R^2 - \frac{3R}{p}, \\
 \frac{\partial R}{\partial q} &= \frac{\partial S}{\partial p} = 3 \frac{RS}{P} - 2 \frac{S}{p}, \\
 \frac{\partial S}{\partial q} &= \frac{\partial T}{\partial p} = 2 \frac{S^2}{P} + \frac{RT}{P} - \frac{T}{p}, \\
 \frac{\partial T}{\partial q} &= 3 \frac{ST}{P}.
 \end{aligned}$$

This differential system (B) gives all the *third* derivatives of  $Z_0$  expressed in terms of the first and second derivatives,  $P, Q, R, S, T$ . Moreover, the conditions of integrability of equations (B), namely

$$\begin{aligned}\frac{\partial}{\partial q} \left( \frac{3}{P} R^2 - \frac{3R}{p} \right) &= \frac{\partial}{\partial p} \left( 3 \frac{RS}{P} - 2 \frac{S}{p} \right), \\ \frac{\partial}{\partial q} \left( 3 \frac{RS}{P} - 2 \frac{S}{p} \right) &= \frac{\partial}{\partial p} \left( 2 \frac{S^2}{P} + \frac{RT}{P} - \frac{T}{p} \right), \\ \frac{\partial}{\partial q} \left( 2 \frac{S^2}{P} + \frac{RT}{P} - \frac{T}{p} \right) &= \frac{\partial}{\partial p} \left( 3 \frac{ST}{P} \right),\end{aligned}$$

are readily found to be satisfied in consequence of (B). Hence the system (B) is completely integrable, so that from this point of view we know that the problem (A) admits *solutions depending on six arbitrary constants*, since, in fact, the values of the unknown function and its five first and second derivatives are arbitrary for initial values of the independent variables.

#### 7. INTEGRATION OF THE DIFFERENTIAL SYSTEM (B)

In order to integrate the system (B) the ordinary method of solution of a completely integrable system could be used, but a more expeditious course is the following. From (B) we take the three equations

$$\frac{\partial R}{\partial p} = \frac{3R^2}{P} - \frac{3R}{p}, \quad \frac{\partial S}{\partial p} = 3 \frac{RS}{P} - \frac{2S}{p}, \quad \frac{\partial T}{\partial q} = 3 \frac{ST}{P},$$

which give by a first integration

$$(20) \quad R = \frac{P^3}{p^3} \psi(q), \quad S = \frac{P^3}{p^2} \theta(q), \quad T = P^3 \phi(p),$$

when  $\psi(q)$  and  $\theta(q)$  are functions of  $q$  alone and  $\phi(p)$  is a function of  $p$  alone. These functions must be such that the system (B) is satisfied. For this we observe in the first place that since  $dP = Rdp + Sdq$ , we have from (20),

$$(21) \quad \frac{dP}{P^3} = \frac{\psi(q) dp}{p^3} + \frac{\theta(q) dq}{p^2}.$$

The condition of integrability of the second member necessitates

$$\theta(q) = -\frac{1}{2}\psi'(q), \quad \left( \psi'(q) \equiv \frac{d\psi}{dq} \right),$$

so that (21) can be integrated in the form

$$\frac{1}{P^2} = \frac{\psi(q)}{p^2} + c,$$

$c$  being constant, or

$$(22) \quad \frac{p^2}{P^2} = \psi(q) + cp^2.$$

Thus we have

$$S = -\frac{1}{2} \frac{P^3}{p^2} \psi'(q),$$

whence

$$\frac{\partial S}{\partial q} = \frac{3}{4} \frac{P^5}{p^4} \psi'^2(q) - \frac{1}{2} \frac{P^3}{p^2} \psi''(q),$$

so that the third of (B) becomes

$$\frac{1}{4} \frac{P^5}{p^4} \psi'^2(q) - \frac{1}{2} \frac{P^3}{p^2} \psi''(q) = \frac{P^5}{p^3} \phi(p) \psi(q) - \frac{P^3}{p} \phi(p).$$

This equation multiplied by  $4p^4/P^5$  is reducible by means of (22) to

$$(23) \quad 2\psi''(q)(\psi(q) + cp^2) - \psi'^2(q) = 4cp^3 \phi(p).$$

We shall find that the cases  $c \neq 0$  and  $c = 0$  have different geometrical significance, and so we treat these cases separately.

1° *Case.*  $c \neq 0$ . If in (23) we take for  $q$  any constant value whatever, then necessarily

$$p^3 \phi(p) = \alpha p^2 + \beta,$$

where  $\alpha$  and  $\beta$  are constants, and from (23) results  $\psi''(q) = 2\alpha$ , so that

$$(24) \quad \psi(q) = \alpha q^2 + 2\gamma q + \delta,$$

where  $\gamma$  and  $\delta$  are new constants. In order that these values shall satisfy (23) it is necessary and sufficient that the constants  $c, \alpha, \beta, \gamma, \delta$  be in the relation

$$(25) \quad \alpha\delta - \gamma^2 = c\beta,$$

whence we have the definitive formulas

$$(26) \quad R = \frac{P^3}{p^3} (\alpha q^2 + 2\gamma q + \delta), \quad S = -\frac{P^3}{p^2} (\alpha q + \gamma),$$

$$T = \frac{P^3}{p^3} (\alpha p^2 + \beta),$$

together with

$$(27) \quad \frac{p^2}{P^2} = \alpha q^2 + 2\gamma q + \delta + cp^2.$$

Now we observe that conversely these expressions, in which the constants are in the relation (25), satisfy all of equations (B), including the last which was not considered above.

2° *Case.*  $c = 0$ . We will show that the expressions (26) hold also in this case when in them we put  $c = 0$ . In fact, (23) reduces now to

$$2\psi(q) \cdot \psi''(q) = \psi'^2(q),$$

of which the integral is given by (24) with  $\alpha\delta - \gamma^2 = 0$ , in conformity with (25). At the same time we find that the first two of (26) hold also. With respect to the third we take  $T = P^3 \phi(p)$  as given by (20), subject to the condition

$$\frac{\partial T}{\partial p} = 2 \frac{S^2}{P} + \frac{RT}{P} - \frac{T}{p},$$

for the determination of  $\phi(p)$ . This gives

$$\begin{aligned} \frac{2P^5}{p^3} (\alpha q^2 + 2\gamma q + \delta) \phi(p) + P^3 \left( \phi'(p) - \frac{\phi(p)}{p} \right) \\ = 2 \frac{P^5}{p^2} (\alpha q + \gamma)^2 - \frac{P^3}{p} \phi(p). \end{aligned}$$

Multiplying by  $p^4/P^5$  and reducing with the aid of (27), we get

$$p^3 \phi'(p) + 3p^2 \phi(p) = 2\alpha p,$$

of which the integral is

$$p^3 \phi(p) = \alpha p^2 + \beta,$$

where  $\beta$  is a constant. This gives the expression (26) for  $T$ , and consequently (26) and (27) hold when  $c = 0$ .

## 8. GEOMETRIC INTERPRETATION OF THE SOLUTION

It remains for us to complete the integration by calculating by quadratures  $Q$  and  $Z_0$  from (26) and (27), which process introduces two new arbitrary constants. Thus we have the complete solution of the problem (A) which depends effectively on six arbitrary constants, as remarked in § 6. We have now to give the geometric interpretation of these results. In doing so, we separate the cases  $c \neq 0$  and  $c = 0$ .

1° *Case.*  $c \neq 0$ . Substituting from (26) and (27) in  $dQ = Sdp + Tdq$ , we have

$$dQ = \frac{-p(\alpha q + \gamma) dp + (\alpha p^2 + \beta) dq}{(\alpha q^2 + 2\gamma q + \delta + cp^2)^{3/2}},$$

of which the integral is

$$Q = \frac{\alpha q + \gamma}{c(\alpha q^2 + 2\gamma q + \delta + cp^2)^{1/2}} + h,$$

$h$  being a constant.

In like manner

$$dZ_0 = Pdp + Qdq = \frac{cpdp + (\alpha q + \gamma) dq}{c(\alpha q^2 + 2\gamma q + \delta + cp^2)^{1/2}} + hdq,$$

whence we have

$$Z_0 = \frac{1}{c} \sqrt{\alpha q^2 + 2\gamma q + \delta + cp^2} + hq + k,$$

where  $k$  is a constant.

Having regard to the formulas of transformation of Legendre (§ 6), we can write the parametric equations of our surface  $S_0$ , expressing the coördinates  $x_0, y_0, z_0$  of a moving point in terms of parameters  $p$  and  $q$ , in the form

$$(28) \quad x_0 = P, \quad y_0 = Q, \quad z_0 = pP + qQ - Z_0,$$

which in consequence of the preceding are reducible to

$$x_0 = \frac{p}{\sqrt{\alpha q^2 + 2\gamma q + \delta + cp^2}}, \quad y_0 = \frac{\alpha q + \gamma}{c \sqrt{\alpha q^2 + 2\gamma q + \delta + cp^2}} + h,$$

$$z_0 + k = \frac{-(\gamma q + \delta)}{c \sqrt{\alpha q^2 + 2\gamma q + \delta + cp^2}}.$$

These equations, because of the relation (25) between the constants, define the *central quadric* with the equation

$$(29) \quad \beta x_0^2 + \delta (y_0 - h)^2 + 2\gamma (y_0 - h)(z_0 + k) + \alpha (z_0 + k)^2 = \frac{\beta}{c},$$

for which the plane  $x = 0$  is a principal plane. Conversely, we note that the presence of six arbitrary constants enables us to identify the quadric (29) with any central quadric *whatever* for which  $x = 0$  is a plane of symmetry. It should be observed that we must take  $\beta \neq 0$ , otherwise the quadric degenerates into a pair of planes.

2° *Case.*  $c = 0$ . Since  $\alpha\delta - \gamma^2 = 0$ , the expression  $\alpha q^2 + 2\gamma q + \delta$  is a perfect square of a linear expression, say  $aq + b$  ( $a$  and  $b$  constants), with  $\alpha = a^2, \gamma = ab, \delta = b^2$ , and the formulas for  $P, R, S, T$  become

$$(30) \quad P = \frac{p}{aq + b}, \quad R = \frac{1}{aq + b}, \quad S = \frac{-ap}{(aq + b)^2}, \quad T = \frac{a^2 p^2 + \beta}{(aq + b)^3}.$$

In order to obtain the expressions for  $Q$  and  $Z_0$ , it is necessary to distinguish between the cases  $a \neq 0$  and  $a = 0$ . If  $a \neq 0$ , we find successively

$$Q = \frac{-(a^2 p^2 + \beta)}{2a(aq + b)^2} + h, \quad Z_0 = \frac{a^2 p^2 + \beta}{2a^2(aq + b)} + hq + k,$$

where  $h$  and  $k$  are constants. Then  $S_0$  is given by

$$x_0 = \frac{p}{aq + b}, \quad y_0 = h - \frac{a^2 p^2 + \beta}{2a(aq + b)^2}, \quad z_0 = \frac{a^2 bp^2 - \beta(2aq + b)}{2a^2(aq + b)^2} - k.$$

Eliminating  $p$  and  $q$ , we find that  $S_0$  is the *paraboloid*



$$a\{a(z_0 + k) + b(y_0 - h)\}^2 + \beta\{2(y_0 - h) + ax_0^2\} = 0,$$

for which  $x = 0$  is the plane of symmetry, and furthermore it can be made to coincide with any paraboloid possessing this property.

Analogous results follow in the case  $a = 0$ . Then we have

$$P = \frac{p}{b}, \quad R = \frac{1}{b}, \quad S = 0, \quad T = \frac{\beta}{b^3},$$

whence

$$Q = \frac{\beta q}{b^3} + h, \quad Z_0 = \frac{p^2}{2b} + \frac{\beta q^2}{2b^3} + hq + k \quad (h, k \text{ constants}).$$

It results that

$$x_0 = \frac{p}{b}, \quad y_0 = \frac{\beta q}{b^3} + h, \quad z_0 = \frac{p^2}{2b} + \frac{\beta q^2}{2b^3} - k,$$

from which it is seen that  $S_0$  is the paraboloid

$$2(z_0 + k) = bx_0^2 + \frac{b^3}{\beta}(y_0 - h)^2.$$

Hence our researches as to the character of  $S_0$  which solves problem (A) yield the final result:

*The surfaces which solve problem A are all the quadrics having the fixed plane for plane of symmetry and only these.*

It remains for us to examine, for each type of quadric, how the  $\infty^1$  facettes  $f'$ , associated with a facette  $f$ , must be distributed. This distribution, as results from formulas (15), § 4, is fully determined; it is quadratic. If we wish to utilize properties already known in the theory of singular transformations  $B_k$  for the deforms of quadrics, we can at once infer, *at least for the general quadrics*, how the facettes  $f'$  must be distributed. But it is our aim, on the contrary, to discover the law of distribution of the facettes as a consequence of the data of the problem. Thus also in a new way we shall establish the foundations of the theory of (singular) transformations  $B_k$ .

#### 9. DISTRIBUTION OF FACETTES IN THE CASE OF THE GENERAL PARABOLOID

We commence with the case of the paraboloid whose parametric equations can be taken in the normal form

$$x_0 = u, \quad y_0 = v, \quad z_0 = \frac{1}{2}(au^2 + bv^2) \quad (a, b \text{ constants}).$$

Now we have

$$p = au, \quad q = bv, \quad r = a, \quad s = 0, \quad t = b, \quad rt - s^2 = ab,$$

and formulas (15) are simply

$$(31) \quad A = a^2, \quad B = abv, \quad C = \frac{a-b}{b}(1+a^2u^2) + abv^2.$$

It results that

$$\nabla = B^2 - AC = \left(1 - \frac{a}{b}\right)(1 + a^2u^2 + b^2v^2) = \left(1 - \frac{a}{b}\right)(1 + p^2 + q^2),$$

which is in conformity with the general formula (18) § 5, with  $k = 1 - (a/b)$ , so that  $k = 0$  only in the case of the paraboloid of rotation, excluded for the present and considered in particular in § 11.

Hence in the case of the general paraboloid ( $a \neq b$ ), the polynomial

$$(32) \quad \Theta^2 = a^2\lambda^2 + 2abv\lambda + \frac{a-b}{b}(1+a^2u^2) + abv^2$$

is not a perfect square, so that the planes of the  $\infty^1$  facettes associated with a facette  $f$  actually envelop a quadric cone (§ 5), whose geometric character is to be determined. It is necessary for this to recur to formulas (16), § 4, which assign values to  $X', Y', Z'$ , proportional to the direction-cosines of the normal to the plane of a facette  $f'$ , and since the form of the cone is independent of the deformation of  $S$ , we can consider  $S$  in the initial configuration  $S_0$ . Thus we have

$$x = u, \quad y = v, \quad z = \frac{1}{2}(au^2 + bv^2), \quad p = au, \quad q = bv,$$

$$X = \frac{-p}{\sqrt{1+p^2+q^2}}, \quad Y = \frac{-q}{\sqrt{1+p^2+q^2}}, \quad Z = \frac{1}{\sqrt{1+p^2+q^2}},$$

so that (16) become

$$(33) \quad \begin{aligned} X' &\equiv (1 + b^2v^2)\lambda - abu^2v - au^2\Theta, \\ Y' &\equiv -abuv\lambda + (1 + a^2u^2)u - buv\Theta, \\ Z' &\equiv au\lambda + buv + u\Theta, \end{aligned}$$

where

$$(33^*) \quad \Theta = \sqrt{a^2\lambda^2 + 2abv\lambda + \frac{a-b}{b}(1+a^2u^2) + abv^2}.$$

The equation of the plane enveloping the cone may be written,  $x, y, z$  indicating current coördinates,

$$(x - u)X' + (y - v)Y' + (z - \frac{1}{2}(au^2 + bv^2))Z' = 0.$$

This cuts the plane  $x = 0$  in the line

$$(34) \quad Y'y + Z'z = uX' + vY' + \frac{1}{2}(au^2 + bv^2)Z',$$

and the section of the cone made by the plane  $x = 0$  will be the conic enveloped by the line (34), as it moves with the variation of  $\lambda$ . We shall find

that this conic remains the same for all values of  $u$  and  $v$ , and coincides with the focal parabola in the plane  $x = 0$ . The equation in point coördinates of this parabola is

$$y^2 = \left(\frac{1}{b} - \frac{1}{a}\right)\left(2z - \frac{1}{a}\right),$$

and therefore its equation in homogeneous line coördinates  $\xi, \eta, \zeta$  is

$$(35) \quad (b - a)\xi^2 + b\eta^2 + 2ab\eta\zeta = 0.$$

From the expressions (33) for  $X', Y', Z'$  and (33\*) for  $\Theta$  it is seen that for all values of  $u, v, \lambda$  the identity

$$(b - a)Y'^2 + bZ'^2 - 2abZ'(uX' + vY' + \frac{1}{2}(au^2 + bv^2)Z') = 0,$$

holds, and therefore the line (34) with coördinates

$$\xi = Y', \quad \eta = Z', \quad \zeta = -(uX' + vY' + \frac{1}{2}(au^2 + bv^2)Z')$$

envelops the focal parabola (35). Consequently we have the result:

*In the case of the general paraboloids, the planes of the facettes  $f'$ , associated with a facette  $f$  of the paraboloid, envelop the cone which from the center of  $f$  projects the focal parabola in the fixed plane containing the centers of the facettes  $f'$ .*

### 10. CASE OF THE GENERAL CENTRAL QUADRIC

We come now to the other case where  $S_0$  is a central quadric with the fixed plane  $x = 0$  for a principal plane. Its parametric equations can be written in the normal form

$$x_0 = u, \quad y_0 = v, \quad z_0 = \sqrt{au^2 + bv^2 + c} \quad (a, b, c \text{ constants}),$$

whence we have

$$p = \frac{au}{z_0}, \quad q = \frac{bv}{z_0}, \quad 1 + p^2 + q^2 = \frac{z_0^2 + a^2u^2 + b^2v^2}{z_0^2},$$

$$r = \frac{a(bv^2 + c)}{z_0^3}, \quad s = -\frac{abuv}{z_0^3}, \quad t = \frac{b(au^2 + c)}{z_0^3}, \quad rt - s^2 = \frac{abc}{z_0^4}.$$

From the general formulas (15), § 4, we deduce at once the following expressions for  $A, B, C$ :

$$(36) \quad A = a \frac{a(a+1)u^2 + b(b+1)v^2 + c(a+1)}{cz_0^2},$$

$$B = av \frac{a(a+1)u^2 + b(b+1)v^2 + c(b+1)}{cz_0^2},$$

$$C = \frac{a(bv^2 + c)[a(a+1)u^2 + b(b+1)v^2 + c] - bc[a(a+1)u^2 + bv^2 + c]}{bcz_0^2},$$

and the discriminant  $\nabla = B^2 - AC$  of the polynomial

$$\Theta^2 = A\lambda^2 + 2B\lambda + C$$

is easily found to be

$$B^2 - AC = k \frac{z_0^2 + a^2 u^2 + b^2 v^2}{z_0^2} = k(1 + p^2 + q^2),$$

where  $k$  has the value

$$(37) \quad k = \frac{a}{bc} (a + 1)(a - b).$$

This result is in conformity with the general formula (18), § 5, and one sees that  $k = 0$  only in case  $a = -1$  or  $a = b$ , in which cases  $S_0$  is a quadric of revolution with the fixed plane  $x = 0$  for a meridian plane. Hence when the quadric is general, *in the sense that it is not a quadric of revolution about an axis contained in the fixed plane*, the planes of the  $\infty^1$  facettes  $f'$ , associated with a facette  $f$ , envelop a quadric cone (§ 5). We have yet to show that *this cone is the one which projects from the center of  $f$  the focal conic in the plane  $x = 0$* .

Proceeding as in the case of the paraboloids we calculate from (16) the values of  $X'$ ,  $Y'$ ,  $Z'$  when the surface  $S$  takes the initial configuration of the quadric  $S_0$ . Multiplying the values of  $X'$ ,  $Y'$ ,  $Z'$  by the factor  $z_0^2$ , we find

$$(38) \quad \begin{aligned} X' &\equiv (z_0^2 + b^2 v^2)\lambda - abu^2 v - au^2 z_0 \Theta, & \Theta &= \sqrt{A\lambda^2 + 2B\lambda + C}, \\ Y' &\equiv -abuv\lambda + (z_0^2 + a^2 u^2)u - buvz_0 \Theta, \\ Z' &\equiv auz_0 \lambda + buvz_0 + uz_0^2 \Theta. \end{aligned}$$

The plane of the facette  $f'$  intersects the plane  $x = 0$  in the line whose coördinates are

$$\xi = Y', \quad \eta = Z', \quad \zeta = -(uX' + vY' + z_0 Z').$$

The equation in point coördinates, of the focal conic in the plane  $x = 0$  is

$$\frac{aby^2}{c(b-a)} + \frac{az^2}{c(a+1)} = 1,$$

and in line coördinates

$$c(b-a)\xi^2 + bc(a+1)\eta^2 - ab\zeta^2 = 0.$$

It is easily verified that the expressions (38) satisfy identically the equation

$$c(b-a)Y'^2 + bc(a+1)Z'^2 - ab(uX' + vY' + z_0 Z')^2 = 0,$$

so that the cone enveloped by the planes of the facettes  $f'$ , associated with  $f$  with center  $u, v, z_0$ , is the one which from this plane projects the focal conic.

## 11. CASE OF QUADRICS OF REVOLUTION

We consider finally the case previously excluded where  $S_0$  is a quadric of revolution and *the fixed plane  $x = 0$  is a meridian plane*. This may be regarded as a limiting case of the general one, and consequently it is easy to understand from geometrical considerations what the law of distribution of the facettes  $f'$  becomes. We know from geometrical considerations that the focal conic degenerates, *as an envelope*, into a pair of points, the two foci of the meridian conic, or *principal foci*. The enveloping cone must break up correspondingly into two pencils of planes whose axes are the two lines projecting the principal focal points from the center of  $f$ . We can readily confirm this result with our formulas. It will be sufficient to consider only one of the two cases, for example that of the central quadric  $S_0$  with  $a = b$ . The formulas (36) become in this case

$$A = \frac{a(a+1)}{c}, \quad B = \frac{a(a+1)}{c}v, \quad C = \frac{a(a+1)}{c}v^2,$$

so that

$$\Theta = \pm \sqrt{\frac{a(a+1)}{c}}(\lambda + v).*$$

The formulas (38) give

$$X' \equiv (z_0^2 + a^2 v^2)\lambda - a^2 u^2 v \mp au^2 z_0 \sqrt{\frac{a(a+1)}{c}}(\lambda + v),$$

$$Y' \equiv -a^2 uv\lambda + (z_0^2 + a^2 u^2)u \mp auvz_0 \sqrt{\frac{a(a+1)}{c}}(\lambda + v),$$

$$Z' \equiv auz_0(\lambda + v) \pm uz_0^2 \sqrt{\frac{a(a+1)}{c}}(\lambda + v),$$

and one sees that between these values subsists the identity

$$uX' + vY' + z_0 Z' = \pm \frac{c}{a} \sqrt{\frac{a(a+1)}{c}} Z'.$$

It results that the planes of the facettes  $f'$ , whose equations are

$$(x - u)X' + (y - v)Y' + (z - z_0)Z' = 0,$$

all pass through one or the other of the points

$$x = 0, \quad y = 0, \quad z = \pm \frac{c}{a} \sqrt{\frac{a(a+1)}{c}},$$

\* If it is desired that the results be real, one must take  $c > 0, a > 0$ , or  $c > 0, a < -1$ . The first case is that of the hyperboloid of revolution of two sheets, the second the prolate ellipsoid.

which are precisely the principal foci. Similar results can be established for the paraboloids, in which case, however, it must be realized that there is only one proper principal focus. Hence we have the result:

*When in the solution of problem (A) one takes a quadric of revolution with fixed plane for a meridian plane, the  $\infty^1$  facettes  $f'$ , associated with a facette  $f$  of the quadric, have their planes distributed in a pencil whose axis is a line through the center of  $f$  projecting one or the other of the two principal foci.*

## 12. THE TRANSFORMED SURFACES $S'$

The researches set forth above have demonstrated that all the solutions of problem (A) are obtained on assuming that  $S_0$  is a quadric which has the fixed plane for a principal plane and by associating with each of its facettes  $f$  a simple infinity of facettes  $f'$ , according to the geometric law described, with reference to the focal conic in the general case, and the two principal foci in the special case. If now the quadric  $S_0$  assumes, by flexure, any form whatever  $S$ , and each facette  $f$  together with its associated  $\infty^1$  facettes  $f'$  are carried along in invariable relation, it will always be true that the  $\infty^3$  facettes  $f'$  will be distributed on  $\infty^1$  surfaces  $S'$ , transforms of  $S$ . Each  $S'$  and the primitive  $S$  are the focal surfaces of the congruence of lines joining corresponding points on these two surfaces. We will find that this is always a  $W$ -congruence, that is on  $S$  and  $S'$  conjugate systems correspond (or, what is the same thing, asymptotic lines correspond). For this it will be sufficient to prove\* that one of the focal surfaces, for instance  $S'$ , admits an infinitesimal deformation in which each of its points  $(x', y', z')$  is displaced parallel to the normal  $(X, Y, Z)$  to  $S$  at the corresponding point. If  $\rho$  denotes the unknown amplitude of this displacement, we must verify that the following three conditions are satisfied:

$$S \frac{\partial(\rho X)}{\partial u} \frac{\partial x'}{\partial u} = 0, \quad S \frac{\partial(\rho X)}{\partial v} \frac{\partial x'}{\partial v} = 0,$$

$$S \left( \frac{\partial(\rho X)}{\partial u} \frac{\partial x'}{\partial v} + \frac{\partial(\rho X)}{\partial v} \frac{\partial x'}{\partial u} \right) = 0.$$

Developing these expressions by means of (10), (11), and (A), § 2, we find that these conditions are reducible to the following two:

$$(39) \quad \frac{\partial \log \rho}{\partial u} = \frac{p(ur - \lambda s)}{u(1 + p^2 + q^2)} - \Delta' \Theta,$$

$$\frac{\partial \log \rho}{\partial v} = \frac{p(us - \lambda t)}{u(1 + p^2 + q^2)} - \Delta'' \Theta.$$

\* *Lezioni*, vol. 2, § 242.

By means of the calculations carried out in § 3 we can show that the conditions of integrability of (39) are satisfied, and consequently our assertion is proved.

We shall not carry on further any study of the transformations of deforms of quadrics from this point of view, but we will record that in the general case *all the transforms  $S'$  are applicable to one another and to the primitive surface  $S$* . We will add that also in the special case when  $S_0$  is a quadric of rotation with the fixed plane for a meridian plane it turns out that all of the transforms  $S'$  are applicable to one another and to a fixed quadric. However, the latter does not in this case coincide with the primitive quadric, but is a quadric (imaginary) of Darboux tangent at one point to the imaginary circle at infinity. The  $W$ -congruence which is obtained in this case, with its two focal surfaces applicable to two different quadrics, was discussed from an entirely different point of view in 1906 in my memoir inserted in volume 22 of the *Rendiconti del Circolo matematico di Palermo*. Thus it is seen that this congruence and those which arise from singular transformations  $B_k$  are coördinated by the geometrical principle set forth in problem (A) whose solution has been effected in this memoir.

PISA, ITALY

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