CONCERNING A SET OF POSTULATES FOR PLANE ANALYSIS SITUS*

BY

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My paper *On the foundations of plane analysis situs*† contains three sets of postulates, Σ_1, Σ_2, and Σ_3, expressed in terms of the undefined notions *point* and *region*. In the present paper I will show that every space S that satisfies Σ_1 or Σ_2 is a number plane, that is to say there exists, between S and a two-dimensional euclidean space S', a one-to-one correspondence that preserves limits.‡ This signifies that if P is a point and M is a point-set in S, and P' and M' are the corresponding point and point-set in S', then P is a limit point of M in the sense defined on page 132 of the above mentioned paper if, and only if, P' is a limit point of M' in the sense that every circle in S' that encloses P' encloses also a point of M' distinct from P'. It follows that Σ_1 and Σ_2 are both categorical with respect to§ *point* and *limit point as defined* on page 132. Moreover between every space S, satisfying Σ_1, and a two-dimensional euclidean space S' there exists a one-to-one correspondence preserving *point* and *region* if in S' the term region is interpreted to mean *Jordan region*. That is to say if the set of points M is a *region* in S then the set M' of corresponding points in S' is the interior of a simple closed curve and conversely. Thus Σ_1 is *absolutely*‖ categorical. The system Σ_2 is satisfied if in ordinary euclidean space of two dimensions the term region is interpreted as signifying Jordan region. It is however also satisfied if in such a space region is so interpreted

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† These Transactions, vol. 17 (1916), pp. 131–164. Hereafter this paper will be referred to as *Foundations*.

‡ This is not true of Σ_3. Indeed there exist spaces satisfying Σ_3 that are neither metrical, descriptive, nor separable. Cf. *Foundations*, pp. 131, 132, and 162. In the paper referred to in the Annals of Mathematics, vol. 16 (1915), p. 131, in the statement of Condition I, "$|y_k' - y_k''| < \epsilon$" is to be omitted in (1) and inserted (with the conjunction "and" prefixed) immediately after "$|x_k' - x_k''| = 0$" in (2). For certain results concerning the relation between Fréchet's Calcul Fonctionnel and spaces that satisfy Σ_1 or Σ_2 cf. E. W. Chittenden, Bulletin of the American Mathematical Society, vol. 23 (1917), p. 390.

§ Cf. my paper *The linear continuum in terms of point and limit*, Annals of Mathematics, loc. cit., p. 127.

‖ Loc. cit., p. 127.

as to apply to every bounded, connected set of points R of connected exterior such that (1) every point of R lies in the interior of some triangle that is contained in R, (2) every point of the boundary of R is a limit point of the exterior of R. Thus, though categorical with respect to *point* and *limit point*, Σ_2 is not *absolutely* categorical.

In my proof that Σ_1 is categorical I will make use of the following theorems.

THEOREM A. *If L is a simple closed curve and M and N are two closed point-sets with no point in common, there does not exist an infinite set of arcs G such that* (1) *each arc of the set G is a subset of L*, (2) *each arc of G contains a point of M and also a point of N*, (3) *no two arcs of G have a segment in common.*

*Proof.** Suppose there exists an infinite set of such arcs $A_1 B_1, A_2 B_2, \cdots$ where $A_1, A_2, \cdots$ are points of M and $B_1, B_2, \cdots$ are points of N. Then there exist points A and B and an infinite subsequence $A_{n_1} B_{n_1}, A_{n_2} B_{n_2}, \cdots$ such that A is a sequential limit point of $A_{n_1}, A_{n_2}, \cdots$ and B is a sequential limit point of $B_{n_1}, B_{n_2}, \cdots$. There exist on L four points C, D, E, and F in the order $ECADFB$. There exists a positive number δ such that if $m > \delta$ then A_{n_m} and B_{n_m} are on the segments CAD and EBF respectively of the curve L. It follows that if p, q, and r are three distinct positive integers greater than δ then two of the arcs $A_{n_p} B_{n_p}, A_{n_q} B_{n_q}, A_{n_r} B_{n_r}$ have in common either a segment from C to E or a segment from D to F. But this is contrary to supposition.

THEOREM B. *If J and L are two simple closed curves and A and B are two distinct points of J each of which is either not on L at all or on some segment* that is common to J and L, then there exists an arc from A to B that lies, except for its endpoints, entirely within J and has not more than a finite number of points in common with L.*‡

Proof. By Theorem 40§ there exists an arc AB that lies, except for its endpoints, entirely within J. If the segment AB and the closed curve L have points in common let M denote the set of all such common points. In view

* Cf. the proof of Theorem 37 on pages 152 and 153 of *Foundations.*

† If X and Y are two points of a closed curve J, there are just two arcs that lie wholly on J and have A and B as endpoints. Each of these arcs is called an *interval* of J. An interval minus its endpoints is a *segment.*

‡ The following example shows that this theorem would not be true if no restriction were imposed on the relative positions of A, B, J, and L other than merely that A and B should lie somewhere on J.

Example. Let J denote the perimeter of the triangle whose vertices are $(0, 0)$, $(2, 1)$, and $(2, -1)$. Let L be the closed curve bounded by the arc from $(0, 0)$ to $(1/\pi, 0)$ on the curve $y = x \sin(1/x)$, and straight line intervals from $(1/\pi, 0)$ to $(2, -2)$, from $(2, -2)$ to $(0, -2)$ and from $(0, -2)$ to $(0, 0)$ respectively. Let A and B be the points $(0, 0)$ and $(2, 1/2)$ respectively. Clearly there is no arc from A to B that lies, except for its endpoints, entirely within J and has only a finite number of points in common with L.

§ Numbered theorems are theorems in my paper *On the foundations of plane analysis situs,* loc. cit.

of the conditions of our hypothesis it is clear that there exists a point A_1 which is the first* point that the segment AB has in common with L. If there exist any points of M that can be joined to A_1 by an arc of L, that lies entirely within J then there must be a last such point. For otherwise there would exist on AB a point X which cannot be joined to A_1 by an arc of L lying entirely within J but which is a limit point of a set of points K that lie on AB and each of which can be so joined to A_1. But in this case there would exist on L a segment YXZ containing X and lying entirely within J. The segment YXZ must† contain at least one point P of the set K. But P could be joined to A_1 by an arc $A_1 P$ of L that lies entirely within J, and the point-set $A_1 P + YXZ$ would be an arc of L lying entirely within J. Thus the supposition that there exists no last point that can be properly‡ joined to A_1 leads to a contradiction. Let B_1 denote the last point on AB that can be properly joined to A_1. By an argument similar to the above proof of the non-existence of X it can be shown that B_1 cannot be a limit point of a set of points of M no one of which can be properly joined to A_1. But B_1 is the last point that can be properly joined to A_1. It follows that on AB there must be a point W, following B_1, such that the segment $B_1 W$ of the arc AB is entirely free of points of L. But L is a closed set of points. It follows that if there are on the segment AB any points of L that follow B_1 then there is a first such point, A_2. There exists§ a point B_2 which is the last point on AB that can be properly joined to A_2. If there are any points of M after B_2 there is|| a point A_3 which is the first such point. Continue this process. There must exist only a finite number of the points $A_1, A_2, A_3, \cdots$. For suppose there were infinitely many such points. Then there must be infinitely many points $B_1, B_2, B_3, \cdots$. Suppose n is a positive integer. The points B_n and A_{n+1} are the extremities of two arcs of L. Let $B_n A_{n+1}$ denote one of these arcs. The arc $B_n A_{n+1}$ does not lie wholly within J. Hence there exists a point X_n which is the first point that it has in common with J. Let $B_n X_n$ denote the arc from B_n to X_n on the arc $B_n A_{n+1}$. Let i and j be two positive integers. If the segments $B_i X_i$ and $B_{i+j} X_{i+j}$ had a point in common then B_{i+j} could be properly joined to B_i. But B_i can be properly joined to A_i. Hence B_{i+j} could be properly joined to A_i. But B_i is the last point of M that can be properly joined to A_i and B_{i+j} follows B_i. It follows that the segments $B_i X_i$ and $B_{i+j} X_{i+j}$ have no point in common. Thus if there

* Cf. *Foundations*, p. 139. Hereafter in this paper, unless the contrary is indicated, by "first point" and "last point" will be meant first and last point respectively in the order from A to B on the arc AB.

† Loc. cit., p. 139.

‡ The phrase "can be properly joined to A_1" is an abbreviation for "is identical with A_1 or can be joined to A_1 by an arc of L that lies entirely within J."

§ Cf. the above proof of the existence of B_1.

|| Cf. the above proof of the existence of A_2.

are infinitely many of the points $A_1, A_2, A_3, \cdots$ then there are infinitely many arcs, all lying on L, such that no two of them have, in common, any point unless possibly a common endpoint, and such that each of them has one endpoint on J and the other on the interval $A_1 O$ of the arc AB where O is the last point before B that AB has in common with L. But by Theorem A a simple closed curve cannot contain an infinite set of arcs of this type. It follows that there must exist only a finite number m of the points $A_1, A_2, A_3, \cdots$. It is clear that O is either A_m or B_m. For every positive integer n, less than m, if $A_n \neq B_n$, A_n can be joined to B_n by an arc $A_n Z_n B_n$ that belongs to L and lies entirely within J. It is easy to see, with the aid of Theorems 43 and 40, that there exists an arc $A_n \bar{Z}_n B_n$ that lies entirely within J and, except for its endpoints, is also entirely within L. Consider the point-set τ composed of $AA_1 + (A_1 \bar{Z} B_1) + B_1 A_2 + (A_2 \bar{Z}_2 B_2) + B_2 A_3 + \cdots + B_{m-1} A_m + (A_m \bar{Z}_m B_m) + OB$ where AA_1, $B_1 A_2$, $B_2 A_3$, $\cdots B_{m-1} A_m$ and OB are intervals of the arc AB and for each $i\,(1 \leqq i \leqq m)$ "$(A_i \bar{Z}_i B_i)$" denotes the arc $A_i \bar{Z} B_i$ or the point A_i according as B_i is not or is identical with A_i. The point-set $B_n A_{n+1}$ $(1 \leqq n \leqq m-1)$ contains in common with L no points except B_n and A_{n+1}. In view of these considerations it is clear that the point-set τ contains, as a subset, an arc from A to B that lies, except for its endpoints, entirely within J and has not more than $2(m+1)$ points in common with L. The truth of Theorem B is therefore established.

THEOREM C. *If the closed curve g has only a finite number of points in common with the closed curve $ABCDA$ and does not contain A, B, C, or D then the interior of $ABCDA$ can be divided by a double ruling* (such that the arcs of one of its two single rulings are parallel to AB and CD and those of the other are parallel to AD and BC) into subdivisions such that the interior of each one of them is either wholly within or wholly without g.*

Indication of proof. There exists a finite set K of arcs of g each of which lies entirely within $ABCDA$ except that its endpoints are on $ABCDA$ and such that every point of g that is within $ABCDA$ lies on an arc of K. If t is an arc of the set K, either (1) t has both endpoints on the same side† of

* If, on the closed curve $ABCDA$, $X_1, X_2, \cdots X_{n-1}, X_n, Y_1, Y_2, \cdots Y_{n-1}, Y_n, X'_n, X'_{n-1}, \cdots X'_2 X'_1, Y'_n, Y'_{n-1}, \cdots Y'_2 Y'_1$ are points in the order $AX_1 X_2 \cdots X_{n-1} X_n BY_1 Y_2 \cdots Y_{n-1} Y_n CX'_n X'_{n-1} \cdots X'_2 X'_1 DY'_n Y'_{n-1} \cdots Y'_2 Y'_1 A$ and $X_1 X'_1$, $X_2 X'_2$, $\cdots X_n X'_n$, $Y_1 Y'_1$, $Y_2 Y'_2$, $\cdots Y_n Y'_n$ are arcs which, except for their endpoints, lie entirely within $ABCDA$, and finally, for every i, j $(i \leqq i \leqq n,\ 1 \leqq j \leqq n)$, $X_i X'_i$ has just one point in common with $Y_j Y'_j$ and no point in common with $X_j X'_j$ (unless $i = j$), then these two sets of arcs are said to constitute a *double ruling* of the interior of $ABCDA$ (or merely a *double ruling* of $ABCDA$). The arcs $X_1 X'_1$, $X_2 X'_2$, $\cdots X_n X'_n$ are said to be parallel to BC and to AD and the arcs $Y_1 Y'_1$, $Y_2 Y'_2$, $\cdots Y_n Y'_n$ are said to be parallel to AB and to CD. Each of these two sets of arcs is a *single ruling* of the double ruling formed by the two sets combined. Of course the word parallel is not used here in the same sense as in ordinary plane geometry.

† The segments AB, BC, CD, and DA of the closed curve $ABCDA$ are called the sides of $ABCDA$. By *the interval AB* of J will be meant that one which does not contain C.

$ABCDA$, (2) t has one endpoint on one side, and the other one on an adjacent side, of $ABCDA$, or (3) t has one endpoint on one side, and the other one on an opposite side, of $ABCDA$. Let us suppose, for example, that there are four arcs $A_1 B_1 C_1 D_1 E_1$, $A_2 B_2 C_2 D_2 E_2$, $A_3 B_3 C_3 D_3 E_3$, and $A_4 B_4 C_4 D_4 E_4$, of type 1, two arcs $A_5 B_5 E_5$ and $A_6 B_6 E_6$, of type 2, and one arc $A_7 E_7$ of type 3, situated as indicated in Fig. 1. In this figure a double ruling τ, satisfying the conditions of Theorem C, is represented. This indication of how a ruling may be constructed in the case of this typical example is given instead of a tedious formal proof covering all possible cases. Here the arcs of τ that are parallel to AB are $K_1 D_1 C_1 B_1 L_1$, $K_2 D_2 C_2 B_2 L_2$, $A_4 B_4 L_3$, $A_3 B_3 L_4$, $E_3 D_3 L_5$, $E_4 D_4 L_6$, $E_7 A_7$, $K_3 B_6 A_6$, and $K_4 B_5 A_5$. Those parallel to BC are $H_1 B_3 C_3 D_3 M_6$, $H_2 B_4 C_4 D_4 M_5$, $E_2 D_2 M_4$, $E_1 D_1 M_3$, $A_1 B_1 M_2$, $A_2 B_2 M_1$, $H_3 B_6 E_6$, and $H_4 B_5 E_5$.

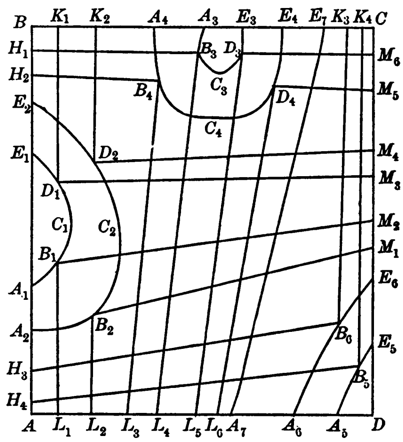

FIG. 1

THEOREM D. *If $ABCDA$ is a closed curve and G is a set of closed curves and each point on or within $ABCDA$ is within some curve of the set G then the interior of $ABCDA$ can be divided by a double ruling (such that the arcs of one of its single rulings are parallel to AB and CD and those of the other are parallel to AD and BC) into subdivisions each of which is within some curve of the set G.*

Proof. There exists a finite set $\overline{G}$ of closed curves g_1, g_2, $\cdots$ g_m such that every one of them is a curve of the set G and such that every point on or within J* is within one of them. Every point of the interval AB of J is within some curve of the set $\overline{G}$. It follows that for each point of AB there is a segment of J, containing that point, lying wholly within some curve of $\overline{G}$ and not containing any side of $ABCDA$. There exists a finite subset, Q, of these segments such that every point of AB is on some segment of Q. The set Q contains as a subset a set of segments $A_1 B_1$, $A_2 B_2$, $\cdots$ $A_n B_n$ with endpoints in the order $A_1 A A_2 B_1 A_3 B_2 A_4 B_3 A_5 B_4 \cdots A_n B_{n-1} B B_n$. There exists a set of points $E_1, E_2, \cdots E_{n-1}$ and arcs $E_1 E_1'$, $E_2 E_2'$, $\cdots E_{n-1} E_{n-1}'$ such that, for every i and j $(1 \leqq i \leqq n-1,\ 1 \leqq j \leqq n-1,\ i \neq j)$, (1) E_i is in the order $A A_{i+1} E_i B_i B$, (2) $E_i E_i'$ lies within every region of the set $\overline{G}$ that contains E_i, (3) every point of $E_i E_i'$ except E_i is within J, (4) $E_i E_i'$ and $E_j E_j'$ have no point in common, (5) not every point of the segment $E_i E_i'$ is on g_1. There exists a set of curves $\bar{g}_1$, $\bar{g}_2$, $\cdots$ g_n, belonging to the set $\overline{G}$, such that $A_i B_i$ $(1 \leqq i \leqq n)$ is entirely within $\bar{g}_i$. Within $\bar{g}_1$ (and

* Hereafter the letter J will be used, at times, as an abbreviation for $ABCDA$.

within J except for the point A_1) there is* an arc $A_1 E_1'$ that has in common with $E_1 E_1'$ only the point E_1'. Within $\bar{g}_2$ and J there is an arc $E_1' E_2'$ that has in common with $E_1 E_1'$ only the point E_1' and has in common with $E_2 E_2'$ only the point E_2' and in common with $A_1 E_1'$ only the point E_1'. There exists a set of arcs $A_1 E_1'$, $E_1' E_2'$, $E_2' E_3'$, $\cdots E_{n-1}' B_n$ such that (1) $E_i' E_{i+1}'$ is within $\bar{g}_{i+1}$ and J $(1 \leqq i \leqq n-2)$, $E_{n-1}' B_n$ is within $\bar{g}_n$ and, except for the point B_n, within J, (2) no two non-consecutive arcs of this set have any point in common and two consecutive ones have only one endpoint in common. It easily follows that there exist (Fig. 2) (1) on $ABCDA$, $4n$ points in the order $AE_1 E_2 \cdots E_n BF_1 F_2 \cdots F_n CL_1 L_2 \cdots L_n DM_1 M_2 \cdots M_n A$, (2) within

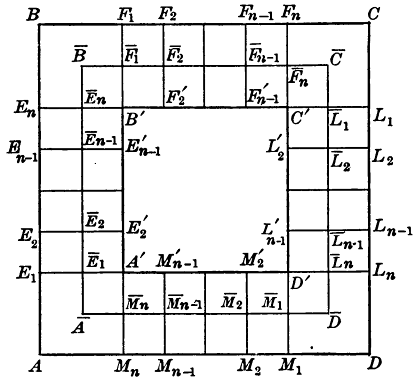

FIG. 2

$ABCDA$, a closed curve J' containing $4(n-1)$ points in the order $A' E_2' E_3' \cdots E_{n-1}' B' F_2' F_3' \cdots F_{n-1}' C' L_2' L_3' \cdots L_{n-1}' D' M_2' M_3' \cdots M_{n-1}' A'$, (3) arcs $A' \bar{E}_1 E_1$, $E_2' \bar{E}_2 E_2$, $E_3' \bar{E}_3 E_3$, $\cdots$, $E_{n-1}' \bar{E}_{n-1} E_{n-1}$, $B' \bar{E}_n E_n$, $B' \bar{F}_1 F_1$, $F' \bar{F}_2 F_2$, $F_3' \bar{F}_3 F_3$, $\cdots$, $F_{n-1}' \bar{F}_{n-1} F_{n-1}$, $C' \bar{F}_n F_n$, $C' \bar{L}_1 L_1$, $L_2' \bar{L}_2 L_2$, $\cdots L_{n-1}' \bar{L}_{n-1} L_{n-1}$, $D' \bar{L}_n L_n$, $D' \bar{M}_1 M_1$, $M_2' \bar{M}_2 M_2$, $\cdots M_{n-1}' \bar{M}_{n-1} M_{n-1}$, $A' \bar{M}_n M_n$ such that every one of these arcs lies, except for its endpoints, entirely between J' and J and no two of them have any point in common except that each of the points A', B', C', and D' is a common endpoint of two of them and such that no one of the points $\bar{E}_1$, $\bar{E}_2$, $\bar{E}_3$, $\cdots \bar{E}_{n-1}$, $\bar{E}_n$, $\bar{F}_1$, $\bar{F}_2$, $\cdots \bar{F}_{n-1}$, $\bar{F}_n$, $\bar{L}_1$, $\bar{L}_2$, $\bar{L}_3$, $\cdots \bar{L}_n$, $\bar{M}_1$, $\bar{M}_2$, $\bar{M}_3$, $\cdots \bar{M}_{n-1}$, $\bar{M}_n$ is on the curve g_1; the domain bounded by J and J' being divided by these arcs into subdivisions each of which lies within some curve of the set $\bar{G}$. By Theorem B there exists a set, α, of arcs $\bar{E}_i \bar{E}_{i+1}$ $(1 \leqq i \leqq n-1)$, $\bar{E}_n \bar{F}_1$, $\bar{F}_i \bar{F}_{i+1}$ $(1 \leqq i \leqq n-1)$, $\bar{F}_n \bar{L}_1$, $\bar{L}_i \bar{L}_{i+1}$ $(1 \leqq i \leqq n-1)$, $\bar{L}_n \bar{M}_1$, $\bar{M}_i \bar{M}_{i+1}$ $(1 \leqq i \leqq n-1)$, $\bar{M}_n \bar{E}_1$, lying except for their endpoints wholly within $E_i E_i' E_{i+1}' E_{i+1}$, $E_n B' F_1 B$, $F_i F_i' F_{i+1}' F_{i+1}$, $F_n C L_1 C'$, $L_i L_i' L_{i+1}' L_{i+1}$, $L_n D' M_1 D$, $M_i M_i' M_{i+1}' M_{i+1}$, and $M_n A' E_1 A$ respectively† no one of these

* Make use of Theorems 43 and 40 and the existence of the curve $ABTNA$ described on page 149 of *Foundations*.

† Here $E_1' = M_n' = A'$, $E_n' = F_1' = B'$, $F_n' = L_1' = C'$ and $L_n' = M_1' = D'$.

arcs having more than a finite number of points in common with g_1. There exist, on the segments $\bar{E}_n \bar{F}_1$, $\bar{F}_n \bar{L}_1$, $\bar{L}_n \bar{M}_1$, and $\bar{M}_n \bar{E}_1$ respectively, points $\bar{B}$, $\bar{C}$, $\bar{D}$, and $\bar{A}$ that are not on g_1. The arcs of α form a closed curve $\bar{A}\bar{B}\bar{C}\bar{D}\bar{A}$ which will, at times, be called $\bar{J}$. This curve does not have more than a finite number of points in common with g_1. Furthermore the domain ω bounded by $\bar{A}\bar{B}\bar{C}\bar{D}\bar{A}$ and $ABCDA$ is subdivided by the arcs $E_i \bar{E}_i$, $F_i \bar{F}_i$, $L_i \bar{L}_i$, $M_i \bar{M}_i$ $(1 \leqq i \leqq n)$ into subdivisions each of which lies wholly within some curve of the set $\bar{G}$. By Theorem C the interior of $\bar{A}\bar{B}\bar{C}\bar{D}\bar{A}$ may be divided by a double ruling $\bar{\tau}$ (such that the arcs of one of its two single rulings are parallel to AB and CD and those of its other one are parallel to BC and AD) into subdivisions the interior of each one of which is either entirely within or entirely without g_1. By the addition of certain arcs having their endpoints on J and $\bar{J}$ and lying, except for their endpoints, entirely between J and $\bar{J}$, it is possible to continue the arcs of $\bar{\tau}$, together with suitably chosen additional arcs, in such a way that there will result a double ruling τ_1 of $ABCDA$ such that every arc of the set α and every arc of the double ruling $\bar{\tau}$ will be a portion of some arc of the double ruling τ_1. The interior of each subdivision of the ruling τ_1 is either within one of the subdivisions into which $\bar{I}$* is divided by the rulings of $\bar{\tau}$ or within one of the subdivisions into which the domain ω is divided by the arcs of the set α. It follows that the interior of each of the subdivisions into which I is divided by τ_1 is either wholly within some curve of the set $\bar{G}$ or wholly without g_1. If any of them are wholly without g_1 then each such subdivision can itself be divided by a double ruling (the arcs of one single ruling of which are parallel to one side,† and the arcs of the other single ruling of which are parallel to an adjacent side of this subdivision) into subdivisions each of which is either wholly within some curve of the set $\bar{G}$ or wholly without g_2. The arcs of these rulings may be extended in such a way that there will result a new double ruling τ_2 of $ABCDA$ such that each arc of τ_1 is also an arc of τ_2 and each subdivision formed by τ_2 is within some subdivision formed by τ_1 and is either wholly without g_1 and wholly without g_2 or wholly within some curve of the set $\bar{G}$. It can be shown in a similar way that there exists a double ruling τ_3 of $ABCDA$ such that the interior of every subdivision formed by τ_3 is either wholly without each of the curves g_1, g_2, and g_3 or wholly within some curve of the set $\bar{G}$. This process may be continued. It follows that there exists a double ruling τ_n of $ABCDA$ such that every one of the subdivisions into which τ_n divides $ABCDA$ is either wholly without every curve of the set $g_1, g_2, \cdots, g_m$ or wholly within one of them. But by hypothesis every point of the interior of $ABCDA$ is

* I and $\bar{I}$ denote the interiors of J and $\bar{J}$ respectively.

† A *side* of one of the subdivisions into which the double ruling τ_1 divides I is a segment of the boundary of that subdivision which lies between two consecutive arcs of one of the single rulings of τ_1 and has its endpoints on those arcs.

within some curve of the set $g_1, g_2, \cdots, g_m$. It follows that every subdivision of I formed by the ruling τ_n is wholly within some curve of this set.

THEOREM E. *If ABCDA is a closed curve there exist two sets of arcs* α_1 *and* α_2 *such that* (1) *each arc of* α_1 *lies wholly within ABCDA except that its endpoints are on AB and CD*, (2) *each arc of* α_2 *lies wholly within ABCDA except that its endpoints are on BC and DA*, (3) *each point on ABCDA, with the exception of A, B, C, and D, is an endpoint either of just one arc of* α_1 *or of just one arc of* α_2, (4) *through each point within ABCDA there is just one arc of* α_1 *and just one arc of* α_2, (5) *each arc of* α_1 *has just one point in common with each arc of* α_2.

Proof. For each positive integer n there exists, about each point P of $ABCDA$ plus its interior, a fundamental* region of subscript greater than n. For each n let G_n denote the set of all such fundamental regions for all such points P. By Theorem D the interior of $ABCDA$ can be divided by a double ruling β_1 into subdivisions each of which is within some region of the set G_1. Each of these subdivisions can be divided by a double ruling into subdivisions each of which is within some region of the set G_2, the arcs of this ruling being so chosen that they can be extended in such a way that there will result a double ruling β_2 of $ABCDA$ such that each arc of β_1 is an arc of β_2 and such that each subdivision of β_2 is within some region of the set G_2. This process may be continued. It follows that there exists an infinite sequence β_1, β_2, $\beta_3, \cdots$ of rulings of $ABCDA$ such that (1) for each n, every arc of the ruling β_n belongs also to the ruling β_{n+1}, (2) every subdivision of the ruling β_n is within some fundamental region of subscript more than n. Let $\left\{ {\bar{\alpha}_2 \atop \bar{\alpha}_1} \right\}$ denote the set of all arcs $\left\{ {s_2 \atop s_1} \right\}$ such that $\left\{ {s_2 \atop s_1} \right\}$ has one endpoint on $\left\{ {BC \atop AB} \right\}$ and the other on $\left\{ {DA \atop CD} \right\}$ and belongs to some β_n. If P is a point within or on $ABCD$, distinct from A, B, C, and D, and not on any arc of the set $\left\{ {\bar{\alpha}_2 \atop \bar{\alpha}_1} \right\}$ it can be proved that there exists through P, and with endpoints on $ABCDA$, just one arc that does not intersect any arc of the set $\left\{ {\bar{\alpha}_2 \atop \bar{\alpha}_1} \right\}$. Let $\left\{ {\alpha_2^* \atop \alpha_1^*} \right\}$ be the set of all such arcs for all such points P. Let $\left\{ {\alpha_2 \atop \alpha_1} \right\}$ denote the set of all arcs of the set $\left\{ {\bar{\alpha}_2 \atop \bar{\alpha}_1} \right\}$ together with all arcs of the set $\left\{ {\alpha_2^* \atop \alpha_1^*} \right\}$. The sets α_1 and α_2 satisfy the conditions of Theorem E.

THEOREM F. *There exists a countably infinite sequence of closed curves* $J_1, J_2, J_3, \cdots$ *such that every point lies within one of them and such that, for every* n, J_{n+1} *encloses* J_n.

Proof. By Theorem 36 every point is within some closed curve. It follows, with the aid of Theorem 12, that there exists, for each n, a finite set G_n of closed curves such that every point of the fundamental region R_n is within some curve of the set G_n and every curve of the set G_n encloses a point of R_n. By Theorem 42 there exists a closed curve J_n' that encloses the interiors of all

* Cf. *Foundations*, p. 133.

the curves of the set G_n and therefore encloses R_n. If A_1' is a point within J_1' and A_2' is a point within J_2' then by Theorem 15 there exists an arc $A_1' A_2'$. There exists a finite set of curves enclosing $J_1' + J_2' + A_1' A_2'$. It follows that there exists a closed curve J_2 enclosing J_1', R_1, and R_2. Similarly there exists a closed curve J_3 enclosing J_2, R_1, R_2, and R_3. This process may be continued. It follows that there exists a countably infinite sequence of closed curves $J_1, J_2, J_3, \cdots$ such that, for every n, J_{n+1} encloses $J_n, R_1, R_2, \cdots, R_n$. But every point lies in some region of the set $R_1, R_2, R_3, \cdots$. Thus the sequence $J_1, J_2, J_3, \cdots$ satisfies the conditions of Theorem F.

THEOREM G. *There exist two sets G_1 and G_2 of open curves such that* (1) *through each point there is just one curve of G_1 and just one curve of G_2*, (2) *each curve of G_1 has just one point in common with each curve of G_2.*

Proof. Let $J_1, J_2, J_3, \cdots$ denote an infinite sequence of closed curves satisfying the conditions of Theorem F. If A_n, B_n, C_n, and D_n are four distinct points on J_n in the order $A_n B_n C_n D_n$ then (Fig. 3) there exist on

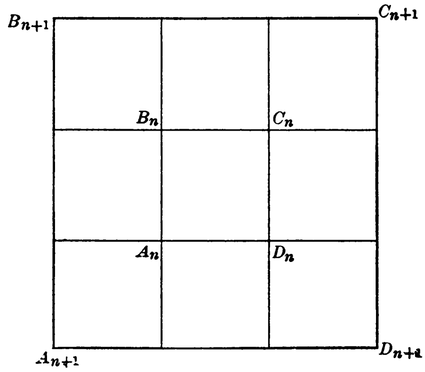

FIG. 3

J_{n+1} four distinct points A_{n+1}, B_{n+1}, C_{n+1}, and D_{n+1} such that J_{n+1} can be divided by a double ruling, with two arcs parallel to $B_{n+1} C_{n+1}$ and $A_{n+1} D_{n+1}$ and the other two parallel to $A_{n+1} B_{n+1}$ and $C_{n+1} D_{n+1}$, into nine subdivisions of which the central one is $A_n B_n C_n D_n A_n$ (J_n). It follows that there exists a set of open curves* $[k_m] (-\infty < m < \infty)$ and another set $[t_n] (-\infty < n < \infty)$ such that (1) if m and n are any integers, positive or negative, k_m has just one point in common with t_n, (2) for every integer n, the curve t_n separates t_{n-1} from t_{n+1} and k_n separates k_{n-1} from k_{n+1}, (3) if P is a point not lying on any open curve of either of these sets then there exist m and n such that P is between k_m and k_{m+1} and also between t_n and t_{n+1}. For every m and n let A_{mn} denote the intersection of k_m with t_n. Then the intervals $A_{mn} A_{m(n+1)}$, $A_{m(n+1)} A_{(m+1)(n+1)}$, $A_{(m+1)(n+1)} A_{(m+1)n}$, and $A_{(m+1)n} A_{mn}$ of the open curves k_m, t_{n+1}, k_{m+1}, and t_n respectively form a closed curve J_{mn}.

* Cf. *Foundations*, p. 159.

By Theorem E there exist two sets of arcs G_{mn} and G'_{mn} such that (1) each arc of G_{mn} is entirely within J_{mn} except that one of its endpoints is between A_{mn} and $A_{(m+1)n}$ on the curve t_n and the other one is between $A_{m(n+1)}$ and $A_{(m+1)(n+1)}$ on the curve t_{n+1}, (2) each arc of G'_{mn} is entirely within J_{mn} except that one of its endpoints is between A_{mn} and $A_{m(n+1)}$ on k_n and the other one is between $A_{(m+1)n}$ and $A_{(m+1)(n+1)}$ on k_{m+1}, (3) through each point within J_{mn} there is just one arc of the set G_{mn} and just one arc of the set G'_{mn}, (4) each point of J_{mn}, with the exception of A_{mn}, $A_{m(n+1)}$, $A_{(m+1)n}$, and $A_{(m+1)(n+1)}$, is an endpoint of just one arc of G_{mn} or of just one arc of G'_{mn}, (5) each arc of G_{mn} has just one point in common with each arc of G'_{mn}. If P_0 is a point on the arc t_0 and lying between the curves k_m and k_{m+1}, P_0 is an endpoint of just one arc g_{m0} of the set G_{m0}. The other endpoint of g_{m0} is a point P_1 lying between k_m and k_{m+1} on the curve t_1. The point P_1 is an endpoint of just one arc g_{m1} of the set G_{m1}. The other endpoint of g_{m1} is a point P_2 lying between k_m and k_{m+1} on the curve t_2. This process may be continued. It follows that there exist two sequences of arcs g_{m0}, g_{m1}, g_{m2}, $\cdots$ and $g_{m(-1)}$, $g_{m(-2)}$, $g_{m(-3)}$, $\cdots$ such that (1) P_0 is an endpoint of g_{m0}, (2) for every integer n, g_{mn} is an arc of the set G_{mn} and the arcs g_{mn} and $g_{m(n+1)}$ have an endpoint in common. The sum of all the arcs g_{m0}, g_{m1}, $\cdots$ $g_{m(-1)}$, $g_{m(-2)}$, $\cdots$ is clearly a continuous open curve, l_{P_0}, passing through the point P_0. Let α denote the set of all the curves k_m for all integers m together with the set of all the curves l_P for all points P on t_0. Similarly if P is a point on k_0 there exists an open curve r_P that passes through P and is the sum of an infinite number of arcs each of which is, for some m and n, an arc of the set G'_{mn}. Let β denote the set of all the curves t_n for all intergers n together with the set of all the curves r_P for all points P on k_0. It is clear that through every point there is just one curve of the set α and just one curve of the set β and furthermore each curve of the set α has just one point in common with each curve of the set β.

Between the points of $\left\{{t_0 \atop k_0}\right\}$ and the set of all real numbers there is a one-to-one continuous correspondence $\left\{{\pi_2 \atop \pi_1}\right\}$ in which the intersection of k_0 and t_0 is associated with the number 0. If P is any point let $\left\{{y_P \atop x_P}\right\}$ denote the number which, in the correspondence $\left\{{\pi_2 \atop \pi_1}\right\}$ is associated with the point in which $\left\{{t_0 \atop k_0}\right\}$ intersects that curve of the set $\left\{{\alpha \atop \beta}\right\}$ which passes through P. Thus to every point P there corresponds a definite pair of real numbers (x_P, y_P) and conversely.

It is clear that the correspondence thus established between S and a number plane is continuous in the sense that in S the point P is a sequential limit point of the sequence of points P_1, P_2, P_3, $\cdots$ if, and only if, $\lim_{n=\infty} x_{P_n} = x_P$ and $\lim_{n=\infty} y_{P_n} = y_P$.

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