ON THE SUMMABILITY OF THE DEVELOPMENTS IN BESSEL'S FUNCTIONS*

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The object of the present paper is to establish sufficient conditions for the summability (Cesàro) at the origin and the uniform summability in the neighborhood of the origin of the developments in Bessel's functions. It has also been the aim of the writer to obtain summability of as low an index as possible without placing any considerable restriction on the function to be developed.

The study of the behavior of the development in the neighborhood of the origin presents much greater difficulties than the study of the development in intervals that do not include the origin. This accounts for the fact that previous discussions of the summability† of the development apply only to intervals of the latter type, and that most of the previous discussions of the convergence of the development are incomplete in the same manner.‡

The difficulty in studying the development in the neighborhood of the origin arises from the fact that the terms in which the asymptotic expansion of the Bessel's functions can be used to advantage begin later and later in the series as we approach the origin. Hence, in studying the series in an interval that reaches up to the origin, the point at which we start using the asymptotic expansion is continually shifting. This gives rise to many complications and accounts for the length of that portion of this paper which

* Presented to the Society, September 4, 1917, and April 12, 1918. Owing to considerations of space some of the results of these papers are reserved for later publication.

† To my knowledge there have been only two such discussions, the first a paper by myself in these Transactions, vol. 10 (1909), pp. 391–435, hereafter referred to as Transactions I, and the second to be found in Ford's *Studies on Divergent Series and Summability*, Chapter V. *Added later*. Since the present paper was written, an article by W. H. Young has appeared (cf. Proceedings of the London Mathematical Society, ser. 2, vol. 18 (1919–20), p. 163), in which both the convergence and the summability are considered, but in this case also the behavior of the development in the neighborhood of the origin is not determined.

‡ As far as I am aware, the only discussion of the convergence of the developments in Bessel's functions which is complete in this respect, is to be found in a paper by myself in these Transactions, vol. 12 (1911), pp. 181–206, to be hereafter referred to as Transactions II.

deals with the uniform summability of the development in the neighborhood of the origin (§§ 7–21).

It may be noted that many of the lemmas obtained, particularly those in sections 7–17, have an interest of their own and many possible applications in other investigations, such as the study of Fourier's series. I have made no attempt to point out explicitly these applications, as I felt that this would detract from the unity of the paper and unnecessarily increase its length.

An asymptotic formula for the roots of a certain equation involving Bessel's functions. §§ 1–2

1. We wish to obtain in this section an asymptotic formula for the positive roots of the equation

$$l\lambda J_\nu'(\lambda) + hJ_\nu(\lambda) = 0, \tag{1}$$

where $J_\nu(\lambda)$ represents a Bessel's function of the νth order, ν is a constant positive or zero, and h and l are any constants not both zero. It has already been shown in Transactions I (cf. equation (57)) that the nth positive root of (1) satisfies the equation

$$\lambda_n = n\pi + q + \frac{\psi_1(n)}{n}, \tag{2}$$

where q is a constant dependent on ν and l, and $\psi_1(n)$ *is used to represent any function of n that remains finite for all values of n*. The object of the following discussion is to arrive at a more precise expression for the last term.

By means of the asymptotic development for $J_\nu(x)$ and $J_{\nu+1}(x)$, equation (1) may be replaced by the equation*

$$\cos(\lambda - \alpha) + \frac{k}{\lambda}\sin(\lambda - \alpha) + \frac{\psi_1(\lambda)}{\lambda^2} = 0, \tag{3}$$

where k is a constant, and α is a constant that has the value $\frac{1}{4}(2\nu + 3)\pi$ or $\frac{1}{4}(2\nu + 1)\pi$ according as the l of equation (1) is different from or is equal to zero. The corresponding values of q in (2) will be given by†

$$q = r\pi + \frac{\pi}{2} + \alpha \qquad (r \text{ an integer or zero}). \tag{4}$$

We now set

$$\lambda = (n + r)\pi + \frac{\pi}{2} + \alpha + h = m\pi + \frac{\pi}{2} + \alpha + h. \tag{5}$$

In view of (2) and (4) the value of h corresponding to $\lambda = \lambda_n$, which we shall

* Cf. Transactions I, Lemma 3.

† Cf. Transactions I, p. 415. In making use of the equations for q a slight modification is necessary to take account of the fact that the α on page 415 has the same value $\frac{1}{4}(2\nu + 1)$, whether l is or is not zero.

designate by h_n, will be of the form $\psi_1(n)/n$. We shall show further that it is of the form $k/n + \psi_1(n)/n^2$.

From (5) and an application of Maclaurin's expansion we have

$$\begin{aligned}\cos(\lambda - \alpha) &= \cos\left(m\pi + \frac{\pi}{2} + h\right) = (-1)^{m+1}\sin h \\ &= (-1)^{m+1}\left[h - \frac{h^3}{6}\cos(\theta_1 h)\right],\end{aligned} \tag{6}$$

$$\begin{aligned}\sin(\lambda - \alpha) &= \sin\left(m\pi + \frac{\pi}{2} + h\right) = (-1)^{m}\cos h \\ &= (-1)^{m}\left[1 - \frac{h^2}{2}\cos(\theta_2 h)\right],\end{aligned} \tag{7}$$

where θ_1 and θ_2 each lie between zero and one. If now we set $h = k/\lambda - c/\lambda^2$ and substitute from (6) and (7) into the left-hand side of (3), it is easy to see that if c and λ are chosen to satisfy the inequalities

$$|\psi_1(\lambda)| + 1 < c < k\lambda, \qquad \lambda > k^3, \tag{8}$$

the left-hand side of (3) will have the sign $(-1)^m$. If we substitute $k/\lambda + c/\lambda^2$ for h, we see that if c and λ satisfy the inequalities (8) the left-hand side of (3) will have the sign $(-1)^{m+1}$. Hence this expression changes sign as h passes from the value $k/\lambda - c/\lambda^2$ to the value $k/\lambda + c/\lambda^2$, and therefore (3) has a root, λ_n, for a value of h, h_n, between these limiting values. Thus we see that h_n is of the form $k/\lambda_n + \psi_1(\lambda_n)/\lambda_n^2$, and consequently, in view of (2), of the form $k/n + \psi_1(n)/n^2$. Therefore, taking account of (4) and (5), we have

$$\lambda_n = n\pi + q + \frac{k}{n} + \frac{\psi_1(n)}{n^2}, \tag{9}$$

the asymptotic formula we desired to obtain.

2. We shall now employ (9) to obtain another formula which will be of use in later reductions. We have from (9)

$$\begin{aligned}\cos(\lambda_n x - \alpha) &= \cos[(n\pi + q)x - \alpha] \\ &+ \cos\left[(n\pi + q)x - \alpha + \frac{kx}{n} + \frac{x\psi_1(n)}{n^2}\right] - \cos[(n\pi + q)x - \alpha].\end{aligned} \tag{10}$$

By means of two elementary trigonometric reductions, the last two terms on the right-hand side of (10) may be put in the form

$$\begin{aligned}-2\sin[(n\pi + q)x - \alpha]\cos\frac{1}{2}\left(\frac{kx}{n} + \frac{x\psi_1(n)}{n^2}\right)\sin\frac{1}{2}\left(\frac{kx}{n} + \frac{x\psi_1(n)}{n^2}\right) \\ -2\cos[(n\pi + q)x - \alpha]\sin^2\frac{1}{2}\left(\frac{kx}{n} + \frac{x\psi_1(n)}{n^2}\right).\end{aligned}$$

If we reduce this expression by expanding the sine and cosine of

$$\frac{1}{2}\left(\frac{kx}{n}+\frac{x\,\psi_1(n)}{n^2}\right)$$

by means of the formulas for sin h and cos h given in (6) and (7), and substitute the result for the last two terms of (10), we have finally

$$(11)\qquad \cos(\lambda_n x-\alpha)=\cos[(n\pi+q)x-\alpha] -\frac{kx}{n}\sin[(n\pi+q)x-\alpha]+\frac{x\,\psi(x,n)}{n^2},$$

where $\psi(x,n)$ *is used to represent a function of* x *and* n *which for all fixed values of* n *is continuous in the interval* $(0\leqq x\leqq 1)$ *and which remains finite when* n *increases indefinitely and* x *varies in that interval.*

Summability at the origin of the development of an arbitrary function in terms of Bessel's functions. §§ 3–6

3. We turn now to the consideration of the development

$$(12)\qquad \sum_{n=1}^{\infty} A_n J_\nu(\lambda_n x),$$

where

$$(13)\qquad A_n=\frac{\int_0^1 x f(x) J_\nu(\lambda_n x)\,dx}{\int_0^1 x[J_\nu(\lambda_n x)]^2\,dx},$$

and $\lambda_1,\lambda_2,\lambda_3,\cdots$ are the successive positive roots of equation (1). We will first establish some lemmas.

Lemma 1. *If in the interval* $(0\leqq x\leqq 1)$, $f(x)$ *has a Lebesgue integral, is continuous at the origin and such that*

$$(1\ )\qquad \left|\frac{f(x)-f(0)}{x^{\frac{1}{2}}}\right|<M \qquad (0<x\leqq c\leqq 1),$$

where M *is a positive constant, we have*

$$(15)\qquad A_n=\left(M_1\sqrt{n}+\frac{M_2}{\sqrt{n}}\right)\int_0^1 \sqrt{x}\,\phi(x)\cos\{(n\pi+q)x-\alpha\}dx +\frac{M_3}{\sqrt{n}}\int_0^1\frac{\phi(x)}{\sqrt{x}}(x^2+b)\sin\{(n\pi+q)x-\alpha\}dx +(-1)^n\frac{M_4}{\sqrt{n}}f(0)+\frac{\nu K f(0)}{\lambda_n}+r_n,$$

where A_n *is defined by* (13), $\phi(x) = f(x) - f(0)$, M_1, M_2, M_3, M_4, q, α, b, *and* K *are constants, and* r_n *is the general term of an absolutely convergent series.*

From the definition of $\phi(x)$ and the substitution $y = \lambda x$, we have

$$(16)\quad \int_0^1 x f(x) J_\nu(\lambda x)\, dx = \int_0^1 x\,\phi(x) J_\nu(\lambda x)\, dx + \frac{f(0)}{\lambda^2}\int_0^\lambda y\, J_\nu(y)\, dy.$$

But, on integrating by parts,

$$(17)\quad \int_0^\lambda y\, J_\nu(y)\, dy = \int_0^\infty \frac{1}{y^\nu} y^{\nu+1} J_\nu(y)\, dy = \lambda J_{\nu+1}(\lambda) + \nu \int_0^\lambda J_{\nu+1}(y)\, dy.$$

Moreover,*

$$(18)\quad \int_0^\lambda J_{\nu+1}(y)\, dy = \int_0^\infty J_{\nu+1}(y)\, dy - \int_\lambda^\infty J_{\nu+1}(y)\, dy = K_1 + \frac{\psi_1(\lambda)}{\lambda^{\frac{1}{2}}},$$

where K_1 is a constant.

From the asymptotic expansion† for $J_\nu(\lambda x)$ we have

$$(19)\quad \int_{c/\lambda}^1 x\,\phi(x) J_\nu(\lambda x)\, dx = \frac{A}{\lambda^{\frac{1}{2}}}\int_{c/\lambda}^1 \sqrt{x}\,\phi(x)\cos(\lambda x - \alpha)\, dx$$
$$+ \frac{A'}{\lambda^{\frac{3}{2}}}\int_{c/\lambda}^1 \frac{\phi(x)}{\sqrt{x}}\sin(\lambda x - \alpha)\, dx + \frac{A}{\lambda^{\frac{3}{2}-\rho}}\int_{c/\lambda}^1 \frac{\phi(x)}{x^{\frac{1}{2}}}\cdot\frac{1}{x^{1-\rho}}\cdot\frac{\psi(\lambda x)}{(\lambda x)^\rho}\, dx$$
$$= \frac{A}{\lambda^{\frac{1}{2}}}\int_0^1 \sqrt{x}\,\phi(x)\cos(\lambda x - \alpha)\, dx$$
$$+ \frac{A'}{\lambda^{\frac{3}{2}}}\int_0^1 \frac{\phi(x)}{\sqrt{x}}\sin(\lambda x - \alpha)\, dx + \frac{\psi_1(\lambda)}{\lambda^{\frac{3}{2}-\rho}} \qquad \begin{pmatrix} A' \text{ a constant} \\ 0 < \rho < \frac{1}{2} \end{pmatrix},$$

since, in view of (14)

$$\int_0^{c/\lambda} \sqrt{x}\,\phi(x)\cos(\lambda x - \alpha)\, dx = \frac{\psi_1(\lambda)}{\lambda^2},$$

$$\int_0^{c/\lambda} \frac{\phi(x)}{\sqrt{x}}\sin(\lambda x - \alpha)\, dx = \frac{\psi_1(\lambda)}{\lambda}.$$

Furthermore

$$(20)\quad \int_0^{c/\lambda} x\,\phi(x) J_\nu(\lambda x)\, dx = \frac{\psi_1(\lambda)}{\lambda^{\frac{3}{2}}}.$$

Hence finally, from (16), (17), (18), (19), and (20)

* Cf. pp. 187–188 of Transactions II.

† Cf. formula 48 of Transactions I.

$$(21)\qquad \int_0^1 x f(x) J^{\nu}(\lambda x)\,dx = \frac{A}{\lambda^{\frac{1}{2}}}\int_0^1 \sqrt{x}\,\phi(x)\cos(\lambda x-\alpha)\,dx + \frac{A'}{\lambda^{\frac{3}{2}}}\int_0^1 \frac{\phi(x)}{\sqrt{x}}\sin(\lambda x-\alpha)\,dx + f(0)\frac{J_{\nu+1}(\lambda)}{\lambda} + \frac{\nu K_2 \dot{f}(0)}{\lambda^2} + \frac{\psi_1(\lambda)}{\lambda^{\frac{3}{2}-\rho}}.$$

Therefore, since*

$$(22)\qquad \int_0^1 x[J^{\nu}(\lambda x)]^2\,dx = \frac{A^2}{2\lambda} + \frac{A_1 \sin 2(\lambda-\alpha)}{\lambda^2} + \frac{\psi_1(\lambda)}{\lambda^3},$$

where A_1 is a constant, we have

$$(23)\qquad A_n = \left(\frac{2\lambda_n^{\frac{1}{2}}}{A} - \frac{4A_1 \sin 2(\lambda_n-\alpha)}{A^3\lambda_n^{\frac{1}{2}}}\right)\int_0^1 \sqrt{x}\,\phi(x)\cos(\lambda_n x-\alpha)\,dx + \frac{2A'}{A^2\lambda_n^{\frac{1}{2}}}\int_0^1 \frac{\phi(x)}{\sqrt{x}}\sin(\lambda_n x-\alpha)\,dx + \frac{2f(0)}{A^2}J_{\nu+1}(\lambda_n) + \frac{2\nu K_1 f(0)}{A^2\lambda_n} + \frac{\psi_1(\lambda_n)}{\lambda_n^{\frac{3}{2}-\rho}}.$$

But from (2)

$$(24)\qquad \lambda_n^{\frac{1}{2}} = \sqrt{n\pi}\left(1+\frac{q}{n\pi}+\frac{\psi_1(n)}{n^2}\right)^{\frac{1}{2}} = \sqrt{n\pi}\left(1+\frac{q}{2n\pi}+\frac{\psi_1(n)}{n^2}\right).$$

Moreover (Transactions II, equation (42))

$$(25)\qquad \frac{1}{\lambda_n^{\frac{1}{2}}} = \frac{1}{\sqrt{n\pi}} + \frac{\psi_1(n)}{n^{\frac{3}{2}}},$$

and (Transactions I, equation (59))

$$(26)\qquad \sin(\lambda_n x-\alpha) = \sin[(n\pi+q)x-\alpha] + \frac{\psi(x,\lambda_n)}{n}.$$

Similarly we obtain

$$(27)\qquad \sin 2(\lambda_n-\alpha) = \sin 2(n\pi+q-\alpha) + \frac{\psi_1(n)}{n} = \sin 2(q-\alpha) + \frac{\psi_1(n)}{n}.$$

Also, from Transactions II, formula (43),

$$(28)\qquad J_{\nu+1}(\lambda_n) = (-1)^n\frac{C}{\sqrt{n}} + \frac{\psi_1(n)}{n^{\frac{3}{2}}} \qquad (C \text{ a constant}).$$

Combining (24), (25), (11), (26), (27), and (28) with (23), we readily obtain for A_n the form† on the right-hand side of (15). Hence our lemma is proved.

* See Transactions I, p. 403, where a somewhat less explicit form for the left-hand side of (22) is obtained. The present form is derived in similar fashion by using one more term in the asymptotic expansions of $J_{\nu}(\lambda)$ and $J_{\nu+1}(\lambda)$.

† For the sake of future reference we note that $M_1 = 2/A$, where A is the A of formula 48 (Transactions I).

4. Lemma 2. *If in the interval* $0 \leqq x \leqq 1$, $\phi(x)$ *has a Lebesgue integral and is furthermore such that*

$$\left| \frac{\phi(x)}{x^{\frac{1}{2}+\rho}} \right| < M \qquad (0 < x \leqq c \leqq 1), \tag{29}$$

where M *is a constant and* ρ *is a constant* > 0, *the series whose general terms are*

$$\begin{aligned} & n \int_0^1 \sqrt{x}\, \phi(x) \cos(qx - \alpha) \cos n\pi x \, dx, \\ & n \int_0^1 \sqrt{x}\, \phi(x) \sin(qx - \alpha) \sin n\pi x \, dx, \end{aligned} \tag{30}$$

where q *and* α *are constants, will be summable* $(C\,1)$.

We will consider first the series whose general term is the first expression in (30). If in that expression we substitute $y = \pi x$, it reduces to

$$\frac{1}{\pi^{\frac{3}{2}}} \int_0^\pi \sqrt{y}\, \phi\left(\frac{y}{\pi}\right) \cos\left(\frac{qy}{\pi} - \alpha\right) n \cos ny \, dy. \tag{31}$$

We shall now obtain the first Cesàro mean for the series we are discussing. We have from a well-known formula*

$$\sum_{m=1}^{m=n} \left(\sum_{r=1}^{r=m} \sin ry \right) = \frac{(n+1)\sin y - \sin(n+1)y}{4 \sin^2 \frac{1}{2} y}. \tag{32}$$

Differentiating both sides of the above equation with regard to y and making a few trigonometric reductions on the right-hand side we obtain

$$\sum_{m=1}^{m=n} \left(\sum_{r=1}^{r=m} r \cos ry \right) = \frac{\sin(n+\frac{1}{2})y}{4 \sin^3 \frac{1}{2} y} - \frac{(n+1) + n\cos(n+1)y}{4 \sin^2 \frac{1}{2} y}. \tag{33}$$

From this equation we obtain at once for the first Cesàro mean of the series whose general term is (31)

$$\begin{aligned} & \frac{1}{4n\pi^{\frac{3}{2}}} \int_0^\pi \frac{\sqrt{y}\, \phi(y/\pi)}{\sin^2 \frac{1}{2} y} \cos\left(\frac{qy}{\pi} - \alpha\right) \frac{\sin(n+\frac{1}{2})y}{\sin \frac{1}{2} y} dy \\ & \qquad - \left(1 + \frac{1}{n}\right) \frac{1}{4\pi^{\frac{3}{2}}} \int_0^\pi \frac{\sqrt{y}\, \phi(y/\pi)}{\sin^2 \frac{1}{2} y} \cos\left(\frac{qy}{\pi} - \alpha\right) dy \\ & \qquad - \frac{1}{4\pi^{\frac{3}{2}}} \int_0^\pi \frac{\sqrt{y}\, \phi(y/\pi)}{\sin^2 \frac{1}{2} y} \cos\left(\frac{qy}{\pi} - \alpha\right) \cos(n+1)y \, dy, \end{aligned} \tag{34}$$

where the convergence of each integral, and hence the justification for writing the expression in the form (34), is obtained from the condition (29).

The second term of (34) is readily seen to approach a limit as n becomes infinite. The third term is a constant multiple of a Fourier's coefficient of a

* Cf. Transactions I, p. 413.

function having a Lebesgue integral. Hence from a theorem due to Lebesgue* we know that it approaches zero as a limit as n becomes infinite. It remains then to examine the first term.

It is readily seen that a positive constant, K, exists, such that for all values of y

$$\left|\frac{\sin (n+\frac{1}{2})y}{n \sin \frac{1}{2}y}\right| < K. \tag{35}$$

Moreover, a positive ϵ being assigned, we may choose δ such that

$$\frac{1}{4\pi^{\frac{3}{2}}}\int_0^{\delta}\left|\frac{\sqrt{y}\,\phi(y/\pi)}{\sin^2 \frac{1}{2}y}\cos\left(\frac{qy}{\pi}-\alpha\right)\right| dy < \frac{\epsilon}{2K}, \tag{36}$$

and δ being thus chosen, we may then select an m such that

$$\frac{1}{4n\pi^{\frac{3}{2}}}\int_{\delta}^{\pi}\left|\frac{\sqrt{y}\,\phi(y/\pi)}{\sin^2 \frac{1}{2}y}\cos\left(\frac{qy}{\pi}-\alpha\right)\frac{\sin (n+\frac{1}{2})y}{\sin \frac{1}{2}y}\right| dy < \frac{\epsilon}{2} \quad (n \geqq m). \tag{37}$$

From (35), (36), and (37) it follows at once that the first term in (34) may be made less than ϵ by choosing $n \geqq m$. Hence this term approaches zero as a limit as n becomes infinite. Thus it follows that (34), or the first Cesàro mean of the series whose general term is the first expression in (30), approaches a limit as n becomes infinite, and the first part of our lemma is proved.

Let us consider now the series whose general term is the second term in (30). If we set $y = \pi x$, this term reduces to

$$\frac{1}{\pi^{\frac{3}{2}}}\int_0^{\pi}\sqrt{y}\,\phi\left(\frac{y}{\pi}\right)\cos\left(\frac{qy}{\pi}-\alpha\right) n \sin ny\, dy.$$

But, from a well-known formula,

$$\sum_{m=1}^{m=n}\left(\sum_{r=1}^{r=m}\cos ry\right) = \frac{\sin^2 \frac{1}{2}(n+1)y}{2\sin^2 \frac{1}{2}y} - \frac{n+1}{2}. \tag{38}$$

Whence, differentiating both sides with respect to y,

$$\sum_{m=1}^{m=n}\left(\sum_{r=1}^{r=m} r \sin ry\right) = \frac{(n+1)\sin (n+1)y}{4\sin^2 \frac{1}{2}y} - \frac{\sin^2 \frac{1}{2}(n+1)y \cos \frac{1}{2}y}{2\sin^3 \frac{1}{2}y}. \tag{39}$$

Thus we have for the first Cesàro mean of the series we are discussing, the expression

$$\begin{aligned}&\left(1+\frac{1}{n}\right)\frac{1}{4\pi^{\frac{3}{2}}}\int_0^{\pi}\frac{\sqrt{y}\,\phi(y/\pi)}{\sin^2 \frac{1}{2}y}\cos\left(\frac{qy}{\pi}-\alpha\right)\sin (n+1)y\, dy - \frac{1}{2n\pi^{\frac{3}{2}}}\\ &\times\int^{\pi}\frac{\sqrt{y}\,\phi(y/\pi)}{\sin^2 \frac{1}{2}y}\cos\left(\frac{qy}{\pi}-\alpha\right)\cos\frac{y}{2}\frac{\sin \frac{1}{2}(n+1)y}{\sin \frac{1}{2}y}\sin \tfrac{1}{2}(n+1)y\, dy.\end{aligned} \tag{40}$$

* Cf. Lebesgue, *Sur les séries trigonométriques*, Annales de l'école normale supérieure, ser. 3, vol. 20 (1903), pp. 471–473.

The integral in the first term is a constant multiple of a Fourier's coefficient of a function having a Lebesgue integral. Hence, by the theorem of Lebesgue quoted above, this term approaches zero as a limit as n becomes infinite. By a discussion entirely analogous to the discussion of the first term of (34) it may be shown that the second term of (40) approaches zero as a limit as n becomes infinite. Hence (40) approaches a limit as n becomes infinite and therefore the series whose general term is the second term of (30) is summable $(C\,1)$. Our lemma is thus completely established.

5. LEMMA 3. *The series whose general terms are*

$$(41)\qquad \begin{aligned} &\int_0^1 \frac{\phi(x)}{\sqrt{x}} \cos(qx - \alpha) \sin n\pi x \, dx, \\ &\int_0^1 \frac{\phi(x)}{\sqrt{x}} \sin(qx - \alpha) \cos n\pi x \, dx, \end{aligned}$$

will be summable $(C\,1)$, *provided* $\phi(x)$ *satisfies the conditions of Lemma* 2.

We will consider first the series whose general term is the first expression in (41). If we set $y = \pi x$, this expression reduces to

$$(42)\qquad \frac{1}{\pi^{\frac{1}{2}}} \int_0^\pi \frac{\phi(y/\pi)}{\sqrt{y}} \cos\left(\frac{qy}{\pi} - \alpha\right) \sin ny \, dy.$$

Making use of (32) we have for the first Cesàro mean of the series whose general term is (42)

$$(43)\qquad \begin{aligned} &\frac{1}{4\sqrt{\pi}}\left(1 + \frac{1}{n}\right) \int_0^\pi \frac{\phi(y/\pi)}{\sqrt{y}\sin\frac{1}{2}y} \cos\left(\frac{qy}{\pi} - \alpha\right) \frac{\sin y}{\sin\frac{1}{2}y} dy \\ &\qquad - \frac{1}{4\sqrt{\pi}} \int_0^\pi \frac{\phi(y/\pi)}{\sqrt{y}\sin\frac{1}{2}y} \cdot \frac{\sin(n+1)y}{n\sin\frac{1}{2}y} \cos\left(\frac{qy}{\pi} - \alpha\right) dy, \end{aligned}$$

the required property of $\phi(x)$ assuring the convergence of each integral involved. The first term in (43) evidently approaches a limit as n becomes infinite. By a discussion entirely analogous to that for the first term of (34) it may be shown that the second term approaches zero as a limit as n becomes infinite. Hence the whole expression (43) approaches a limit as n becomes infinite, and the series whose general term is the first term of (41) is summable $(C\,1)$.

Consider now the series whose general term is the second expression in (41). By means of the transformation $y = \pi x$ this expression reduces to

$$(44)\qquad \frac{1}{\sqrt{\pi}} \int_0^\pi \frac{\phi(y/\pi)}{\sqrt{y}} \sin\left(\frac{qy}{\pi} - \alpha\right) \cos ny \, dy.$$

Making use of (38) we have for the first Cesàro mean of the series whose general term is (44)

$$
(45)\quad \begin{aligned} &\frac{1}{2\sqrt{\pi}}\int_0^{\pi} \frac{\phi(y/\pi)}{\sqrt{y}\sin\frac{1}{2}y}\cdot\frac{\sin\frac{1}{2}(n+1)y}{n\sin\frac{1}{2}y}\sin\tfrac{1}{2}(n+1)y\sin\left(\frac{qy}{\pi}-\alpha\right)dy \\ &\qquad -\frac{1}{2\sqrt{\pi}}\left(1+\frac{1}{n}\right)\int_0^{\pi}\frac{\phi(y/\pi)}{\sqrt{y}}\sin\left(\frac{qy}{\pi}-\alpha\right)dy, \end{aligned}
$$

where again the convergence of the integrals involved is secured by the condition on $\phi(x)$. The second term of (45) obviously approaches a limit as n becomes infinite, and the first term may be shown to approach zero as a limit as n becomes infinite by a discussion completely analogous to that for the first term of (34). Hence the whole expression (45) approaches a limit as n becomes infinite, and the series whose general term is the second expression in (41) is summable $(C\,1)$.

Our lemma is thus completely established.

6. We are now ready to prove the following theorem:

THEOREM I. *If $f(x)$ is such that $\phi(x) = f(x) - f(0)$, satisfies the conditions of Lemma 2, the development of $f(x)$ in Bessel's functions of order ν $(\nu \geqq 0)$ will be summable $(C\,\frac{1}{2})$ at the origin.*

For the case $\nu > 0$ the theorem is trivial, since $J_\nu(0) = 0$ for $\nu > 0$ and henc all the terms of the series are zero.

For $\nu = 0$ the general term of the series (12) becomes A_n since $J_0(0) = 1$, and therefore it may be reduced to the expression on the right-hand side of (15). From Lemmas 2 and 3* and a theor m† due to M. Riesz it follows that the first and second terms of this expression are the general terms of series summable $(C\,\frac{1}{2})$. The third and fifth terms are general terms of convergent series and the fourth term reduces to zero. Hence each term is the general term of a series summable $(C\,\frac{1}{2})$ and consequently the whole expression is the general term of such a series. Our theorem is therefore established.

A GENERALIZATION OF THE RIEMANN-LEBESGUE THEOREM AND A FURTHER PRELIMINARY LEMMA. § 7

7. We will next obtain certain sufficient conditions for the uniform summability $(C\,\frac{1}{2})$, in the neighborhood of the origin, of the series (12). We find it necessary to establish a series of lemmas before proceeding to the main theorem.

* In applying Lemma 3, we note that if $\phi(x)$ satisfies the conditions of Lemma 2, so also does the function $\phi(x)(x^2+b)$.

† Cf. Paris Comptes Rendus, 12 June, 1911, also Hardy-Riesz, *The general theory of dirichlet's series.*

LEMMA 4. *If $f(y)$ is integrable (Lebesgue) in the interval (c, d) and $\phi(x, y)$ is a continuous function of x and y in the region $(a \leqq x \leqq b;\ c \leqq y \leqq d)$, then the integrals*

$$(46)\qquad \int_c^d f(y)\phi(x, y)\cos ny\, dy,\ \int_c^d f(y)\phi(x, y)\sin ny\, dy$$

will approach zero uniformly for all values of x in (a, b) as n becomes infinite.

This lemma is a generalization of Lebesgue's theorem that the Fourier's constants of an integrable function approach zero as a limit.* The proof follows the same general lines as the proof of that theorem.

We will consider only the first integral in (46), as the proof for the second integral is entirely analogous. Since $f(y)$ is integrable we can choose M sufficiently great that for the set of points E of (c, d) for which $|f(y)| > M$

$$(47)\qquad \left|\int_E f(y)\phi(x, y)\cos ny\, dy\right| < \frac{\epsilon}{4} \qquad \left(\begin{matrix} a \leqq x \leqq b \\ n = 1, 2, 3, \cdots \end{matrix}\right),$$

where ϵ is an arbitrary positive quantity.

Let E_1 be the set of points of (c, d) complementary to E. Since $f(y)$is bounded on E_1, this set may be divided into a finite number of measurable sets $e_1, e_2, \cdots, e_p$, in each of which the oscillation of $f(y)$ is less than

$$(48)\qquad \delta = \frac{\epsilon}{4(M + K)(c - d)},$$

where K is the maximum absolute value of $\phi(x, y)$. Moreover, each set e_q may be enclosed in a finite or infinite set of intervals of measure $m(e_q) + \eta$, where

$$(49)\qquad \eta = \frac{\epsilon}{8MKp}.$$

Let I represent the totality of the intervals enclosing the sets e_q and $f_1(y)$ a function of y agreeing with $f(y)$ on E_1, and on the other points of each set of intervals in which an e_q is enclosed defined to have the value of $f(y)$ at some point of e_q. Then

$$(50)\qquad \left|\int_{E_1} f(y)\phi(x, y)\cos ny\, dy - \int_I f_1(y)\phi(x, y)\cos ny\, dy\right| < MKp\eta = \frac{\epsilon}{8}.$$

Moreover, since $\phi(x, y)$ is continuous in a closed region and therefore uni-

* L. c. § 4; also Hobson, *Theory of functions of a real variable*, pp. 674–675. Lebesgue's theorem is a generalization to the case of integrals of his type of the corresponding theorem due to Riemann.

formly continuous, we can subdivide the intervals of I in such a way that in each of the new intervals the oscillation of $\phi(x, y)$ is less than δ for any value of x in (a, b). Then in each of these intervals the oscillation of $f_1(y)\phi)x, y)$ is less than $(M + K)\delta$, for we have for any two values of y, y_1 and y_2, in one of these intervals

$$
(51)\qquad \begin{aligned} &|f_1(y_2)\phi(x, y_2) - f_1(y_1)\phi(x, y_1)| \\ &\quad \leqq |f_1(y_2)| \cdot |\phi(x, y_2) - \phi(x, y_1)| \\ &\qquad + |\phi(x, y_1)| \cdot |f_1(y_2) - f_1(y_1)| < (M + K)\delta. \end{aligned}
$$

Furthermore, of the subdivided intervals corresponding to each set e_q, we may choose a certain number r_q in descending order of magnitude so that the sum of the remaining intervals is less than η, η being defined by (49). Then we have

$$
(52)\qquad \begin{aligned} &\left| \int_I f_1(y)\phi(x, y)\cos ny\,dy - \sum_{q=1}^{q=p}\sum_{i=1}^{i=r_q}\int_{l_{q,i}} f_1(y)\phi(x, y)\cos ny\,dy \right| \\ &\qquad < MKp\eta = \frac{\epsilon}{8}, \end{aligned}
$$

where $l_{q,i}$ represents the ith interval of the r_q intervals associated with e_q.

If we represent by $c_{q,i}(x)$ the value of $f_1(y)\phi(x, y)$ for some particular value of y in the interval $l_{q,i}$, we have, in view of (51) and (48),

$$
(53)\qquad \begin{aligned} &\left| \sum_{q=1}^{q=p}\sum_{i=1}^{i=r_q}\left\{ \int_{l_{q,i}} f_1(y)\phi(x, y)\cos ny\,dy - c_{q,i}(x)\int_{l_{q,i}}\cos ny\,dy \right\} \right| \\ &\qquad < (M + K)(c - d)\delta = \frac{\epsilon}{4} \qquad (a \leqq x \leqq b). \end{aligned}
$$

Also, for all values of x in (a, b),

$$
(54)\qquad \left| \sum_{x=1}^{q=p}\sum_{i=1}^{i=r_q} c_{q,i}(x)\int_{l_{q,i}}\cos ny\,dy \right| < \frac{2MK(r_1 + r_2 + \cdots + r_q)}{n}.
$$

If now we choose a value m of n, large enough to make the right-hand side of (54) less than $\frac{1}{4}\epsilon$, we have on combining (47), (50), (52), (53), and (54)

$$
\left| \int_c^d f(y)\phi(x, y)\cos ny\,dy \right| < \epsilon \qquad \left(\begin{matrix} n \geqq m \\ a \leqq x \leqq b \end{matrix} \right).
$$

Our lemma is therefore proved.

LEMMA 5. *If $\phi(x, y)$ is integrable (Lebesgue) with regard to y in the interval $(a \leqq y \leqq b)$ for all values of x such that $(c \leqq x \leqq d)$ and for any positive ϵ*

$$\int_a^e |\phi(x, y)| dy < \epsilon \qquad (c \leqq x \leqq d),$$

when e is properly chosen, and if furthermore $F(x, y, n)$ is a function that remains finite for all values of x and y in the region $(a \leqq y \leqq b;\ c \leqq x \leqq d)$ and all values of n, and is such that

$$\lim_{n \to \infty} F(x, y, n) = 0$$

uniformly for all values of x and y in the region $(a + \delta \leqq y \leqq b;\ c \leqq x \leqq d)$, where δ is any quantity > 0, then

$$\lim_{n \to \infty} \int_a^b \phi(x, y) F(x, y, n) dy = 0$$

uniformly for all values of x such that $(c \leqq x \leqq d)$.

A positive ϵ being assigned, we may in view of the conditions of the lemma choose δ such that

$$(55) \qquad \left| \int_a^{a+\delta} \phi(x, y) F(x, y, n) dy \right| < \frac{\epsilon}{2} \qquad (c \leqq x \leqq d).$$

Then, δ being fixed, we can select an m such that

$$(56) \qquad \left| \int_{a+\delta}^{b} \phi(x, y) F(x, y, n) dy \right| < \frac{\epsilon}{2} \qquad \left(\begin{matrix} n \geqq m \\ c \leqq x \leqq d \end{matrix} \right).$$

Combining (55) and (56) we obtain

$$\left| \int_a^b \phi(x, y) F(x, y, n) dy \right| < \epsilon \qquad \left(\begin{matrix} n \geqq m \\ c \leqq x \leqq d \end{matrix} \right),$$

and our lemma is proved.

Lemmas on the uniform summability of certain types of trigonometric series. §§ 8–12

8. Lemma 6. *If $g(y)$ is integrable (Lebesgue) in the interval $(0 \leqq y \leqq \pi)$, is continuous in the interval $(0 \leqq y \leqq c)$, and is such that $g(y)/y^{1+\rho}$, where $\rho > 0$, remains finite as $y \to 0$, and if furthermore the integral*

$$(57) \qquad \int_0^u \frac{g(x + 2t) - 2g(x) + g(x - 2t)}{t^2} dt$$

exists and approaches zero uniformly with u for all values of x in the interval $(0 < x \leqq c)$, then the series whose general terms are

$$(58) \qquad n \cos nx \int_0^\pi G(y) \cos ny\, dy, \qquad n \sin nx \int_0^\pi G(y) \sin ny\, dy,$$

where

$$G(y) = g(y) - y\frac{g(x)}{x}, \tag{59}$$

will be uniformly summable for all values of x in the interval $(0 < x \leqq c)$.

We will prove the lemma only for the series corresponding to the first term in (58), since the proof for the other case is entirely analogous. This term may be written in the form

$$\frac{1}{2}\int_0^\pi G(y)\,n\cos n(y+x)\,dy + \frac{1}{2}\int_0^\pi G(y)\,n\cos n(y-x)\,dy. \tag{60}$$

In view of (33) the first Cesàro mean for the series whose general term is the first term of (60), is given by

$$\begin{aligned}
&\frac{1}{8}\int_0^\pi \frac{g(y)}{y^2}\cdot\frac{y^2}{\sin^2\frac{1}{2}(y+x)}\cdot\frac{\sin(n+\frac{1}{2})(y+x)}{n\sin\frac{1}{2}(y+x)}\,dy\\
&\quad-\frac{g(x)}{8x\sin^\rho\frac{1}{2}x}\int_0^\pi\frac{\sin^\rho\frac{1}{2}x}{\sin^\rho\frac{1}{2}(y+x)}\cdot\frac{y}{\sin^{2-\rho}\frac{1}{2}(y+x)}\cdot\frac{\sin(n+\frac{1}{2})(y+x)}{n\sin\frac{1}{2}(y+x)}\,dy\\
&\quad-\frac{1}{8}\left(1+\frac{1}{n}\right)\left[\int_0^\pi\frac{g(y)}{y^2}\cdot\frac{y^2}{\sin^2\frac{1}{2}(y+x)}\,dy\right.\\
&\qquad\qquad\left.-\frac{g(x)}{x\sin^\rho\frac{1}{2}x}\int_0^\pi\frac{\sin^\rho\frac{1}{2}x}{\sin^\rho\frac{1}{2}(y+x)}\cdot\frac{y}{\sin^{2-\rho}\frac{1}{2}(y+x)}\,dy\right]\\
&\quad-\frac{1}{8}\int_0^\pi\frac{g(y)}{y^2}\cdot\frac{y^2}{\sin^2\frac{1}{2}(y+x)}\cdot\cos(n+1)(y+x)\,dy+\frac{g(x)}{8x\sin^\rho\frac{1}{2}x}\\
&\quad\times\int_0^\pi\frac{\sin^\rho x}{\sin^\rho\frac{1}{2}(y+x)}\cdot\frac{y^{2-\rho}}{\sin^{2-\rho}\frac{1}{2}(y+x)}\cdot\frac{1}{y^{1-\rho}}\cos(n+1)(y+x)\,dy.
\end{aligned}\tag{61}$$

If we break up the integral in the second term of (60) into $\int_0^{2x}$ and $\int_{2x}^{\pi}$, the first Cesàro mean for the series whose general term is the part of this term corresponding to the second integral may be written in the form (61) provided we replace $(y+x)$ wherever it occurs by $(y-x)$, and use for the lower limits of the integrals $2x$ instead of zero. The discussion of the various terms of this expression is then found to be entirely analogous to the discussion of (61) as written above. We shall therefore carry through the latter discussion and omit the former one.

It follows from Lemma 5 that as n becomes infinite the first and second terms of (61) approach zero uniformly for all values of x in the interval $(0 < x \leqq c)$. That the third term approaches a limit uniformly for all values of x in the same interval as n becomes infinite is readily apparent. If in the fourth and fifth terms we expand $\cos(n+1)(y+x)$ in terms of functions of $(n+1)y$ and $(n+1)x$, we may show by an application of Lemma 4 that these terms approach zero as a limit uniformly in the interval $(0 < x \leqq c)$ as n becomes infinite. Thus it has been shown that the first term of (60) is the general term of a series that is uniformly summable $(C\,1)$ in the interval $(0 < x \leqq c)$.

We turn now to the discussion of the second term. We pointed out above that the discussion of the portion of this term corresponding to $\int_{2x}^{\pi}$ is analogous to the discussion of the first term. It remains then to consider the portion of the second term corresponding to $\int_0^{2x}$. We have for the first Cesàro mean of the series whose general term is this portion

$$(62) \qquad \int_0^{2x} G(y) \frac{\begin{array}{c}\sin(n+\frac{1}{2})(y-x) - (n+1)\sin\frac{1}{2}(y-x) \\ - n\cos(n+1)(y-x)\sin\frac{1}{2}(y-x)\end{array}}{8n\sin^3\frac{1}{2}(y-x)}\,dy.$$

If in (62) we break up the interval of integration into $(0, x)$ and $(x, 2x)$, set $y - x = -2t$ in the first interval and $y - x = 2t$ in the second, and then recombine and make use of (59), we obtain

$$(63) \qquad \begin{aligned} &\frac{1}{4}\int_0^{x/2} \frac{g(x+2t) - 2g(x) + g(x-2t)}{t^2}\cdot\frac{t^2}{\sin^2 t}\cdot\frac{\sin(2n+1)t}{n\sin t}\,dt \\ &-\frac{1}{4}\left(1+\frac{1}{n}\right)\int_0^{x/2} \frac{g(x+2t) - 2g(x) + g(x-2t)}{t^2}\cdot\frac{t^2}{\sin^2 t}\,dt \\ &-\frac{1}{4}\int_0^{x/2} \frac{g(x+2t) - 2g(x) + g(x-2t)}{t^2}\cdot\frac{t^2}{\sin^2 t}\cdot\cos 2(n+1)t\,dt, \end{aligned}$$

where the convergence of the various integrals, and hence the justification for breaking up into three terms, follows from the requirement on $g(y)$ that the integral (57) exist. In view of the further requirement that this integral approach zero with u uniformly for all values of x in the interval $(0 < x \leqq c)$ and Lemma 5, it follows that the first term of (63) approaches zero uniformly for all values of x in the same interval as n becomes infinite. That the second term approaches a limit uniformly for values of x in the same interval as n becomes infinite is readily apparent. It remains to discuss the third term.

Given a positive ϵ, we may in view of the condition imposed on the integral (57) choose $\delta < \frac{1}{2}x$ and so small that

$$(64) \qquad \left|\frac{1}{4}\int_0^{\delta} \frac{g(x+2t) - 2g(x) + g(x-2t)}{t^2}\cdot\frac{t^2}{\sin^2 t}\cos 2(n+1)t\,dt\right| < \frac{\epsilon}{2}.$$

If in the portion of the third term of (63) corresponding to $\int_{\delta}^{x/2}$ we reverse the transformations previously made, it becomes

$$
(65)\qquad \begin{aligned} -\frac{1}{8}\int_0^{x-2\delta} G(y)\frac{1}{\sin^2\frac{1}{2}(y-x)}\cos(n+1)(y-x)\,dy \\ -\frac{1}{8}\int_{x+2\delta}^{2x} G(y)\frac{1}{\sin^2\frac{1}{2}(y-x)}\cos(n+1)(y-x)\,dy. \end{aligned}
$$

If in each term of (65) we expand $\cos(n+1)(y-x)$ in terms of functions of $(n+1)y$ and $(n+1)x$, it follows from Lemma 4 that for large enough values of n (65) is less in absolute value than $\frac{1}{2}\epsilon$ for all values of x in $(0 < x \leqq c)$. Hence the third term of (63) approaches zero uniformly for all values of x in this interval as n becomes infinite.

We have now shown that each term of (63) approaches a limit uniformly for all values of x in the interval $(0 < x \leqq c)$ as n becomes infinite. Hence (63) and therefore (62) has this same property. It follows then that the series whose general term is the portion of the second term of (60) corresponding to $\int_0^{2x}$ is uniformly summable $(C\,1)$ in the interval $(0 < x \leqq c)$. As we have pointed out before the proof of the uniform summability of the series whose general term is the portion corresponding to $\int_{2x}^{\pi}$ is analogous to the proof of the corresponding fact for the first term of (60). Hence each term of (60), and therefore (60) itself or the first term of (58) is the general term of a series that is uniformly summable $(C\,1)$ in the interval $(0 < x \leqq c)$. As pointed out before the proof for the second term of (58) is analogous, and therefore our lemma may be regarded as established.

9. Lemma 7. *If $g(y)$ satisfies the first three conditions of Lemma* 6, *and furthermore the integral*

$$
(66)\qquad \int_0^u \frac{g(x+2t)-g(x-2t)}{t}\,dt
$$

exists, and the function

$$
(67)\qquad g_1(x, u) = \frac{1}{u}\int_0^u \frac{g(x+2t)-g(x-2t)}{t}\,dt
$$

is such that its total variation in the interval $(0 \leqq u \leqq \gamma)$ approaches zero with γ uniformly for all values of x in the interval $(0 < x \leqq c)$, then the series whose general terms are

$$
(68)\qquad n\sin nx\int_0^\pi G(y)\cos ny\,dy, \qquad n\cos nx\int_0^\pi G(y)\sin ny\,dy,
$$

where $G(y)$ is defi ed by (59), *will be uniformly summable for all values of x in the interval $(0 < x \leqq c)$.*

We will prove the lemma only for the series corresponding to the first term in (68), since the proof for the second term is entirely analogous. The first term may be written in the form

$$(69)\quad \frac{1}{2}\int_0^\pi G(y)n\sin n(y+x)\,dy - \frac{1}{2}\int_0^\pi G(y)n\sin n(y-x)\,dy.$$

In view of (39) the first Cesàro mean for the series whose general term is the first term of (69) is given by

$$\frac{1}{8}\left(1+\frac{1}{n}\right)\int_0^\pi \frac{g(y)}{y^2}\cdot\frac{y^2}{\sin^2\frac{1}{2}(y+x)}\cdot\sin(n+1)(y+x)\,dy$$

$$-\frac{1}{8}\left(1+\frac{1}{n}\right)\frac{g(x)}{x\sin^\rho\frac{1}{2}x}\int_0^\pi\frac{\sin^\rho\frac{1}{2}x}{\sin^\rho\frac{1}{2}(y+x)}\cdot\frac{y^{2-\rho}}{\sin^{2-\rho}\frac{1}{2}(y+x)}$$

$$\times\frac{1}{y^{1-\rho}}\sin(n+1)(y+x)\,dy$$

$$(70)\qquad -\frac{1}{4}\int_0^\pi\frac{g(y)}{y^2}\cdot\frac{y^2}{\sin^2\frac{1}{2}(y+x)}$$

$$\times\frac{\sin(n+1)\frac{1}{2}(y+x)}{n\sin\frac{1}{2}(y+x)}\sin(n+1)\tfrac{1}{2}(y+x)\cos\tfrac{1}{2}(y+x)\,dy$$

$$+\frac{g(x)}{x\sin^\rho\frac{1}{2}x}\int_0^\pi\frac{\sin^\rho\frac{1}{2}x}{\sin^\rho\frac{1}{2}(y+x)}\cdot\frac{y}{\sin^{2-\rho}\frac{1}{2}(y+x)}$$

$$\times\frac{\sin(n+1)\frac{1}{2}(y+x)}{n\sin\frac{1}{2}(y+x)}\sin(n+1)\tfrac{1}{2}(y+x)\cos\tfrac{1}{2}(y+x)\,dy.$$

If we break up the integral in the second term of (69) into $\int_0^{2x}$ and $\int_{2x}^\pi$, the first Cesàro mean for the series whose general term is the part of this term corresponding to the second integral may be written in the form (70) provided we replace $(y+x)$ wherever it occurs by $(y-x)$, and use for the lower limits of the integrals $2x$ instead of zero. The discussion of the various terms of this expression is then found to be entirely analogous to the discussion of (70) as written above. We shall therefore give only the latter discussion.

It follows from Lemma 4 that the first and second terms of (70) approach zero uniformly for all values of x in the interval $(0 < x \leqq c)$ as n becomes infinite. By the use of Lemma 5 we may establish the same property for the third and fourth terms. It follows then that the series whose general term is the first term of (69) is uniformly summable $(C\,1)$ in the interval $(0 < x \leqq c)$.

We turn then to the second term. As pointed out above the discussion of the portion of this term corresponding to $\int_{2x}^\pi$ is analogous to the discussion of the first term. It remains to consider the portion corresponding to $\int_0^{2x}$. We have for the first Cesàro mean of the series whose general term is this portion

$$(71)\quad \frac{1}{8}\int_0^{2x} G(y)\, \frac{(n+1)\sin(n+1)(y-x)\sin\frac{1}{2}(y-x) - 2\sin^2(n+1)\frac{1}{2}(y-x)\cos\frac{1}{2}(y-x)}{n\sin^3\frac{1}{2}(y-x)}\, dy.$$

If in (71) we break up the interval of integration into $(0, x)$ and $(x, 2x)$, set $y - x = -2t$ in the first interval and $y - x = 2t$ in the second, and then recombine and make use of (59), we obtain

$$(72)\quad \begin{aligned} &\frac{1}{4}\left(1+\frac{1}{n}\right)\int_0^{x/2}\{g(x+2t) - g(x-2t)\}\frac{\sin 2(n+1)t}{\sin^2 t}\,dt \\ &-\left(1+\frac{1}{n}\right)\frac{g(x)}{x}\int_0^{x/2}\frac{t}{\sin t}\cdot\frac{\sin 2(n+1)t}{\sin t}\,dt \\ &-\frac{1}{2n}\int_0^{x/2}\{g(x+2t) - g(x-2t)\}\cos t\,\frac{\sin^2(n+1)t}{\sin^3 t}\,dt \\ &+\frac{2g(x)}{nx}\int_0^{x/2}\frac{t}{\sin t}\cdot\frac{\sin^2(n+1)t}{\sin^2 t}\,t\,dt. \end{aligned}$$

From the fact that $g(x)/x$ approaches zero with x and the well-known properties of Dirichlet's integral and Fejér's integral it follows without difficulty that the second and fourth terms approach a limit uniformly for all values of x in the interval $(0 < x \leqq c)$. It remains to consider the first and third terms.

We may replace the first term by

$$(73)\quad \frac{1}{4}\left(1+\frac{1}{n}\right)\int_0^{x/2}\frac{g(x+2t) - g(x-2t)}{t}\cdot\frac{\sin 2(n+1)t}{t}\,dt,$$

for the difference between the two expressions has the form

$$(74)\quad \frac{1}{4}\left(1+\frac{1}{n}\right)\int_0^{x/2}\{g(x+2t) - g(x-2t)\}\frac{t^2 - \sin t^2}{t^2\sin^2 t}\sin 2(n+1)t\,dt,$$

which expression, in view of a theorem due to Hobson,* approaches zero uniformly for all values of x in $(0 < x \leqq c)$. That (73) approaches a limit uniformly in the same interval may be shown by a discussion analogous to that given by de la Vallée Poussin in establishing a certain sufficient condition for the convergence of Fourier's series.† The function $g_1(x, u)$ defined by (67) corresponds to the function $F(u)$ introduced in de la Vallée Poussin's discussion, and the modifications necessary to establish uniformity

*Proceedings of the London Mathematical Society, ser. 2, vol. 5 (1907), p. 277; also *Theory of functions of a real variable*, p. 683.

† Cf. *Un nouveau cas de convergence des séries de Fourier*, Palermo Rendiconti, vol. 31 (1911), p. 296.

of approach are readily apparent. It was in order to secure this uniformity that the restriction upon $F(u)$ made by de la Vallée Poussin was replaced in the hypothesis of our lemma by a corresponding restriction of a uniform character.

We turn now to the discussion of the third term of (72). We may replace it by

$$(75) \qquad -\frac{1}{2n}\int_0^{x/2} \{g(x+2t) - g(x-2t)\} \frac{\sin^2 (n+1)t}{t^3}\, dt,$$

since the difference between the two expressions,

$$(76) \qquad -\frac{1}{2}\int_0^{x/2} \{g(x+2t) - g(x-2t)\} \frac{\sin^3 t - t^3 \cos t}{t^3 \sin^2 t} \cdot \frac{\sin (n+1)t}{n \sin t} \cdot \sin (n+1)t\, dt,$$

is readily seen from Lemma 5 to approach zero uniformly for all values of x in $(0 < x \leqq c)$. It remains to establish the same for (75).

We break up the interval of integration in (75) into $(0, \alpha)$ and $(\alpha, \frac{1}{2}x)$, where α is any positive number $< \frac{1}{2}x$, and consider first the part of (75) corresponding to the former interval. If in this part we integrate by parts, making use of (67), we obtain

$$(77) \qquad \begin{aligned} &-\frac{2}{n} g_1(x, 2\alpha) \frac{\sin^2 (n+1)\alpha}{\alpha} \\ &\quad + 2\left(1+\frac{1}{n}\right)\int_0^{\alpha} g_1(x, 2t) \frac{\sin 2(n+1)t}{t}\, dt \\ &\quad - \frac{4}{n}\int_0^{\alpha} g_1(x, 2t) \frac{\sin^2 (n+1)t}{t^2}\, dt. \end{aligned}$$

That the second term of (77) approaches a limit uniformly for all values of x in $(0 < x \leqq c)$ as $n \to \infty$, provided we properly choose α, follows readily from the condition imposed on $g_1(x, u)$. The proof follows the lines of the well-known treatment of Dirichlet's integral in the theory of the convergence of Fourier's series, the necessary modifications being obvious. The third term of (77) may be replaced by

$$(78) \qquad \frac{4}{n}\int_0^{\alpha} g_1(x, 2t) \frac{\sin^2 (n+1)t}{\sin^2 t}\, dt,$$

since their difference, for a proper choice of α, is readily seen to approach zero uniformly in $(0 < x \leqq c)$ as n becomes infinite. That (78) approaches a limit uniformly for values of x in the same interval may be shown by a discussion analogous to Fejér's proof of the summability of Fourier's series,* the modifications to establish uniformity being apparent. The quantity α having been suitably chosen so as to secure the uniform approach to their limits of the second and third terms of (77), we may for this choice of α make the first term of (77) uniformly small for all values of x in $(0 < x \leqq c)$ by choosing n sufficiently large.

Thus we see that each term of (77), and therefore the whole expression, or the part of (75) corresponding to $\int_0^\alpha$, approaches a limit uniformly in $(0 < x \leqq c)$ as $n \to \infty$, provided α is properly chosen. It is then easy to see that for this fixed value of α the part of (75) corresponding to $\int_\alpha^{x/2}$ may be made as small as we please by choosing n sufficiently large. Thus it follows that (75), or the third term of (72), approaches a limit uniformly in $(0 < x \leqq c)$ as $n \to \infty$. Since we had previously established this same property for the other three terms of (72), it follows that the whole expression, or (71), has this property. Hence the part of the second term of (69) corresponding to $\int_0^{2x}$ is the general term of a series that is uniformly summable $(C1)$ in $(0 < x \leqq c)$. Since the same is true for the part corresponding to $\int_{2x}^{\pi}$ it follows that the whole term has this property. Since we had previously shown the same for the first term, it follows that (69), or the first term of (68) has this same property. The proof for the second term being analogous, our lemma may be regarded as established.

10. LEMMA 8. *If $g(y)$ satisfies the first three conditions of Lemma 6, the series whose general terms are*

$$(80) \qquad \cos nx \int_0^\pi \frac{G(y)}{y} \cos ny\, dy, \qquad \sin nx \int_0^\pi \frac{G(y)}{y} \sin ny\, dy,$$

where $G(y)$ is defined by (59), will be uniformly summable $(C\,1)$ for all values of x in the interval $(0 < x \leqq c)$.

We will prove the lemma only for the series corresponding to the first term in (80), since the proof for the other case is entirely analogous. That term may be written in the form

$$(81) \qquad \frac{1}{2}\int_0^\pi \frac{G(y)}{y} \cos n(y+x)\, dy + \frac{1}{2}\int_0^\pi \frac{G(y)}{y} \cos n(y-x)\, dy.$$

In view of (38) the first Cesàro mean for the series whose general term is the first or second term of (81), is given by

* Mathematische Annalen, vol. 58 (1903–04), pp. 54–59. A more compact discussion in which the conditions on the function are in agreement with our present restrictions, may be found in my symposium paper, Bulletin of the American Mathematical Society, ser. 2, vol. 25 (1919), pp. 259–260.

$$
(82)\qquad \begin{aligned}&\frac{1}{4n}\int_0^\pi \frac{g(y)}{y}\cdot\frac{\sin^2\frac{1}{2}(n+1)(y\pm x)}{\sin^2\frac{1}{2}(y\pm x)}\,dy-\frac{g(x)}{4nx}\\ &\times\int_0^\pi\frac{\sin^2\frac{1}{2}(n+1)(y\pm x)}{\sin^2\frac{1}{2}(y\pm x)}\,dy-\frac{1}{4}\left(1+\frac{1}{n}\right)\left\{\int_0^\pi\frac{g(y)}{y}\,dy-\pi\frac{g(x)}{x}\right\},\end{aligned}
$$

where the upper sign corresponds to the first term, the lower to the second.

It is readily seen that the third term in (82) approaches a limit uniformly for all values of x in the interval $(0 < x \leqq c)$ as n becomes infinite. It remains to consider the first and second terms. If in the second term, the upper sign being used, we set $\frac{1}{2}(y + x) = t$, it takes the form

$$
(83)\qquad \frac{g(x)}{2nx}\int_{x/2}^{\frac{\pi}{2}+\frac{x}{2}}\frac{\sin^2(n+1)t}{\sin^2 t}\,dt.
$$

From the second condition imposed on $g(y)$, it is apparent that $g(x)/x$ approaches zero with x. Hence, given an arbitrary positive ϵ, we may choose a corresponding δ such that (83) is less in absolute value than ϵ for all values of x in the interval $(0 < x \leqq \delta)$ and all values of n. Then, δ being fixed, we may choose an m such that (83) is less in absolute value than ϵ for all values of x in the interval $(\delta \leqq x \leqq c)$, provided $n \geqq m$. Thus we see that (83), and hence the second term of (82), the upper sign being used, approaches zero uniformly for all values of x in the interval $(0 < x \leqq c)$ as n becomes infinite.

We turn now to the consideration of the first term of (82). Since $g(y)/y$ approaches zero with y, we may choose a δ corresponding to an arbitrary positive ϵ, such that

$$
(84)\qquad \left|\frac{1}{4n}\int_0^\delta\frac{g(y)}{y}\cdot\frac{\sin^2\frac{1}{2}(n+1)(y+x)}{\sin^2\frac{1}{2}(y+x)}\,dy\right|<\frac{\epsilon}{2}\quad\begin{pmatrix}0<x\leqq c\\ n=1,2,3,\dots\end{pmatrix}.
$$

Then, δ being fixed, we may choose m such that

$$
(85)\qquad \left|\frac{1}{4n}\int_\delta^\pi\frac{g(y)}{y}\cdot\frac{\sin^2\frac{1}{2}(n+1)(y+x)}{\sin^2\frac{1}{2}(y+x)}\,dy\right|<\frac{\epsilon}{2}\quad\begin{pmatrix}0<x\leqq c\\ n\geqq m\end{pmatrix}.
$$

Combining (84) and (85), we see that the first term of (82), the upper sign being used, approaches zero as a limit as n becomes infinite, uniformly for all values of x in the interval $(0 < x \leqq c)$.

We have now shown that the third term of (82) and the first and second terms, the upper sign being used, each approach a limit uniformly for all values of x in $(0 < x \leqq c)$, as n becomes infinite. Hence the first term of (81) is the general term of a series that is uniformly summable $(C\,1)$ in the interval $(0 < x \leqq c)$. It remains to establish the same for the second term.

If in the first and second terms of (82), the lower sign being used, we make the transformation $y - x = 2t$, and then combine them into a single term, they take the form

$$(86) \qquad \frac{1}{2n}\int_{-x/2}^{\pi/2-x/2}\left\{\frac{g(x+2t)}{x+2t}-\frac{g(x)}{x}\right\}\frac{\sin^2(n+1)t}{\sin^2 t}\,dt.$$

Since $g(x)/x$ is continuous in the closed interval $(0 \leqq x \leqq c)$, and therefore uniformly continuous there, we may choose a δ corresponding to an arbitrary positive ϵ, such that

$$(87) \quad \left|\frac{1}{2n}\int_{-v}^{u}\left\{\frac{g(x+2t)}{x+2t}-\frac{g(x)}{x}\right\}\frac{\sin^2(n+1)t}{\sin^2 t}\,dt\right| < \frac{\epsilon}{2} \quad \left(\begin{array}{c} 0 < x \leqq c \\ n = 1, 2, 3, \cdots \\ 0 < v \leqq \delta \\ 0 < u \leqq \delta \end{array}\right).$$

Then, δ being fixed, we may choose an m such that

$$(88) \quad \left|\frac{1}{2n}\int_{-x/2}^{-\delta}\left\{\frac{g(x+2t)}{x+2t}-\frac{g(x)}{x}\right\}\frac{\sin^2(n+1)t}{\sin^2 t}\,dt\right| < \frac{\epsilon}{4}^{*}$$

$$(89) \quad \left|\frac{1}{2n}\int_{\delta}^{\pi-x/2}\left\{\frac{g(x+2t)}{x+2t}-\frac{g(x)}{x}\right\}\frac{\sin^2(n+1)t}{\sin^2 t}\,dt\right| < \frac{\epsilon}{4} \qquad \left(\begin{array}{c} 0 < x \leqq c \\ n \geqq m \end{array}\right).$$

Combining (87), (88), and (89), we see that the first and second terms of (82), the lower sign being used, approach zero as a limit uniformly in $(0 < x \leqq c)$ as n becomes infinite. Since the third term also approaches a limit uniformly in this same interval, it follows that the second term of (81) is the general term of a series that is uniformly summable $(C\,1)$ in that interval. As we have previously shown that the first term is the general term of such a series, it follows that the whole expression (81) or the first term in (80) has the same property. Hence our lemma is proved for this term and, as pointed out before, the proof for the second term is analogous.

11. LEMMA 9. *If $g(y)$ satisfies the conditions of Lemma 7, the series whose general terms are*

$$(90) \qquad \cos nx \int_0^{\pi}\frac{G(y)}{y}\sin ny\,dy, \qquad \sin nx\int_0^{\pi}\frac{G(y)}{y}\cos ny\,dy,$$

where $G(y)$ is defined by (59), *will be uniformly summable* $(C\,1)$ *for all values of x in the interval* $(0 < x \leqq c)$.

We will consider only the first term in (90), the proof for the second term being analogous. The first term may be written in the form

$$(91) \qquad \frac{1}{2}\int_0^{\pi}\frac{G(y)}{y}\sin n(y+x)\,dy + \frac{1}{2}\int_0^{\pi}\frac{G(y)}{y}\sin n(y-x)\,dy.$$

In view of (32) the first Cesàro mean for the series whose general term is the

* If $\frac{1}{2}x < \delta$, (88) is a consequence of (87) for all values of n; if $\frac{1}{2}x \geqq \delta$ we choose m so as to make both (88) and (89) hold.

first term of (91), is given by

$$
\begin{aligned}
&\frac{1}{4}\left(1+\frac{1}{n}\right)\int_0^{\pi}\frac{g(y)}{y^2}\cdot\frac{y}{\sin\frac{1}{2}(y+x)}\cdot\cos\tfrac{1}{2}(y+x)\,dy \\
&\quad -\frac{1}{4}\left(1+\frac{1}{n}\right)\frac{g(x)}{x^{1+\rho}}\int_0^{\pi}\frac{x^\rho}{(y+x)^\rho}\cdot\frac{y^{1-\rho}}{(y+x)^{1-\rho}} \\
&\qquad\qquad \times\frac{y+x}{\sin\frac{1}{2}(y+x)}\cdot\cos\tfrac{1}{2}(y+x)\cdot\frac{1}{y^{1-\rho}}\,dy \\
(92)\qquad &\quad -\frac{1}{8}\int_0^{\pi}\frac{g(y)}{y^2}\cdot\frac{y}{\sin\frac{1}{2}(y+x)}\cdot\frac{\sin(n+1)(y+x)}{n\sin\frac{1}{2}(y+x)}\,dy \\
&\quad +\frac{1}{8}\frac{g(x)}{x^{1+\rho}}\int_0^{\pi}\frac{x^\rho}{(y+x)^\rho}\cdot\frac{y^{1-\rho}}{(y+x)^{1-\rho}} \\
&\qquad\qquad \times\frac{y+x}{\sin\frac{1}{2}(y+x)}\cdot\frac{\sin(n+1)(y+x)}{n\sin\frac{1}{2}(y+x)}\cdot\frac{1}{y^{1-\rho}}\,dy,
\end{aligned}
$$

where the ρ is the ρ of the third condition in Lemma 6. From the conditions imposed on $g(y)$ it follows readily that each term of (92) approaches a limit uniformly for all values of x in $(0 < x \leqq c)$, as n becomes infinite. Hence the first term of (91) is the general term of a series that is uniformly summable $(C\,1)$ in the above interval. It remains to establish the same for the second term.

The first Cesàro mean for the series whose general term is the second term, may be written in the form

$$
\begin{aligned}
&\frac{1}{4}\left(1+\frac{1}{n}\right)\int_0^{\pi}\left\{\frac{g(y)}{y}-\frac{g(x)}{x}\right\}\cot\tfrac{1}{2}(y-x)\,dy \\
(93)\qquad &\qquad -\frac{1}{8n}\int_0^{\pi}\left\{\frac{g(y)}{y}-\frac{g(x)}{x}\right\}\frac{\sin(n+1)(y-x)}{\sin^2\frac{1}{2}(y-x)}\,dy.
\end{aligned}
$$

If we break up the interval of integration for each integral in (93) into two parts $(0, 2x)$ and $(2x, \pi)$, the part of (93) which corresponds to the second interval may be put in a form that differs from (92) only by having $2x$ for the lower limit of the integrals, instead of zero, and by having $(y - x)$ wherever $(y + x)$ occurs in (92). This part of (93) is readily seen to approach a limit uniformly in $(0 < x \leqq c)$ as n becomes infinite. It remains to consider the other part.

If in this latter part we set $y - x = 2t$, it may be written in the form,

$$
\begin{aligned}
&\frac{1}{2}\left(1+\frac{1}{n}\right)\int_{-\frac{x}{2}}^{x/2}\left\{\frac{g(x+2t)}{x+2t}-\frac{g(x)}{x}\right\}\cot t\,dt \\
(94)\qquad &\qquad -\frac{1}{4n}\int_{-\frac{x}{2}}^{x/2}\left\{\frac{g(x+2t)}{x+2t}-\frac{g(x)}{x}\right\}\frac{\sin 2(n+1)t}{\sin^2 t}\,dt.
\end{aligned}
$$

If we break up the intervals of integration of the integrals in (94) into $(-\frac{1}{2}x, 0)$ and $(0, \frac{1}{2}x)$, set $t = -t$ in the integrals over the first interval and then recombine, we obtain

$$\begin{aligned}(95)\quad &\frac{1}{2}\left(1+\frac{1}{n}\right)\int_0^{x/2}\left\{\frac{g(x+2t)}{x+2t}-\frac{g(x-2t)}{x-2t}\right\}\cot t\,dt\\ &\qquad -\frac{1}{4n}\int_0^{x/2}\left\{\frac{g(x+2t)}{x+2t}-\frac{g(x-2t)}{x-2t}\right\}\frac{\sin 2(n+1)t}{\sin^2 t}dt.\end{aligned}$$

The first term of (95) may be written in the form

$$\begin{aligned}(96)\quad &\frac{1}{2}\left(1+\frac{1}{n}\right)\int_0^{x/2}\frac{1}{x+2t}\{g(x+2t)-g(x-2t)\}\cot t\,dt\\ &\qquad -2\left(1+\frac{1}{n}\right)\frac{1}{x}\int_0^{x/2}\frac{x}{x+2t}\cdot\frac{g(x-2t)}{x-2t}\cdot t\cot t\,dt.\end{aligned}$$

It is readily seen that the second term of (96) approaches a limit uniformly for all values of x in $(0 < x \leqq c)$ as n becomes infinite. It remains to establish the same for the first term. That term may be written in the form

$$\begin{aligned}(97)\quad &\frac{n+1}{2n}\cdot\frac{1}{x}\int_0^{x/2}\frac{g(x+2t)-g(x-2t)}{t}dt\\ &\qquad +\frac{n+1}{2n}\int_0^{x/2}\{g(x+2t)-g(x-2t)\}\frac{t\cos t-\sin t}{(x+2t)t\sin t}dt\\ &\qquad -\frac{n+1}{n}\cdot\frac{1}{x}\int_0^{x/2}\left\{\frac{g(x+2t)}{x+2t}-\frac{g(x-2t)}{x+2t}\right\}dt.\end{aligned}$$

It follows from the conditions imposed upon $g(y)$ that each term of (97) approaches a limit uniformly in $(0 < x \leqq c)$ as n becomes infinite. Hence (96), and therefore the first term of (95), has this same property. We have now to prove the same for the second term of (95).

This term may be written in the form

$$\begin{aligned}(98)\quad &-\frac{1}{4n}\int_0^{x/2}\frac{1}{x+2t}\{g(x+2t)-g(x-2t)\}\frac{\sin 2(n+1)t}{\sin^2 t}dt\\ &\qquad +\frac{1}{x}\int_0^{x/2}\frac{x}{x+2t}\cdot\frac{g(x-2t)}{x-2t}\cdot\frac{t}{\sin t}\cdot\frac{\sin 2(n+1)t}{n\sin t}dt.\end{aligned}$$

The second term of (98) may readily be shown to approach zero uniformly in $(0 < x \leqq c)$ as n becomes infinite. It remains to consider the first term. This term may be put in the form

$$
\begin{aligned}
& -\frac{1}{4x}\int_0^{x/2}\frac{g(x+2t)-g(x-2t)}{t}\cdot\frac{\sin 2(n+1)t}{n\sin t}\,dt \\
(99)\qquad & -\frac{1}{4}\int_0^{x/2}\{g(x+2t)-g(x-2t)\}\frac{t-\sin t}{(x+2t)t\sin t}\cdot\frac{\sin 2(n+1)t}{n\sin t}\,dt \\
& +\frac{1}{2x}\int_0^{x/2}\left\{\frac{g(x+2t)}{x+2t}-\frac{g(x-2t)}{x+2t}\right\}\frac{\sin 2(n+1)t}{n\sin t}\,dt.
\end{aligned}
$$

The second and third terms of (99) are readily seen to approach zero as a limit uniformly in $(0 < x \leqq c)$ as n becomes infinite. The first term requires further consideration.

If in this term we integrate by parts, making use of (67), we obtain

$$
\begin{aligned}
& -\frac{1}{4nx}\left[4tg_1(x, 2t)\frac{\sin 2(n+1)t}{\sin t}\right]_0^{x/2} \\
& +\frac{1}{4nx}\int_0^{x/2}4tg_1(x, 2t)\left[\frac{2(n+1)\cos 2(n+1)t}{\sin t} - \frac{\sin 2(n+1)t\cos t}{\sin^2 t}\right]dt \\
(100)\qquad & = -\frac{g_1(x, x)\sin(n+1)x}{2n\sin\frac{1}{2}x} \\
& +2\left(1+\frac{1}{n}\right)\frac{1}{x}\int_0^{x/2}g_1(x, 2t)\frac{t}{\sin t}\cos 2(n+1)t\,dt \\
& -\frac{1}{nx}\int_0^{x/2}g_1(x, 2t)\frac{t\cos t}{\sin t}\cdot\frac{\sin 2(n+1)t}{\sin t}\,dt.
\end{aligned}
$$

We are going to show that the right-hand side of (100) approaches zero uniformly for all values of x in the interval $(0 \leqq x < c)$ as n becomes infinite. We must first establish an additional property of $g_1(x, u)$.

If we extend the region of definition of $g(y)$ to the left of the origin by setting $g(-y) = -g(y)$, it s apparent from the third condition imposed on $g(y)$ in Lemma 6 that it possesses a zero derivative at the origin. Hence $g_1(0, u)$ approaches zero with u, and since $g_1(x, u)$ is a continuous function of x for a fixed $u > 0$ and approaches a limit uniformly for all values of x in $(0 < x \leqq c)$ as u approaches zero, it follows that we may make $g_1(x, u)$ as small as we please if we choose x and u sufficiently small. Therefore we may choose a positive δ corresponding to an arbitrary positive ϵ, such that the right-hand side of (100) is less in absolute value than ϵ for all values of x in $(0 < x \leqq \delta)$ and all values of n. Then being fixed, we may choose m so large that this same expression is less in absolute value than ϵ for $(\delta < x$

$\leqq c$) and values of $n \geqq m$.* Hence it follows that the right-hand side of (100), and therefore the first term of (99), approaches zero uniformly in $(0 < x \leqq c)$ as n becomes infinite. Therefore (99) and hence (98), or the second term of (95) has this same property.

We have now shown that each term of (95) approaches a limit uniformly in $(0 < x \leqq c)$ as n becomes infinite. Hence (95), and therefore (94), has this property. Since (94) is part of (93) and the other part has this property, it follows that (93) has it. Therefore the second term of (91) is the general term of a series that is uniformly summable $(C\,1)$ in $(0 < x \leqq c)$. Since we previously showed that the first term was the general term of such a series, it follows that (91), and therefore the first term in (90), has this same property. Hence our lemma is proved for the first term of (90) and, as pointed out before, the proof for the second term is analogous.

12. Lemma 10. *The series whose general terms are*

$$(101) \qquad (-1)^n \cos n\pi x, \qquad (-1)^n \sin n\pi x,$$

are uniformly summable $(C\,1)$ *in the interval* $(0 \leqq x \leqq c < 1)$.

We may find the sum of the first n terms of each of these two series by first finding the sum of the first n terms of the series whose general term is $(-1)^n e^{in\pi x}$ and then taking the real and imaginary parts respectively for the two sums we wish to obtain. We thus find for the two sums in question the expressions

$$(102) \qquad \begin{gathered} -\frac{1}{2} + (-1)^n \frac{\cos n\pi x + \cos (n+1)\pi x}{2(1+\cos \pi x)}, \\ \frac{-\sin \pi x + (-1)^n \{\sin n\pi x + \sin (n+1)\pi x\}}{2(1+\cos \pi x)}. \end{gathered}$$

From these expressions we readily obtain for the arithmetic mean of the first n sums in each of the two cases the following results

$$(103) \qquad \begin{gathered} -\frac{1}{2} - \frac{\cos \pi x + (-1)^{n+1} \cos (n+1)\pi x}{2n(1+\cos \pi x)}, \\ -\frac{\sin \pi x}{2(1+\cos \pi x)} - \frac{\sin \pi x + (-1)^{n+1} \sin (n+1)\pi x}{2n(1+\cos \pi x)}. \end{gathered}$$

It is easy to see that each of the two expressions in (103) approaches a limit uniformly in the interval $(0 \leqq x \leqq c < 1)$. Our lemma is therefore proved.

Lemma 11. *The series whose general terms are*

$$(104) \qquad n \cos nx \int_0^\pi y \sin ny\, dy, \qquad n \sin nx \int_0^\pi y \sin ny\, dy$$

are uniformly summable in the interval $(0 \leqq x \leqq c < \pi)$.

* In connection with the second term on the right-hand side of (100) we need to apply here Lemma 4.

By means of the transformation $ny = z$, the integrals in (104) may be reduced to a form that is readily integrated, whence it is seen that the two expressions in (104) reduce to

$$\pi[(-1)^{n+1}\cos nx], \qquad \pi[(-1)^{n+1}\sin nx].$$

It follows from Lemma 10 and a change of variable that each of these two expressions is the general term of a series that is uniformly summable in the interval $(0 \leqq x \leqq c < \pi)$. Hence the expressions in (104) have the same property, and our lemma is proved.

Lemmas on the uniform convergence of certain types of trigonometric series. §§ 13–14

13. Lemma 12. *The series whose general terms are*

$$(108) \qquad n\cos nx \int_0^\pi y\cos ny\,dy, \qquad n\sin nx \int_0^\pi y\cos ny\,dy$$

are uniformly convergent in the interval $(0 \leqq x \leqq c < \pi)$.

By means of the transformation used in Lemma 11 we are able to evaluate the integral in each of the terms of (108). These terms reduce thus to the form

$$(109) \qquad (-1)^n\frac{\cos nx}{n}, \qquad (-1)^n\frac{\sin nx}{n}.$$

We have shown in Lemma 10 that the series whose general terms are the expressions in (101) are uniformly summable $(C\,1)$ in the interval $(0 \leqq x \leqq c < 1)$. From a combination of this fact, a change of variable, and the theorem of M. Riesz referred to in § 6, we are able to infer at once the truth of our lemma.

14. Lemma 13. *The series whose general term is*

$$(110) \qquad x^\rho \cos nx \int_0^\pi \sin ny\,dy,$$

where ρ *is any positive constant, is uniformly convergent in the interval* $(0 \leqq x \leqq c < \pi)$.

On evaluating the integral in (110), this expression reduces to the form $(-2x^\rho\cos nx)/n$ when n is odd, and to zero when n is even. Hence the proof of our lemma resolves itself into establishing the uniform convergence of the series

$$(111) \qquad x^\rho\cos x + \frac{x^\rho\cos 3x}{3} + \frac{x^\rho\cos 5x}{5} + \cdots$$

in the interval in question.

We consider first the series

$$\sin x + \sin 3x + \sin 5x + \cdots. \tag{112}$$

By the device of summing the first n terms of the geometric series $e^{ix} + e^{3ix} + e^{5ix} + \cdots$ and taking the pure imaginary part of that sum we obtain for the sum of the first n terms of (112), provided x is such that $\sin x \neq 0$,

$$\frac{1}{2} - \frac{\cos 2nx}{2 \sin x}. \tag{113}$$

But since

$$\int_x^{\pi/2} \sin nx dx = \frac{\cos nx}{n} \qquad (n = 1, 3, 5, \cdots), \tag{114}$$

we hav

$$\sum_{m=0}^{m=n-1} x^\rho \frac{\cos (2m+1)x}{2m+1} = x^\rho \left(\frac{\pi}{4} - \frac{x}{2}\right) - x^\rho \int_x^{\pi/2} \frac{\cos 2nx}{2 \sin x} dx. \tag{115}$$

It is apparent from the theorem of Lebesgue referred to in § 4 that the second term on the right-hand side of (115) approaches zero for any value of $x > 0$. If we can show that for any given positive ϵ we can choose n so large that this term is less in absolute value than ϵ for all values of x in $(0 < x \leqq c < \pi)$, our lemma will be established.*

We have

$$\left| x^\rho \int_x^{\pi/2} \frac{\cos 2nx}{2 \sin x} dx \right| < x^\rho \int_x^{\pi/2} \frac{dx}{x} = x^\rho \left[\log \frac{\pi}{2} - \log x \right]. \tag{116}$$

We may therefore choose a δ such that the second term of (115) is less in absolute value than ϵ when $(0 < x \leqq \delta)$, for all values of n. Then, δ being fixed, we may choose an m so large that this same term is less in absolute value than ϵ when $(\delta < x \leqq c < \pi)$ and $n \geqq m$.† Hence, as pointed out before, our lemma is established.

LEMMA 14. *The series whose general term is*

$$x^\rho \sin nx \int_0^\pi \sin ny \, dy, \tag{117}$$

where ρ is any positive constant, is uniformly convergent in the interval $(0 \leqq x \leqq c < \pi)$.

* The convergence of the series for $x = 0$ is obvious, since each term is equal to zero in that case.

† This follows from the relationship

$$\int_x^{\pi/2} \frac{\cos 2nx}{\sin x} dx = -\frac{\sin 2nx}{4n \sin x} + \frac{1}{4n} \int_x^{\pi/2} \frac{\sin 2nx}{\sin^2 x} \cos x \, dx,$$

obtained from an integration by parts.

If we evaluate the integral in (117), we find that the proof of our lemma reduces to proving the uniform convergence of the series

$$x^\rho \sin x + \frac{x^\rho \sin 3x}{3} + \frac{x^\rho \sin 5x}{5} + \cdots \tag{118}$$

in the interval in question.

We consider first the series

$$\cos x + \cos 3x + \cos 5x + \cdots . \tag{119}$$

By the device of the previous lemma we obtain for the sum of the first n terms of (119)

$$\frac{\sin 2nx}{2 \sin x}. \tag{120}$$

But since

$$\int_x^{\pi/2} \cos nx\, dx = \left[\frac{\sin nx}{n}\right]_x^{\pi/2} = \frac{(-1)^{(n-1)/2}}{n} - \frac{\sin nx}{n} \qquad (n = 1, 3, 5, \cdots), \tag{121}$$

we have

$$\sum_{n=0}^{m=n-1}\left[x^\rho \frac{(-1)^m}{2m+1} - x^\rho \frac{\sin(2m+1)x}{2m+1}\right] = x^\rho \int_x^{\pi/2} \frac{\sin 2nx}{2 \sin x}\, dx. \tag{122}$$

Equation (122) furnishes a fresh proof of the well-known fact* that the series (118) without the factor x^ρ, converges to a constant value for all values of x between 0 and π. We shall use it further to prove the convergence of (118) to be uniform in the interval $(0 \leqq x \leqq c < \pi)$.

We have from (122)

$$\begin{aligned} \left|\sum_{m=0}^{m=n-1} x^\rho \frac{(-1)^m}{2m+1} - \sum_{m=0}^{m=n-1} x^\rho \frac{\sin(2m+1)x}{2m+1}\right| &< x^\rho \int_x^{\pi/2} \frac{1}{x}\, dx \\ &= x^\rho \left[\log \frac{\pi}{2} - \log x\right]. \end{aligned} \tag{123}$$

Given an arbitrary positive ϵ, we may choose δ so that the right-hand side of (123) is less in absolute value than $\frac{1}{2}\epsilon$ for all values of x such that $(0 < x \leqq \delta)$. We may then choose an m_1 so that for the same values of x and all values of $n \geqq m_1$ the first summation on the left-hand side of (123) differs from the value to which it converges, $\frac{1}{4}\pi x^\rho$, by less than $\frac{1}{2}\epsilon$. Then we have

$$\left|\tfrac{1}{4}\pi x^\rho - \sum_{m=0}^{m=n-1} x^\rho \frac{\sin(2m+1)x}{2m+1}\right| < \epsilon \qquad \begin{pmatrix} 0 < x \leqq \delta \\ n \geqq m_1 \end{pmatrix}. \tag{124}$$

Now, δ being fixed, we may choose an m_2 so large that the right-hand side of (122) is less in absolute value than $\frac{1}{2}\epsilon$ for all values of x in $(\delta \leqq x \leqq c)$ and

* Cf. Byerly, *Fourier's Series*, etc., p. 39.

all values of $n \geqq m_2$,* while the summation corresponding to the first term on the left-hand side of (122) differs from $\frac{1}{4}\pi x^\rho$ by a quantity less in absolute value than $\frac{1}{2}\epsilon$. Then we have

$$\left| \frac{1}{4}\pi x^\rho - \sum_{m=0}^{m=n-1} x^\rho \frac{\sin(2m+1)x}{2m+1} \right| < \epsilon \qquad \left(\begin{matrix} \delta \leqq x \leqq c \\ n \geqq m_2 \end{matrix} \right). \tag{125}$$

Taking m as the larger of the two numbers m_1 and m_2, we have from a combination of (124) and (125)

$$\left| \frac{1}{4}\pi x^\rho - \sum_{m=0}^{m=n-1} x^\rho \frac{\sin(2m+1)x}{2m+1} \right| < \epsilon \qquad \left(\begin{matrix} 0 < x \leqq c < \pi \\ n \geqq m \end{matrix} \right). \tag{126}$$

Our lemma is therefore proved since the convergence of the series for $x = 0$ is obvious.

Further lemmas on the uniform summability of certain types of series. §§ 15–17

15. Lemma 15. *If $\phi(y)$ satisfies the conditions of Lemma 2 and if furthermore $\chi(y) = \sqrt{y}\,\phi(y)$ satisfies the fourth condition imposed on $g(y)$ in Lemma 6 and the second and third conditions imposed on $g(y)$ in Lemma 7, then the functions*

$$\begin{matrix} \sqrt{y}\,\phi(y)\cos(qy-\alpha), & \sqrt{y}\,\phi(y)\sin(qy-\alpha), \\ \sqrt{y}\,\phi(y)\,y^2\cos(qy-\alpha), & \sqrt{y}\,\phi(y)\,y^2\sin(qy-\alpha), \end{matrix} \tag{130}$$

will satisfy the conditions imposed on $g(y)$ in Lemmas 6 and 7.

We will consider only the first function of (130) since the proof for the other functions is analogous. From the fact that $\phi(y)$ satisfies the conditions of Lemma 2, it is readily apparent that the first function of (130) satisfies the first three conditions of Lemma 6 and therefore the first condition of Lemma 7. If we represent this function by $F(y)$, we have

$$\begin{aligned} F(y+2t) &- 2F(y) + F(y-2t) \\ &= \cos(qy-\alpha)[\chi(y+2t) - 2\chi(y) + \chi(y-2t)] \\ &\quad - 2\sin qt \sin(qy+qt-\alpha)[\chi(y+2t) - \chi(y-2t)] \\ &\quad - 4\sin^2 qt \cos(qy-\alpha)\chi(y-2t). \end{aligned} \tag{131}$$

$$\begin{aligned} F(y+2t) - F(y-2t) &= \cos(qy-\alpha)[\chi(y+2t) - \chi(y-2t)] \\ &\quad - 2\sin qt \sin(qy-qt-\alpha)\chi(y-2t) \\ &\quad - 2\sin qt \sin(qy+qt-\alpha)\chi(y+2t). \end{aligned} \tag{132}$$

From these two equations and the conditions imposed on $\chi(y)$ it is readily

* That this is possible follows from an integration by parts in the integral on the right-hand side of (122) similar to that given in a previous footnote for the integral on the right-hand side of (115).

inferred that $F(y)$, or the first function in (130), satisfies the fourth condition imposed on $g(y)$ in Lemma 6 and the second and third conditions imposed on $g(y)$ in Lemma 7. Hence the present lemma is established.

We find it convenient to introduce here the following notation,

$$\Phi_n(x) = A\cos[(n\pi + q)x - \alpha] + B[\sin(n\pi + q)x - \alpha]. \tag{133}$$

We are now ready for the proof of the following lemma:

LEMMA 16. *The series whose general terms are*

$$\begin{aligned} &n\Phi_n(x)\int_0^1 \sqrt{y}\,\phi(y)\cos(qy-\alpha)\cos n\pi y\,dy,\\ &n\Phi_n(x)\int_0^1 \sqrt{y}\,\phi(y)\sin(qy-\alpha)\sin n\pi y\,dy,\\ &\Phi_n(x)\int_0^1 \frac{\phi(y)}{\sqrt{y}}(y^2+b)\sin(qy-\alpha)\cos n\pi y\,dy,\\ &\Phi_n(x)\int_0^1 \frac{\phi(y)}{\sqrt{y}}(y^2+b)\cos(qy-\alpha)\sin n\pi y\,dy, \end{aligned} \tag{134}$$

will be uniformly summable $(C\,1)$ *in the interval* $(0 < x \leqq c < 1)$, *provided* $\phi(y)$ *satisfies the conditions of Lemma* 15.

We will carry through the proof only for the first term in (134), since the proof for the other terms is analogous. If we substitute in this term for $\Phi_n(x)$ its value, it takes the form

$$\begin{aligned} &\{A\cos(qx-\alpha) + B\sin(qx-\alpha)\}n\cos n\pi x\\ &\qquad\times\int_0^1 \sqrt{y}\,\phi(y)\cos(qy-\alpha)\cos n\pi y\,dy\\ &\qquad+\{B\cos(qx-\alpha) - A\sin(qx-\alpha)\}n\sin n\pi x\\ &\qquad\times\int_0^1 \sqrt{y}\,\phi(y)\cos(qy-\alpha)\cos n\pi y\,dy. \end{aligned} \tag{135}$$

Since $\phi(y)$ satisfies the conditions of Lemma 15, it follows from that lemma that $\sqrt{y}\,\phi(y)\cos(qy-\alpha)$ satisfies the conditions of Lemmas 6 and 7. Hence from these lemmas and a change of variable we infer that the series whose general terms are

$$\begin{aligned} &n\cos n\pi x\int_0^1 \sqrt{y}\,\phi(y)\cos(qy-\alpha)\cos n\pi y\,dy\\ &\qquad-\frac{\phi(x)\cos(qx-\alpha)}{\sqrt{x}}\,n\cos n\pi x\int_0^1 y\cos n\pi y\,dy,\\ &n\sin n\pi x\int_0^1 \sqrt{y}\,\phi(y)\cos(qy-\alpha)\cos n\pi y\,dy\\ &\qquad-\frac{\phi(x)\cos(qx-\alpha)}{\sqrt{x}}\,n\sin n\pi x\int_0^1 y\cos n\pi y\,dy, \end{aligned} \tag{136}$$

are uniformly summable $(C\,1)$ in the interval $(0 < x \leqq c < 1)$. From Lemma 12 and a change of variable we infer the same property for the series whose general terms are the second terms of the expressions in (136). Hence it follows that the series whose general terms are the first terms in (136) have this property and therefore that the series whose general term is (135) has the same property. Thus our lemma is proved.

16. LEMMA 17. *If the series $\sum u_n(y)$ is uniformly summable $(C\,1)$ or uniformly convergent in the interval $(b < y \leqq d)$, then the series $\sum u_n(y) f_n(x)$, where*

$$(137) \qquad \begin{aligned} f_n(x) &= 1 && (n = 1, 2, \cdots, m+1); \\ f_n(x) &= \frac{a^{\frac{1}{2}}}{n^{\frac{1}{2}} x^{\frac{1}{2}}} && (n = m+2, m+3, \cdots), \end{aligned}$$

a being a positive constant and m being used to represent the largest integer such that $(0 < mx < a)$, will be uniformly summable $(C\,1)$ or uniformly convergent respectively, in the region $(b < y \leqq d;\ 0 < x \leqq c)$, where c is any positive constant.

For the case of convergence the proof follows readily from Abel's lemma.

For the case of summability the lemma may be proved by showing that the functions (137) satisfy, throughout the interval $(0 < x \leqq c)$ the conditions of a theorem* due to G. H. Hardy. These conditions require that a positive constant K exist such that

$$(138) \qquad \sum_{n=\mu}^{n=\nu} (n+1) \left| \Delta^2 f_n(x) \right| < K, \qquad \sum_{n=\mu}^{n=\nu} \left| \Delta f_n \right| < K,$$

for all values of μ and ν and all values of x in $(0 < x \leqq c)$. It will only be necessary to establish the first inequality in (138), since by virtue of a lemma† due to Bromwich the second is a consequence of the first.

It is readily seen that the first inequality is equivalent to

$$(139) \qquad \sum_{n=1}^{\infty} (n+1) \left| \Delta^2 f_n(x) \right| < K \qquad (0 < x \leqq c).$$

Moreover,

$$(140) \qquad \begin{aligned} \sum_{n=1}^{\infty} (n+1) \left| \Delta^2 f_n(x) \right| &= (m+1)\left(1 - \frac{a^{1/2}}{(m+2)^{1/2} x^{1/2}}\right) \\ &\quad + (m+2) \left| \left(1 - \frac{a^{1/2}}{(m+2)^{1/2} x^{1/2}}\right) - \left(\frac{a^{1/2}}{(m+2)^{1/2} x^{1/2}} - \frac{a^{1/2}}{(m+3)^{1/2} x^{1/2}}\right) \right| \\ &\quad + \frac{a^{1/2}}{x^{1/2}} \sum_{n=m+1}^{\infty} (n+1) \left\{ \frac{1}{n^{1/2}} - \frac{1}{2(n+1)^{1/2}} + \frac{1}{(n+2)^{1/2}} \right\}. \end{aligned}$$

---

* Cf. Proceedings of the London Mathematical Society, ser. 2, vol. 4 (1906), p. 263, Theorem 2*a* 1. Hardy does not prove the theorem for the case where the original series has variable terms, but the necessary modifications are obvious.

† Cf. Mathematische Annalen, vol. 65 (1908), p. 361.

But we have, since $mx < a \leqq (m+1)x$

$$(141)\quad 1 - \frac{a^{1/2}}{(m+2)^{1/2}x^{1/2}} < \frac{a^{1/2}}{m^{1/2}x^{1/2}} - \frac{a^{1/2}}{(m+2)^{1/2}x^{1/2}} = \frac{a^{1/2}\{(m+2)^{1/2} - m^{1/2}\}}{m^{1/2}(m+2)^{1/2}x^{1/2}} < \frac{K_1}{(m+1)},$$

$$(142)\quad \frac{1}{(m+2)^{1/2}x^{1/2}} - \frac{1}{(m+3)^{1/2}x^{1/2}} = \frac{(m+3)^{1/2} - (m+2)^{1/2}}{(m+2)^{1/2}(m+3)^{1/2}x^{1/2}} < \frac{K_2}{m+1},$$

$$(143)\quad \frac{1}{x^{1/2}}\sum_{n=m+2}^{\infty}(n+1)\left\{\frac{1}{n^{1/2}} - \frac{1}{2(n+1)^{1/2}} + \frac{1}{(n+2)^{1/2}}\right\} < \frac{K_3}{x^{1/2}}\sum_{n=m+2}^{\infty}\frac{1}{(n+1)^{3/2}} < \frac{K_3}{x^{1/2}}\int_{m+1}^{\infty}\frac{dx}{(x+1)^{3/2}} = \frac{K_3}{x^{1/2}(m+2)^{1/2}} < \frac{K_3}{a},$$

where K_1, K_2, and K_3 are positive constants. Combining these three inequalities with (140), the inequality (139) follows at once and our lemma is proved.

17. Before proceeding to the proof of the next lemma we find it convenient to make certain reductions and introduce certain new notations. From the asymptotic expansion* for $J_\nu(\lambda x)$, equations (9), (11), and

$$(144)\quad \sin(\lambda_n x - \alpha) = \sin[(n\pi + q)x - \alpha] + \frac{x\psi(x, n)}{n},$$

which is readily derived in the same manner as (11), we obtain

$$(145)\quad J_\nu(\lambda_n x) = \phi_n(x) + \frac{1}{n^{3/2}x^{1/2}}\{k_2 \cos[(n\pi + q)x - \alpha] + k_4 x[\sin(n\pi + q)x - \alpha]\} + \frac{k_5}{n^{5/2}x^{3/2}}\sin[(n\pi + q)x - \alpha] + \frac{\psi(x, n)}{n^{5/2}x^{1/2}} + \frac{\psi(x, n)}{n^{1/2}x^{3/2}} + \frac{\psi(\lambda_n x)}{\lambda^{5/2}x^{5/2}},$$

where the k's represent constants and

$$(146)\quad \phi_n(x) = \frac{k_1}{n^{1/2}x^{1/2}}\cos[(n\pi + q)x - \alpha] + \frac{k_3}{n^{3/2}x^{3/2}}\sin[(n\pi + q)x - \alpha].\dagger$$

We also set

* Cf. formula 48 of Transactions I.

† Here also the k's represent constants.

$$(147)\quad \bar{\phi}_n(x) = \frac{k_1}{a^{1/2}} \cos [(n\pi + q)x - \alpha] + \frac{k_3}{a^{3/2}} \sin [(n\pi + q)x - \alpha].$$

We are now ready for the proof of the following lemma:

LEMMA 18. *The series*

$$(148)\quad \begin{gathered} \sum_{j=1}^{j=m+1} \bar{\phi}_j(x) \int_0^1 \sqrt{y}\,\phi(y) \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} (qy - \alpha) j \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} j\pi y \, dy \\ + \sum_{j=m+2}^{\infty} \phi_j(x) \int_0^1 \sqrt{y}\,\phi(y) \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} (qy - \alpha) j \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} j\pi y \, dy \\ \sum_{j=1}^{j=m+1} \bar{\phi}_j(x) \int_0^1 \frac{\phi(y)}{\sqrt{y}} (y^2 + b) \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} (qy - \alpha) \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} j\pi y \, dy \\ + \sum_{j=m+2}^{\infty} \phi_j(x) \int_0^1 \frac{\phi(y)}{\sqrt{y}} (y^2 + b) \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} (qy - \alpha) \begin{smallmatrix}\cos\\ \sin\end{smallmatrix} j\pi y \, dy, \end{gathered}$$

where m has the same significance as in Lemma 17, will be summable $(C\,1)$ in the interval $(0 < x \leqq c < 1)$, provided $\phi(y)$ satisfies the conditions of Lemma 15.

We will consider only the first series in (148), since the proof for the other series is analogous. It is readily seen that the former series may be obtained by introducing the functions $f_n(x)$ defined by (137) as factors of the successive terms of the series

$$(149)\quad \sum_{j=1}^{\infty} \bar{\phi}_j(x) \int_0 \sqrt{y}\,\phi(y) \cos(qy - \alpha) j \cos j\pi y \, dy.$$

But, by Lemma 16, this latter series is uniformly summable $(C\,1)$ in the interval $(0 < x \leqq c < 1)$, and hence by Lemma 17 the first series in (148) is uniformly summable $(C\,1)$ in this same interval.

LEMMA 19. *The series whose general terms are*

$$(150)\quad \begin{gathered} nJ_\nu(\lambda_n x) \int_0^1 \sqrt{y}\,\phi(y) \begin{matrix}\cos(qy - \alpha)\cos n\pi y\\ \sin(qy - \alpha)\sin n\pi y\end{matrix} dy, \\ nJ_\nu(\lambda_n x) \int_0^1 \frac{\phi(y)}{\sqrt{y}} (y^2 + b) \begin{matrix}\sin(qy - \alpha)\cos n\pi y\\ \cos(qy - \alpha)\sin n\pi y\end{matrix} dy, \end{gathered}$$

will be uniformly summable $(C\,\tfrac{1}{2})$ in the interval $(0 < c_1 \leqq x \leqq c_2 < 1)$, provided $\phi(y)$ satisfies the conditions of Lemma 15 in the interval $(0 < y \leqq c_2)$.

We will consider only the first expression in (150), since the treatment of the other cases is analogous. If we substitute for $J_\nu(\lambda_n x)$ the value given by (145), the resulting expression may be written in the form

$$
(151) \quad \begin{aligned}
&\frac{k_1 n + k_2}{n^{1/2} x^{1/2}} \cos(qx - \alpha) \cos n\pi x \int_0^1 \sqrt{y}\, \phi(y) \cos(qy - \alpha) \cos n\pi y \, dy \\
&\quad - \frac{k_1 n + k_2}{n^{1/2} x^{1/2}} \sin(qx - \alpha) \sin n\pi x \int_0^1 y\, \phi(y) \cos(qy - \alpha) \cos n\pi y \, dy \\
&\quad + \frac{k_3 + k_4 x^2}{n^{1/2} x^{3/2}} \{\sin(qx - \alpha) \cos n\pi x + \cos(qx - \alpha) \sin n\pi x\} \\
&\qquad \times \int_0^1 \sqrt{y}\, \phi(y) \cos(qy - \alpha) \cos n\pi y \, dy + \frac{\psi(x, n)}{n^{3/2} x^{5/2}} .
\end{aligned}
$$

The last term of (151) is obviously the general term of a series that is uniformly convergent in the interval in question and therefore uniformly summable $(C\,\frac{1}{2})$ there. It remains to consider the other terms.

We know from Lemmas 15, 6, and 7 that the series whose general terms are

$$
(152) \quad \begin{aligned}
&n \cos n\pi x \int_0^1 \left(\sqrt{y}\, \phi(y) \cos(qy - \alpha) - y \frac{\phi(x) \cos(qx - \alpha)}{\sqrt{x}} \right) \cos n\pi y \, dy , \\
&n \sin n\pi x \int_0^1 \left(\sqrt{y}\, \phi(y) \cos(qy - \alpha) - y \frac{\phi(x) \cos(qx - \alpha)}{\sqrt{x}} \right) \cos n\pi y \, dy ,
\end{aligned}
$$

are uniformly summable $(C\,1)$ in the interval $(0 < x \leqq c_2)$. We know from Lemmas 15 and 12 that the series whose general terms are

$$
(153) \quad n \cos n\pi x \int_0^1 y \cos n\pi y \, dy , \qquad n \sin n\pi x \int_0^1 y \cos n\pi y \, dy ,
$$

are uniformly summable $(C\,1)$ in that same interval. Combining these facts and making use of the theorem of Riesz referred to in § 6, we readily infer that the first, second, and third terms of (151) are the general terms of series that are summable $(C\,\frac{1}{2})$ in the interval $(0 < c_1 \leqq x \leqq c_2)$. Since the fourth term has this same property, so also does the whole expression (151), or the first expression in (150). As the treatment of the other expressions is analogous, our lemma may be regarded as established.

Lemmas on the behavior of certain summations of terms. §§ 18–19

18. Lemma 20. *If a is the first positive root of the equation*

$$
(154) \quad \frac{k_1}{v^{1/2}} \cos(\pi v - \alpha) + \frac{k_3}{v^{3/2}} \sin(\pi v - \alpha) = 0,
$$

where k_1 and k_3 are the k's of equation (146), *then corresponding to an arbitrary, positive ϵ, an integer μ and a positive δ exist such that the expressions*

$$\begin{aligned}(155)\quad &\sum_{j=m+2}^{j=n} \frac{n-j+1}{n} \overline{\phi}_j(x) \int_0^1 \sqrt{y}\,\phi(y) \frac{\cos}{\sin}(qy-\alpha) j \frac{\cos}{\sin} j\pi y\, dy, \\ &\sum_{j=m+2}^{j=n} \frac{n-j+1}{n} \overline{\phi}_j(x) \int_0^1 \frac{\phi(y)}{\sqrt{y}} (y^2+b) \frac{\sin}{\cos}(qy-\alpha) \frac{\cos}{\sin} j\pi y\, dy,\end{aligned}$$

where $\overline{\phi}_j(x)$ is defined by (146) *and m is the greatest integer such that* $(0 < mx < a)$,* *are less in absolute value than ϵ throughout the interval* $(0 < x \leqq \delta)$ *when $n \geqq \mu$, provided $\phi(y)$ satisfies the conditions of Lemma* 15.

We find it convenient to begin by deriving certain properties of the functions $\overline{\phi}_n(x)$ defined by (147), the a of that equation being taken as the a of the present lemma. Since a is a root of (154) and since $mx < a \leqq (m+1)x$, we have

$$(156)\quad \begin{aligned}|\overline{\phi}_{m+1}(x)| \leqq &\left|\frac{2k_1}{a^{1/2}} \sin\left[\frac{\pi\{(m+1)x+a\}+qx}{2} - \alpha\right] \times \sin\frac{\pi\{(m+1)x-a\}+qx}{2}\right| \\ &+ \left|\frac{2k_3}{a^{3/2}} \cos\left[\frac{\pi\{(m+1)x+a\}+qx}{2} - \alpha\right] \times \sin\frac{\pi\{(m+1)x-a\}+qx}{2}\right| < \frac{B_1}{m},\end{aligned}$$

where B_1 is a positive constant, since the second factor in each term of the second member of (156) is easily seen to be of order $1/m$. Similarly we may establish

$$(157)\quad |\overline{\phi}_{m+2}(x)| < \frac{B_2}{m},$$

where B_2 is a positive constant.

If we set

$$(158)\quad h(v) = \frac{k_1}{a^{1/2}} \cos(\pi v - \alpha) + \frac{k_3}{a^{3/2}} \sin(\pi v - \alpha),$$

we have from the definition of $\overline{\phi}_j(x)$

$$(159)\quad \begin{aligned}\overline{\phi}_j(x) = h(jx) &- \frac{2k_1}{a^{1/2}} \sin\ (j\pi + \tfrac{1}{2}q)x - \alpha] \sin\frac{qx}{2} \\ &+ \frac{2k_3}{a^{3/2}} \cos[(j\pi + \tfrac{1}{2}q)x - \alpha] \sin\frac{qx}{2},\end{aligned}$$

* It should be understood that when $(m+2) > n$, there are no terms in the expressions (155).

whence

(160) $$\Delta\bar{\phi}_j(x) = \Delta h(jx) + x^2 \psi_1(x, j) \quad \left(\begin{matrix} |\psi_1(x,j)| < B_3 \\ j = 1, 2, 3, \cdots \end{matrix}\right),$$

(161) $$\Delta^2 \bar{\phi}_j(x) = \Delta^2 h(jx) + x^3 \psi_2(x, j) \quad \left(\begin{matrix} |\psi_2(x,j)| < B_4 \\ j = 1, 2, 3, \cdots \end{matrix}\right),$$

B_3 and B_4 being positive constants. Furthermore we have

(162) $$|h(jx) - h((j+1)x)| < B_5 x \qquad (j = 1, 2, 3, \cdots),$$

(163) $$|h(jx)| < B_6 \qquad (j = 1, 2, 3, \cdots),$$

where B_5 and B_6 are positive constants.

We will give the proof only for the first expression in (155), since the discussion for the other cases is analogous. Let us represent by $S(x)$ the value to which the first series in (148) is summable, and let us consider the series obtained from that series by substituting for the first term that same term minus $S(x)$. This series will be uniformly summable $(C\,1)$ to the value zero in the interval $(0 < x \leqq c)$. Hence if we use $\epsilon_j(x)$ to represent $S_j(x)/j$, where $S_j(x)$ stands for the sum of the first j sums of j terms of this series, it is apparent that $\epsilon_j(x)$ approaches zero uniformly in this interval and therefore, corresponding to an arbitrary positive ϵ, we may choose $p > 2$ and such that

(164) $$|\epsilon_j(x)| < \eta = \frac{\epsilon}{2\{1 + 2B_1 + B_2 + B_4 a^3 + 2(r+1)(B_5 a + B_6) + 2[B_3 a^2 + 2(r_1 + 1)B_6]\}} \quad \left(\begin{matrix} j \geqq p \\ 0 < x \leqq c \end{matrix}\right),$$

where the B's are the B's of (156), (157), (160), (161), (162), and (163), and r and r_1 represent respectively the number of times that $h''(v)$ and $h'(v)$ change sign in the interval $(0 < v < 2a)$. Then, p being fixed, we choose $\delta < a/p$ and $< c$, and so small that

(165) $$\left|\sum_{j=1}^{j=p}\left\{ j\,\epsilon_j(x)\,\Delta^2\bar{\phi}_j(x) + \frac{j}{n}\epsilon_j(x)\,\Delta\bar{\phi}_j(x)\right\}\right| < \eta \quad \left(\begin{matrix} 0 < x \leqq \delta \\ n = 1, 2, 3, \cdots \end{matrix}\right),$$

which we may do in view of the fact that $\Delta^2\bar{\phi}_j(x)$ and $\Delta\bar{\phi}_j(x)$ approach zero with x uniformly for all values of j, and $\epsilon_j(x)$ remains finite for all values of j and all values of x in the interval $(0 < x \leqq c)$.

Since the series we are considering is uniformly summable $(C\,1)$ to zero in the interval $(0 < x \leqq c)$, the expression

(166) $$\begin{aligned} &\left\{\bar{\phi}_1(x)\int_0^1 \sqrt{y}\,\phi(y)\cos(qy - \alpha)\cos\pi y\,dy - S(x)\right\} \\ &+ \sum_{j=2}^{j=m+1}\frac{n-j+1}{n}\bar{\phi}_j(x)\int_0^1 \sqrt{y}\,\phi(y)\cos(qy-\alpha)j\cos j\pi y\,dy \\ &+ \sum_{j=m+2}^{j=n}\frac{n-j+1}{n}\phi_j(x)\int_0^1 \sqrt{y}\,\phi(y)\cos(qy-\alpha)j\cos j\pi y\,dy \end{aligned}$$

can be made less in absolute value than $\frac{1}{2}\epsilon$ for all values of n greater than a certain fixed integer and for all values of x in $(0 < x \leqq \delta)$. If we can show that the same is true for the sum of the first two terms of (166), it will follow that the third term can be made less in absolute value than ϵ for all values of x in $(0 < x \leqq \delta)$ and all values of n greater than a fixed integer, and our lemma will be proved.

The qth term of the expression formed from the first two terms of (166) may be replaced by $(q\epsilon_q(x) - 2(q-1)\epsilon_{q-1}(x) + (q-2)\epsilon_{q-2}(x))$, where we write $\epsilon_{-1}(x) = \epsilon_{-2}(x) = 0$. If we make this substitution, drop the parentheses, and rearrange the terms so that terms involving the same subscript for the ϵ's are brought together, we obtain

$$\begin{aligned}(167)\qquad &\sum_{j=1}^{j=m} j\,\epsilon_j(x)\Delta^2\left(\frac{n-j+1}{n}\bar{\phi}_j(x)\right)\\ &+ (m+1)\epsilon_{m+1}(x)\frac{n-m}{n}\bar{\phi}_{m+1}(x) - m\,\epsilon_m(x)\frac{n-m-1}{n}\bar{\phi}_{m+2}(x).\end{aligned}$$

Moreover, since

$$(168)\qquad \Delta^2\frac{n-j+1}{n}\bar{\phi}_j(x) = \frac{2}{n}\Delta\bar{\phi}_{j+1}(x) + \frac{n-j+1}{n}\Delta^2\bar{\phi}_j(x),$$

the first term of (167) takes the form

$$(169)\qquad \sum_{j=1}^{j=m}\left\{ j\,\epsilon_j(x)\frac{n-j+1}{n}\Delta^2\bar{\phi}_j(x) + \frac{2j}{n}\epsilon_j(x)\Delta\bar{\phi}_{j+1}(x)\right\}.$$

We have from (161)

$$\begin{aligned}(170)\qquad \sum_{j=p+1}^{j=m} j\,\epsilon_j(x)\Delta^2\bar{\phi}_j(x) &= \sum_{j=p+1}^{j=m} j\,\epsilon_j(x)\Delta^2 h(jx)\\ &\quad + x^3\sum_{j=p+1}^{j=m} j\,\epsilon_j(x)\psi_2(x,j).\end{aligned}$$

But

$$(171)\qquad \left| x^3\sum_{j=p+1}^{m=1} j\,\epsilon_j(x)\psi_2(x,j)\right| < B_4\,\eta x^3\sum_{j=p+1}^{j=m} j < B_4\,\eta m^2 x^3 < B_4\,a^3\,\eta,$$

and if $p_1, p_2, \cdots, p_s$ represent the values of j at which $\Delta^2 h(jx)$ changes sign and we set $p+1 = p_0$ and $m = p_{s+1}$, we have from (162) and (163)

$$\begin{aligned}(172)\qquad &\left|\sum_{j=p+1}^{j=m} j\,\epsilon_j(x)\Delta^2 h(jx)\right| \leqq \eta\sum_{i=0}^{i=s}\left|\sum_{j=p_i}^{j=p_{i+1}} j\,\Delta^2 h(jx)\right|\\ &= \eta\sum_{i=0}^{i=s}\big| p_i\{h(p_i x) - h((p_i+1)x)\} + h((p_i+1)x)\\ &\quad - p_{i+1}\{h((p_{i+1}+1)x) - h((p_{i+1}+2)x)\} - h((p_{i+1}+1)x)\big|\\ &< \eta(s+1)2(B_5 a + B_6) \leqq 2(r+1)(B_5 a + B_6)\eta,\end{aligned}$$

since obviously $s \leqq r$. Combining (170), (171), and (172), we have

$$(173)\quad \left| \sum_{j=p+1}^{j=m} j\,\epsilon_j(x)\,\bar{\Delta}^2\,\phi_j(x) \right| < \{B_4\,a^3 + 2(r+1)(B_5\,a + B_6)\}\eta \qquad (0 < x \leqq c).$$

By means of a similar discussion, making use of (160) and (163), we obtain for the second term in (169) the inequality

$$(174)\quad \frac{2}{n}\left| \sum_{j=p+1}^{j=m} j\,\epsilon_j(x)\,\bar{\Delta}\phi_{j+1}(x) \right| < 2\{B_3 a^2 + 2(r_1+1)B_5\}\eta \quad (0 < x \leqq c).$$

We have, moreover, from (156) and (157)

$$(175)\quad |(m+1)\,\epsilon_{m+1}(x)\,\bar{\phi}_{m+1}(x)| < 2B_1\,\eta, \qquad |m\epsilon_m(x)\,\bar{\phi}_{m+2}(x)| < B_2\,\eta.$$

Combining (175) with (174), (173), (169), and (165), we find that (167) is less in absolute value than $\frac{1}{2}\,\epsilon$ for values of x in the interval $(0 < x \leqq \delta)$. Thus, as pointed out before, our lemma is proved.

19. LEMMA 21. *Given an arbitrary positive* ϵ, *we can choose a* μ *and a* δ *such that the expressions*

$$(176)\quad \begin{aligned} &\sum_{j=m+2}^{j=n} \frac{n-j+1}{n} F_j(x) \int_0^1 \sqrt{y}\,\phi(y) \,{\cos \atop \sin}\,(qy-\alpha)\,j\,{\cos \atop \sin}\,j\pi y\,dy,\\ &\sum_{j=m+2}^{j=n} \frac{n-j+1}{n} F_j(x) \int_0^1 \frac{\phi(y)}{\sqrt{y}}\,(y^2+b)\,{\sin \atop \cos}\,(qy-\alpha)\,{\cos \atop \sin}\,j\pi y\,dy, \end{aligned}$$

where

$$(177)\quad F_j(x) = \frac{1}{n^{3/2}\,x^{1/2}}\{k_2 \cos[(n\pi+q)x-\alpha] + k_4\,x\sin[(n\pi+q)x-\alpha]\},$$

and m is defined as in Lemma 20,* *are less in absolute value than* ϵ *in the interval* $(0 < x \leqq \delta)$ *when* $n \geqq \mu$, *provided* $\phi(y)$ *satisfies the conditions of Lemma* 15.

We will carry through the proof only for the first expression in (176), since the proof for the other cases is analogous.

We begin by defining

$$(178)\quad \bar{F}_j(x) = \frac{1}{na^{1/2}}\{k_2 \cos[(n\pi+q)x-\alpha] + k_4\,x\sin[(n\pi+q)x-\alpha]\}.$$

The series

$$(179)\quad \sum_{j=1}^{\infty} \bar{F}_j(x) \int_0^1 \sqrt{y}\,\phi(y)\cos(qy-\alpha)\,j\cos j\pi y\,dy$$

is by Lemma 16 and the theorem of Riesz referred to in § 6, uniformly convergent in the interval $(0 < x \leqq c)$. Hence by Lemma 17 the series

* See also footnote to the statement of this lemma.

$$\sum_{j=1}^{j=m+1} \overline{F}_j(x) \int_0^1 \sqrt{y}\,\phi(y)\cos(qy-\alpha)j\cos j\pi y\,dy \tag{180}$$
$$+ \sum_{j=m+2}^{\infty} F_j(x) \int_0^1 \sqrt{y}\,\phi(y)\cos(qy-\alpha)j\cos j\pi y\,dy$$

is uniformly convergent in the interval $(0 < x \leqq c)$. We choose μ so large that in the series (180)

$$\left|\sum_{j=p}^{j=n} u_j(x)\right| < \epsilon, \qquad (n > p \geqq \mu), \tag{181}$$

$u_j(x)$ representing the jth term of (180). Then we choose $\delta \leqq a/\mu$, so that for any point in $(0 < x \leqq \delta)$, $(m+2) \geqq a/x \geqq a/\delta \geqq \mu$. Under these conditions we have

$$\left|\sum_{j=m+2}^{j=n} u_j(x)\right| < \epsilon \qquad \binom{0 < x \leqq \delta}{n \geqq \mu}. \tag{182}$$

From (182) and Abel's lemma, it follows that the first expression in (176) satisfies the desired inequality.

Uniform summability of the development in the neighborhood of the origin. §§ 20–21

20. Before taking up the proof of the next lemma, we find it convenient to make some further reductions. From (145), (177), the identity

$$\Delta^2 g_n f_n = g_n \Delta^2 f_n + 2\Delta g_n \Delta f_{n+1} + f_{n+2}\Delta^2 g_n, \tag{183}$$

and the use of the Law of the Mean for derivatives, we have

$$\Delta^2 J_\nu(\lambda_n x) = \Delta^2 \phi_n(x) + \Delta^2 F_n(x) + R_n(x), \tag{184}$$

where

$$R_n(x) = \frac{\psi(x,n)}{n^{5/2}x^{1/2}} + \frac{\psi(x,n)}{n^{7/2}x^{3/2}} + \frac{\psi(x,n)}{n^{9/2}x^{5/2}}. \tag{185}$$

Similarly, using the identity

$$\Delta g_n f_n = g_n \Delta f_n + f_{n+1}\Delta g_n, \tag{186}$$

we obtain

$$\Delta J_\nu(\lambda_n x) = \Delta\phi_n(x) + \Delta F_n(x) + S_n(x), \tag{187}$$

where

$$S_n(x) = \frac{\psi(x,n)}{n^{5/2}x^{3/2}} + \frac{\psi(x,n)}{n^{7/2}x^{5/2}}. \tag{188}$$

We will also find it convenient to derive a few preliminary inequalities that will be of use in the proof of the next lemma. From the definitions of $\phi_n(x)$ and $\overline{\phi}_n(x)$, given in (146) and (147), and the inequalities (156) and (157), we obtain for values of m and x such that $mx < a \leqq (m+1)x$, where a is defined as in Lemma 20, the following relations

(189) $\quad |m\phi_{m+2}(x)| < D_1, \qquad |(m+1)\phi_{m+1}(x)| < D_2,$

where D_1 and D_2 are positive constants. Furthermore, from the definition of $F_j(x)$, given by (177), we readily obtain for values of m and x such that $mx < a \leqq (m+1)x$

(190) $\quad |mF_{m+2}(x)| < D_1', \qquad |(m+1)F_{m+1}(x)| < D_2',$

where D_1' and D_2' are positive constants. Finally we have

(191) $\quad |\phi_n(x)| < K_1, \qquad |F_n(x)| < K_2 \qquad (nx \geqq a);$

where K_1 and K_2 are positive constants.

LEMMA 22. *The series whose general terms are*

$$(192)\qquad \begin{aligned} &n J_\nu(\lambda_n x)\int_0^1 \sqrt{y}\,\phi(y)\,{\cos \atop \sin}(qy-\alpha)\,{\cos \atop \sin}\,n\pi y\,dy,\\ &J_\nu(\lambda_n x)\int_0^1 \frac{\phi(y)}{\sqrt{y}}(y^2+b)\,{\sin \atop \cos}(qy-\alpha)\,{\cos \atop \sin}\,n\pi y\,dy,\end{aligned}$$

will be uniformly summable $(C\,1)$ *in the interval* $(0 < x \leqq c)$, *provided* $\phi(y)$ *satisfies the conditions of Lemma* 15.

We will consider only the series whose general term is the first term in (189), since the proof for the other cases is analogous. The series whose general term is n times the integral in the first term of (189) is summable $(C\,1)$ by Lemma 2. Let S be the value to which it is summable and let us replace the first term of the series we are considering by that same term minus $SJ_\nu(\lambda_1 x)$. If we can establish the uniform summability of the resulting series in the interval $(0 < x \leqq c)$, it is thereby proved that the original series has the same property.

Let us represent by ϵ_j the expression $(S_j/j - S)$, where S_j indicates the sum of the first j sums of j terms for the series summable to S. Then, given an arbitrary positive ϵ, we choose p so large that

$$(193)\qquad |\epsilon_j| < \eta = \frac{\epsilon}{4\{1+(2r+2r_1+5)C+2(r+1)C_1da+\frac{4}{5}a^{-5/2}(5a^2+10a+12)M+D_1+D_1'+D_2+D_2'+K_1+K_2\}} \qquad (j \geqq p),$$

where r and r_1 represent the number of times that $J_\nu''(x)$ and $J_\nu'(x)$ respectively change signs in the interval $(0, 2a)$, a having the same significance as in Lemma 20, the D's and K's are as defined by (189), (190), and (191), d is the upper limit of $(\lambda_{n+1} - \lambda_n)$, M is the upper limit of the functions $\psi(x, n)$ in (185) and (188), and C and C_1 represent the maximum absolute value of $J_\nu(x)$ and $J_\nu'(x)$. Now, p being fixed, we choose $\mu_1 > p$ and a positive $\delta < c$, such that the first expressions in (155) and (176) are each less in absolute value than $\frac{1}{8}\epsilon$ throughout the interval $(0 < x \leqq \delta)$ for $n \geqq \mu_1$, and so that

$$\left|\frac{1}{n}\sum_{j=1}^{j=p-1}\epsilon_j\{(n-j+1)\Delta^2 J_\nu(\lambda_j x)+2\Delta J_\nu(\lambda_j x)\}\right|<\eta \quad (0<x\leqq\delta). \tag{194}$$

By Lemma 19 the series we are considering is uniformly summable $(C\frac{1}{2})$, and therefore $(C\,1)$, in the interval $(\delta\leqq x\leqq c)$. Hence we may choose μ_2 so large that

$$|\sigma_{n_1}(x)-\sigma_{n_2}(x)|<\epsilon \qquad \left(\begin{matrix} n_2>n_1\geqq\mu_2 \\ \delta\leqq x\leqq c\end{matrix}\right), \tag{195}$$

where $\sigma_j(x)$ represents the arithmetic mean of the first j sums of the first j terms of the series in question. It remains to be shown that the same inequality holds for the interval $(0<x<\delta)$.

We have

$$\sigma_n(x)=(u_1-S)J_\nu(\lambda_1 x)+\sum_{j=2}^{j=n}\frac{n-j+1}{n}u_j J_\nu(\lambda_j x),$$

where

$$u_j=\int_0^1\sqrt{y}\,\phi(y)\cos(qy-\alpha)\cos j\pi y\,dy.$$

Furthermore, if in the above expression for $\sigma_n(x)$ we replace u_j by its value $(\epsilon_j-2\epsilon_{j-1}+\epsilon_{j-2})$,* drop parentheses, and rearrange terms according to subscripts of ϵ, we obtain

$$\sigma_n(x)=\frac{1}{n}\sum_{j=1}^{j=n}j\,\epsilon_j\{(n-j+1)\Delta^2 J_\nu(\lambda_j x)+2\Delta J_\nu(\lambda_{j+1}x)\} + \epsilon_n J_\nu(\lambda_{n+2}x). \tag{196}$$

Three cases will arise here, according as m, the greatest integer $<a/x$, where a is the a of Lemma 20, is $<p$, lies between p and n, or is $\geqq n$. We will discuss only the second case, since this discussion virtually includes the discussion of the other two cases.

If we represent by $p_1, p_2, \cdots, p_s$ the values of j between p and m for which $\Delta^2 J_\nu(\lambda_j x)$ changes sign, and we set $p=p_0$, $m=p_{s+1}$, we have

$$\begin{aligned}\left|\sum_{j=p}^{j=m}j\,\epsilon_j\frac{n-j+1}{n}\Delta^2 J_\nu(\lambda_j x)\right| &\leqq \eta\sum_{i=0}^{i=s}\left|\sum_{j=p_i}^{j=p_{i+1}}j\frac{n-j+1}{n}\Delta^2 J_\nu(\lambda_j x)\right| \\ &< \eta\sum_{i=0}^{i=s}\left|\sum_{j=p_i}^{j=p_{i+1}}j\,\Delta^2 J_\nu(\lambda_j x)\right| \\ &= \eta\sum_{i=0}^{i=s}|\,p_i\{J_\nu(\lambda_{p_i}x)-J_\nu(\lambda_{p_i+1}x)\}+J_\nu(\lambda_{p_i+1}x) \\ &\quad -p_{i+1}\{J_\nu(\lambda_{p_{i+1}+1}x)-J_\nu(\lambda_{p_{i+1}+2}x)\}-J_\nu(\lambda_{p_{i+1}+1}x)| \\ &< 2(s+1)(C+C_1 da)\eta\leqq 2(r+1)(C+C_1 da)\eta \\ &\qquad (0<x\leqq\delta).\end{aligned} \tag{197}$$

* In the case of the first term we replace (u_1-S) by $\epsilon_1-2\epsilon_{-1}+\epsilon_{-2}$. It is understood, of course, that $\epsilon_{-1}=\epsilon_{-2}=0$.

Also, if we represent by $p'_1, p'_2, \cdots, p'_{s_1}$, the values of j between p and m at which $\Delta J_\nu(\lambda_{j+1}x)$ changes sign, and we set $p'_0 = p$, $p'_{s_1+1} = m$, we have

$$
\begin{aligned}
\left|\sum_{j=p}^{j=m} \frac{j}{n}\epsilon_j \Delta J_\nu(\lambda_{j+1}x)\right| &< \eta \sum_{i=0}^{i=s_1}\left|\sum_{j=p'_i}^{j=p'_{i+1}} \Delta J_\nu(\lambda_{j+1}x)\right| \\
&= \eta \sum_{i=0}^{i=s_1}\left| J_\nu(\lambda_{p'_i+1}x) - J_\nu(\lambda_{p'_{i+1}+1}x)\right| \\
&< 2(s_1+1)C\eta \leqq 2(r_1+1)C\eta \qquad (0 < x \leqq \delta).
\end{aligned} \tag{198}
$$

Moreover, we have from (184) and (187)

$$
\begin{aligned}
&\frac{1}{n}\sum_{j=m+1}^{j=n} j\,\epsilon_j\{(n-j+1)\Delta^2 J_\nu(\lambda_j x) + 2\Delta J_\nu(\lambda_{j+1}x)\} \\
&\quad = \frac{1}{n}\sum_{j=m+1}^{j=n} j\,\epsilon_j\{(n-j+1)[\Delta^2\phi_j(x) + \Delta^2 F_j(x)] \\
&\qquad\qquad + 2[\Delta\phi_{j+1}(x) + \Delta F_{j+1}(x)]\} \\
&\quad + \sum_{j=m+1}^{j=n}\frac{n-j+1}{n}\epsilon_j\, j\, R_j(x) + 2\sum_{j=m+2}^{j=n}\epsilon_j\frac{j}{n}S_j(x).
\end{aligned} \tag{199}
$$

But, from (185)

$$
\begin{aligned}
\left|\sum_{j=m+1}^{j=n}\frac{n-j+1}{n}\epsilon_j\, j\, R_j(x)\right| &< \eta M\sum_{j=m+2}^{j=n}\left[\frac{1}{j^{3/2}x^{1/2}} + \frac{1}{j^{5/2}x^{3/2}} + \frac{1}{j^{7/2}x^{5/2}}\right] \\
&< \eta M\left[\frac{4}{(m+1)^{1/2}x^{1/2}} + \frac{8}{3(m+1)^{3/2}x^{3/2}} + \frac{16}{5(m+1)^{5/2}x^{5/2}}\right] \\
&< \frac{4(15a^2+10a+12)M}{15a^{5/2}}\eta.
\end{aligned} \tag{200}
$$

Also, from (188),

$$
\begin{aligned}
\left|2\sum_{j=m+1}^{j=n}\epsilon_j\frac{j}{n}S_j(x)\right| &< 2\eta M\sum_{j=m+2}^{j=n}\left[\frac{1}{j^{5/2}x^{3/2}} + \frac{1}{j^{7/2}x^{5/2}}\right] \\
&< 2\eta M\left[\frac{8}{3(m+1)^{3/2}x^{3/2}} + \frac{16}{5(m+1)^{5/2}x^{5/2}}\right] < \frac{16(5a+6)M}{15a^{5/2}}\eta.
\end{aligned} \tag{201}
$$

It remains to consider the first summation on the right-hand side of (199). If we rearrange this summation according to subscripts of ϕ and F, it takes the form

$$
\begin{aligned}
&\sum_{j=m+2}^{j=n}\frac{n-j+1}{n}\{\phi_j(x) + F_j(x)\}\int_0^1 \sqrt{y}\,\phi(y)\cos(qy-\alpha)\cos j\pi y\, dy \\
&\quad - \frac{n-m-1}{n}m\epsilon_m\{\phi_{m+2}(x) + F_{m+2}(x)\} \\
&\quad + \frac{n-m}{n}(m+1)\epsilon_{m+1}\{\phi_{m+1}(x) + F_{m+1}(x)\} \\
&\qquad\qquad - \epsilon_n\{\phi_{n+2}(x) + F_{n+2}(x)\}.
\end{aligned} \tag{202}
$$

We have already chosen μ_1 and δ so that for values of x in the interval $(0 < x \leqq \delta)$ and values of $n \geqq \mu_1$ the summation in (202) should be less in absolute value than $\frac{1}{4}\epsilon$. Hence, making use of the inequalities (189), (190), and (191), we see that the expression (202) is for values of x in the interval $(0 < x \leqq \delta)$ and values of $n \geqq \mu_1$, less in absolute value than

$$\tfrac{1}{4}\epsilon + (D_1 + D_1' + D_2 + D_2' + K_1 + K_2)\eta .$$

Combining this fact with (196), (194), (197), (198), (199), (200), and (201), we obtain

$$|\sigma_n(x)| < \frac{\epsilon}{4} + \{1 + (2r + 2r_1 + 5)C + 2(r+1)C_1 da + \tfrac{4}{5}a^{-5/2}(5a^2 + 10a + 12)M + D_1 + D_1' + D_2 + D_2' + K_1 + K_2\}\eta = \frac{\epsilon}{2} \qquad \begin{pmatrix} 0 < x \leqq \delta \\ n \geqq \mu_1 \end{pmatrix},$$

or

$$|\sigma_{n_1}(x) - \sigma_{n_2}(x)| < \epsilon \qquad \begin{pmatrix} 0 < x \leqq \delta \\ n_2 > n_1 \geqq \mu_1 \end{pmatrix}. \tag{203}$$

Combining (203) with (195), we have

$$|\sigma_{n_1}(x) - \sigma_{n_2}(x)| < \epsilon \qquad \begin{pmatrix} 0 < x \leqq c \\ n_2 > n_1 \geqq \mu \end{pmatrix}, \tag{204}$$

where we have chosen for μ the larger of μ_1 and μ_2. Our lemma is therefore proved.

21. We are now ready to prove the following theorem:

THEOREM II. *If $f(x)$ is such that $\phi(x) = f(x) - f(0)$ satisfies the conditions of Lemma 15, the development of $f(x)$ in Bessel's functions of order $\nu\,(\nu \geqq 0)$ will be uniformly summable $(C\,\frac{1}{2})$ in the interval $(0 \leqq x \leqq c)$, provided $\nu = 0$ or $f(0) = 0$.*

In view of Lemma 1 the general term of the development (12) may be written in the form

$$\left(M_1\sqrt{n} + \frac{M_2}{\sqrt{n}}\right)J_\nu(\lambda_n x)\int_0^1 \sqrt{x}\,\phi(x)\cos\{(n\pi + q)x - \alpha\}dx$$
$$+ \frac{M_3}{\sqrt{n}}J_\nu(\lambda_n x)\int_0^1 \frac{\phi(x)}{\sqrt{x}}(x^2 + b)\sin\{(n\pi + q)x - \alpha\}dx \tag{205}$$
$$+ (-1)^n M_4 f(0)\frac{J_\nu(\lambda_n x)}{\sqrt{n}} + \frac{\nu K f(0)}{\lambda_n}J_\nu(\lambda_n x) + r_n J_\nu(\lambda_n x),$$

where r_n is the general term of an absolutely convergent series. We see at once from Lemmas 22, 2, and 3, and the theorem of M. Riesz referred to in the proof of Theorem I, that the first two terms of (205) are the general terms of series that are uniformly summable $(C\,\frac{1}{2})$ in the interval $(0 \leqq x \leqq c)$. The last term is obviously the general term of a series that has the same property and the fourth term is zero under the conditions of our theorem. The uniform convergence in $(0 \leqq x \leqq c)$ of the series of which the third term is the general term has been established in the course of the proof of Lemma 2 of Transactions II. For if we take $c = 1$ in the first expression of (51) of that paper, this expression reduces to the third term of (205), except for a constant factor.

Since each term of (205) is the general term of a series that is uniformly summable $(C\,\frac{1}{2})$ in $(0 \leqq x \leqq c)$, the whole expression has the same property, and our theorem is proved.

Value of the development. §§ 22–24

22. We will now show that under the conditions imposed on $f(x)$ in the previous sections the value to which the development is summable at the origin and uniformly summable in the neighborhood of the origin, will be $f(x)$. We begin by proving some lemmas.

Lemma 23. *If the function $F(x)$ has a Lebesgue integral in the interval $(0 \leqq x \leqq \pi)$, the series whose general terms are*

$$(206)\qquad \begin{aligned} &\frac{1}{n}\left\{\begin{matrix}\cos\\ \sin\end{matrix} nx \int_0^\pi F(x) \begin{matrix}\sin\\ \cos\end{matrix} nx\,dx\right\} \\ &\frac{1}{n}\left\{\begin{matrix}\cos\\ \sin\end{matrix} nx \int_0^\pi F(x) \begin{matrix}\cos\\ \sin\end{matrix} nx\,dx\right\} \end{aligned}$$

will be summable $(C, k > 0)$ almost everywhere in the interval $(0 \leqq x \leqq \pi)$ and may be multiplied by a function of bounded variation and integrated term by term in this interval.

Since, by virtue of the Riemann-Lebesgue theorem* the integrals in (206) approach zero as n becomes infinite, the expressions (206) are of the form $a_n \cos nx$, $b_n \sin nx$, where

$$\lim_{n\to\infty} na_n = 0 = \lim_{n\to\infty} nb_n.$$

Hence the various terms of (206) are the general terms of the Fourier's series of summable functions, and consequently in view of well-known theorems are summable $(C, k > 0)$ almost everywhere† and may be multiplied by a function of bounded variation and integrated term by term over any interval.‡

* See footnote, § 7.

† Cf. Hardy, Proceedings of the London Mathematical Society, ser. 2, vol. 12 (1913), p. 365.

‡ Cf. Young, Proceedings of the London Mathematical Society, ser. 2, vol. 9 (1910), p. 449.

LEMMA 24. *The series whose general term is*

$$(207)\qquad \cos(qx-\alpha)\cos n\pi x\int_0^1 \sqrt{x}f(x)\sin(qx-\alpha)\sin n\pi x\,dx + \sin(qx-\alpha)\sin n\pi x\int_0^1 \sqrt{x}f(x)\cos(qx-\alpha)\cos n\pi x\,dx,$$

where $f(x)$ is integrable (Lebesgue) in the interval $(0, \pi)$ and q and α have the same significance as in equation (15), *will be uniformly convergent in the interval* $(0 \leqq x \leqq \pi)$.

The expression (207) may be written in the form (cf. Transactions I, page 412).

$$(208)\qquad \frac{1}{2}\int_0^1 \sqrt{x'}f(x')\sin[q(x+x')-2\alpha]\sin n\pi(x+x')\,dx' + \frac{1}{2}\int_0^1 \sqrt{x'}f(x')\sin q(x-x')\sin n\pi(x-x')\,dx'.$$

Since, however,

$$\sum_1^{m=n}\sin mz = \frac{\sin z - \sin(n+1)z + \sin nz}{2(1-\cos z)} = \frac{\sin z}{4\sin^2\frac{1}{2}z} - \frac{\cos(n+\frac{1}{2})z}{4\sin\frac{1}{2}z},$$

the sum of the first n terms of the series whose general term is the second term of (208) may be written in the form

$$\frac{1}{8}\int_0^1 \sqrt{x'}f(x')\frac{\sin q(x-x')\sin\pi(x-x')}{\sin^2\frac{1}{2}\pi(x-x')}\,dx'$$

$$(210)\qquad -\frac{1}{8}\int_0^1 \sqrt{x'}f(x')\frac{\sin q(x-x')}{\sin\frac{1}{2}\pi(x-x')}\cos(n+\tfrac{1}{2})x\cos(n+\tfrac{1}{2})x'\,dx'$$

$$+\frac{1}{8}\int_0^1 \sqrt{x'}f(x')\frac{\sin q(x-x')}{\sin\frac{1}{2}\pi(x-x')}\sin(n+\tfrac{1}{2})x\sin(n+\tfrac{1}{2})x'\,dx'.$$

The first term of this expression is independent of n, and in view of Lemma 4, the second and third terms approach zero uniformly in $(0 \leqq x \leqq 1)$ as n becomes infinite. Hence the series whose general term is the second term of (208) is uniformly convergent in this interval. It remains to prove the same for the series whose general term is the first term in (208).

From (209) we have for the sum of the first n terms of this series

$$\frac{1}{8}\int_0^1 \sqrt{x'}f(x')\frac{\sin[q(x+x')-2\alpha]\sin\pi(x+x')}{\sin^2\frac{1}{2}\pi(x+x')}\,dx'$$

$$-\frac{1}{8}\int_0^1 \sqrt{x'}f(x')\frac{\sin[q(x+x')-2\alpha]}{\sin\frac{1}{2}\pi(x+x')}\cos(n+\tfrac{1}{2})x\cos(n+\tfrac{1}{2})x'\,dx'$$

$$+\frac{1}{8}\int_0^1 \sqrt{x'}f(x')\frac{\sin[q(x+x')-2\alpha]}{\sin\frac{1}{2}\pi(x+x')}\sin(n+\tfrac{1}{2})x\sin(n+\tfrac{1}{2})x'\,dx'.$$

This expression may be shown to approach a limit uniformly in $(0 \leqq x \leqq 1)$ as n becomes infinite, in the same way that this property was proved for (210), provided we know that the expression $\sin[q(x + x') - 2\alpha]/\sin \frac{1}{2}\pi(x + x')$ approaches a finite limit as $x + x'$ approaches 2. But this has been shown to be the case on page 415 of Transactions I. Our lemma is therefore proved.

23. LEMMA 25. *If $f(x)$ satisfies the conditions of Lemma* 1, *the series* (12) *is summable* $(C, k > 0)$ *almost everywhere in* $(0 < x \leqq 1)$ *and may be multiplied by a function of bounded variation and integrated term by term over any interval lying in this interval.*

Since $f(x)$ satisfies the conditions of Lemma 1, A_n may be replaced by the expression on the right-hand side of (15). Making this substitution for A_n and replacing $J_\nu(\lambda_n x)$ by its value as given by (145) and (146), we obtain for the general term of (12) an expression which, in view of Lemmas 23 and 24 and the theorems about the Fourier's development of a summable function referred to in the last footnotes, is the general term of a series that is summable $(C, k > 0)$ almost everywhere in the interval $(0 < x \leqq 1)$ and can be multiplied by a function of bounded variation and integrated term by term over any interval in this interval. Thus our lemma is proved.

LEMMA 26. *If $f(x)$ satisfies the conditions of Theorem II, the series*

$$\sum_{n=1}^{\infty} A_n \frac{J_\nu(\lambda_n x)}{\lambda_n} \tag{211}$$

will converge uniformly in the interval $(0 \leqq x \leqq c)$, *provided* $\nu = 0$ *or* $f(0) = 0$.

We know from Theorem II that under the conditions of the present lemma the series whose general term is $A_n J_\nu(\lambda_n x)$ will be uniformly summable $(C, \frac{1}{2})$ in the interval $(0 \leqq x \leqq c)$. Since, in view of (9), we have

$$\frac{1}{\lambda_n} = \frac{1}{n\pi} + \frac{\psi_1(n)}{n^2},$$

it follows readily from this fact, the theorem of M. Riesz used in the proof of Theorem I and the expression for A_n obtained in Lemma 1, that the series (211) is uniformly convergent in the same interval.

In order to avoid the appearance of an exceptional case when the l and h of equation (1) are such that $l\nu + h = 0$, we follow the method employed by us on previous occasions* and replace the series (12) by the series

$$\sum_{n=1}^{\infty} \bar{A}_n F_\nu(\lambda_n x), \tag{212}$$

where, as before,

* Cf., for example, Transactions I, pages 418–421.

$$F_\nu(\lambda, x) = \frac{1}{\lambda^\nu} J_\nu(\lambda x), \qquad \bar{A}_n = \lambda_n^\nu A_n, \tag{213}$$

and the λ's are the roots, positive or zero, of the equation

$$\left[l \frac{\partial}{\partial x} F_\nu(\lambda, x) + h F_\nu(\lambda, x) \right]_{x=1} = 0, \tag{214}$$

arranged in increasing order of magnitude. Since the series (212) is identical with the series (12) except when $l\nu + h = 0$, and in that case only differs from it in the possession of an extra term that is finite and continuous,* all that we have proved with regard to the summability, uniform summability and term by term integrability with regard to (12), holds good with regard to (212). Hence we infer from Lemma 25 that if $f(x)$ satisfies the conditions of Lemma 1, the series (212) will be summable $(C, k > 0)$ almost everywhere in $(0 < x \leqq 1)$ and may be multiplied by a function of bounded variation and integrated term by term over any interval in that interval.

We are now ready to prove the following lemma:

Lemma 27. *If $f(x)$ satisfies the conditions of Theorem II and $\chi(x)$ represents the function to which the series* (212) *is summable* $(C, k > 0)$ *almost everywhere in the interval* $(0 < x \leqq 1)$, *we have*

$$\int_0^1 x\chi(x) F_\nu(\lambda_k, x)\, dx = \int_0^1 x f(x) F_\nu(\lambda_k, x)\, dx \quad (k = 1, 2, 3, \cdots). \tag{215}$$

Since, as we have just pointed out, the series (212) may be multiplied by a function of bounded variation and integrated term by term, we have the equalities

$$\int_\delta^1 x\chi(x) F_\nu(\lambda_k, x)\, dx = \sum_{n=1}^{\infty} \bar{A}_n \int_\delta^1 x F_\nu(\lambda_n, x) F_\nu(\lambda_k, x)\, dx \tag{216}$$
$$(k = 1, 2, 3, \cdots),$$

where δ is any positive number < 1. But we have (cf. Transactions I, equation (113)) for all values of $n \neq k$

$$\begin{aligned} &\int_\delta^1 x F_\nu(\lambda_n, x) F_\nu(\lambda_k, x)\, dx \\ &= \frac{1}{\lambda_n^2 - \lambda_k} \Big[x\{\lambda_n^2 F_\nu(\lambda_k, x) F_{\nu+1}(\lambda_n, x) - \lambda_k^2 F_\nu(\lambda_n, x) F_{\nu+1}(\lambda_k, x)\} \Big]_\delta^1 . \end{aligned} \tag{217}$$

Moreover, since equation (1) may be thrown into the form $(\nu l + h) J_\nu(\lambda) = l\lambda J_{\nu+1}(\lambda)$ (cf. Transactions I, page 397), it follows that $\lambda J_{\nu+1}(\lambda)/J_\nu(\lambda)$ is constant for all values of λ which are roots of (214) and hence we see from (213) that the right-hand side of (217) vanishes at the upper limit. At the

* L. c., page 421.

lower limit it takes the form

$$(218)\quad \frac{\delta}{\lambda_n^\nu \lambda_k^\nu(\lambda_n^2 - \lambda_k^2)}[J_\nu(\lambda_k \delta)\lambda_n J_{\nu+1}(\lambda_n \delta) - \lambda_k J_{\nu+1}(\lambda_k \delta) J_\nu(\lambda_n \delta)],$$

and since from (213) $\overline{A}_n = \lambda_n^\nu A_n$, and from Lemma 1, $A_n = \sqrt{n}\psi_1(n)$, the general term of the series on the right-hand side of (216) may be written in the form

$$(219)\quad \frac{\delta}{\lambda_k^\nu}\left[J_\nu(\lambda_k \delta) A_n \frac{J_{\nu+1}(\lambda_n \delta)}{\lambda_n}\right] + \frac{\psi_1(n)}{n^{3/2}}.$$

We know from Lemma 26 that the expression (219) is the general term of a series that is uniformly convergent in the interval $(0 \leqq x \leqq c)$. Hence the series on the right-hand side of (216) converges uniformly in this interval, and therefore we may let δ approach zero. But in view of the fact that for the different values of λ_k that enter, the functions $\sqrt{x} F_\nu(\lambda_k, x)$ are orthogonal to each other, all the terms but one on the right-hand side of (216) drop out, and we obtain equation (215).

24. We are now ready to prove the following theorem:

THEOREM III. *If $f(x)$ is such that $\phi(x) = f(x) - f(0)$ satisfies the conditions of Lemma* 15, *and if $\lambda_1, \lambda_2, \lambda_3, \cdots$ are the roots, positive or zero, of equation* (214) *arranged in increasing order of magnitude, then the series* (212) *will be uniformly summable $(C, \frac{1}{2})$ to $f(x)$ in the interval $(0 \leqq x \leqq c)$, provided $\nu = 0$ or $f(0) = 0$.*

We know from Theorem II that under the conditions of the present theorem, the series (12), and therefore the series (212), is uniformly summable $(C, \frac{1}{2})$ in the interval in question. It remains to be shown that its value there is $f(x)$.

We have from Lemma 27

$$(220)\quad \int_0^1 x\{\chi(x) - f(x)\}F_\nu(\lambda_k x)\,dx = 0 \qquad (k = 1, 2, 3, \cdots),$$

where $\chi(x)$ is the function to which (212) is summable $(C, k > 0)$ almost everywhere in the interval $(0 < x \leqq 1)$. If we add to the definition of $\chi(x)$ by supposing it to be equal to the value to which the series (212) is summable $(C, \frac{1}{2})$ at any points in the interval $(0 \leqq x \leqq c)$ at which the series is not summable $(C, k > 0)$, the above equality is not altered, since we have at most modified the integrand at points forming a set of measure zero. But this further definition will serve to make $\chi(x)$ continuous in the interval $(0 \leqq x \leqq c)$, since the series (212) is uniformly summable $(C, \frac{1}{2})$ in that interval. Hence $\chi(x) - f(x)$ is continuous there, and in view of a lemma proved by the writer in a previous paper* $\chi(x) - f(x) = 0$, or $\chi(x) = f(x)$, at all points of the interval $(0 \leqq x \leqq c)$. Our theorem is therefore proved.

* Cf. Bulletin of the American Mathematical Society, 2d series, vol. 23 (1916), p. 25.

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