OSCILLATION THEOREMS IN THE COMPLEX DOMAIN*

BY

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1. Introduction

The aim of the present paper is to throw some light on the question of the distribution in the complex domain of the zeros of functions satisfying linear homogeneous differential equations of the second order. The real zeros of such functions are well known from the works of numerous mathematicians from Sturm and Liouville down to living writers, but our knowledge of the complex zeros is very deficient. It is only in special cases that progress has been made.

The number of complex zeros of a hypergeometric function in the case of real parameters has been determined by Hurwitz, Van Vleck and Schafheitlin. Hurwitz has also investigated Bessel functions. The same field has been covered by Macdonald, Porter and Schafheitlin.

Hurwitz used in his paper on Bessel functions‡ certain integral equalities analogous to those frequently used for establishing the reality of the characteristic values in a boundary problem.§ In his thesis || the present writer used similar equalities for the study of the zeros of Legendre functions.

The present paper contains a systematic study of integral equalities, called *Green's transforms,* which are adjoined to linear differential equations of the second order. It is shown that these equalities give information concerning the distribution of the zeros of a function satisfying such an equation. From the knowledge that a particular solution of the equation in question vanishes at a point in the complex plane, is real on an interval, or similar information, we are able to assign certain regions of the plane, containing the point or the interval where this particular solution cannot vanish. In such a fashion we can estab-

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‡ Mathematische Annalen, vol. 33 (1889), pp. 246–266.

§ Cf. for instance, Bôcher, *Leçons sur les Méthodes de Sturm,* Paris, 1917, pp. 73–76.

|| *Some problems concerning spherical harmonics,* Arkiv för Matematik, Astronomi och Fysik, vol. 13 (1918), No. 17.

lish the existence of various *zero-free regions.* It is often possible by simple means to make these zero-free domains cover the greater part of the plane, so that the zeros of the solution are distributed over a comparatively small region.

The investigation is completed by a study of the asymptotic distribution of the zeros by means of a singular integral equation of Volterra's type. This latter method often gives information on the general disposition of the zeros and forms a valuable complement to the method of assigning zero-free regions.

In the greater part of the paper we treat a special differential equation, $w'' - w/z = 0$, for purposes of illustration. The solutions of this equation are expressible in terms of Bessel functions of the first order.

The author has applied his methods with success to the study of several special differential equations such as, for instance, Bessel's, Legendre's,* Mathieu's and Weber's equations. The results will be published in later papers.

2. Green's transform

2.1. Transformation formulas. We can always assume that our differential equation is written in self-adjoint form

$$\frac{d}{dz}\left[K(z)\,\frac{dw}{dz}\right] + G(z)w = 0. \tag{2.11}$$

The functions $G(z)$ and $K(z)$ we suppose to be analytic in a region T of the z-plane, where, further, $K(z)$ does not vanish. We can replace the equation (2.11) by the system

$$\frac{dw_1}{dz} = \frac{1}{K(z)}\,w_2; \qquad \frac{dw_2}{dz} = -\,G(z)w_1; \tag{2.12}$$

where we have put

$$w_1 = w; \qquad w_2 = K(z)\,\frac{dw}{dz}. \tag{2.121}$$

* *On the zeros of Legendre functions,* Arkiv för Matematik, Astronomi och Fysik, vol. 17 (1922), No. 22.

If we further replace the first equation in (2.12) by its conjugate, the system becomes

$$d\bar{w}_1 = \frac{1}{\overline{K(z)}}\,\bar{w}_2 d\bar{z}, \tag{2.122}$$
$$dw_2 = -G(z)w_1 dz.$$

Multiply the first equation by w_2, the second by $\bar{w}_1$, add, and integrate the result between the limits z_1 and z_2. Thus we get

$$\Big[\bar{w}_1 w_2\Big]_{z_1}^{z_2} - \int_{z_1}^{z_2} |w_2|^2 \frac{d\bar{z}}{\overline{K(z)}} + \int_{z_1}^{z_2} |w_1|^2 G(z)dz = 0. \tag{2.13}$$

Here we have assumed z_1 and z_2 to be points in T and also the path of integration to be entirely in T. This expression (2.13), which is of fundamental importance for our coming work, we shall call the *Green's transform* of the differential equation on account of its similarity to a well known formula of Green for real variables.

If we write

$$\frac{dz}{K(z)} = d\mathbf{K}_1 + i\,d\mathbf{K}_2 = d\mathbf{K}, \tag{2.14}$$
$$G(z)dz = d\mathbf{\Gamma}_1 + i\,d\mathbf{\Gamma}_2 = d\mathbf{\Gamma},$$

we have

$$\Big[\bar{w}_1 w_2\Big]_{z_1}^{z_2} - \int_{z_1}^{z_2} |w_2|^2\, \overline{d\mathbf{K}} + \int_{z_1}^{z_2} |w_1|^2\, d\mathbf{\Gamma} = 0, \tag{2.15}$$

or, if the real and the imaginary parts be taken,

$$\mathfrak{R}\Big[\bar{w}_1 w_2\Big]_{z_1}^{z_2} - \int_{z_1}^{z_2} |w_2|^2\, d\mathbf{K}_1 + \int_{z_1}^{z_2} |w_1|^2\, d\mathbf{\Gamma}_1 = 0, \tag{2.151}$$

$$\mathfrak{Im}\Big[\bar{w}_1 w_2\Big]_{z_1}^{z_2} + \int_{z_1}^{z_2} |w_2|^2\, d\mathbf{K}_2 + \int_{z_1}^{z_2} |w_1|^2\, d\mathbf{\Gamma}_2 = 0. \tag{2.152}$$

2.2. **Change of variables.** Let us introduce a new independent variable in system (2.12) by putting

$$dz = f(Z)\, dZ, \tag{2.21}$$

whereby the system is transformed into

(2.22)
$$dw_1 = \frac{1}{k(Z)}\, w_2 dZ,$$
$$dw_2 = -\, g(Z) w_1 dZ,$$

with

$$k(Z) = \frac{K(z)}{f(Z)}, \quad g(Z) = G(z) f(Z).$$

The Green's transform goes over into

(2.23)
$$\left[\bar{w}_1 w_2\right]_{Z_1}^{Z_2} - \int_{Z_1}^{Z_2} |w_2|^2\, d\kappa + \int_{Z_1}^{Z_2} |w_1|^2\, d\gamma = 0,$$

where

(2.231)
$$\frac{dZ}{k(Z)} = d\kappa,$$
$$g(Z) dZ = d\gamma.$$

Thus we infer that *the Green's transform is invariant under a transformation of the independent variable.*

By special choice of Z we can obtain simpler forms of the transform. Our first choice will be

(2.24)
$$Z = \mathbf{K}(z).$$

If we put

$$J(Z) = G(z) K(z),$$

we obtain

(2.241)
$$\left[\bar{w}_1 w_2\right]_{Z_1}^{Z_2} - \int_{Z_1}^{Z_2} |w_2|^2\, dZ + \int_{Z_1}^{Z_2} |w_1|^2 J(Z)\, dZ = 0,$$

to which formula corresponds a differential equation

(2.242)
$$\frac{d^2 w}{dZ^2} + J(Z) w = 0.$$

This special type of differential equation is the one with which we shall be concerned mostly in this paper.

We can of course also choose

(2.25)
$$Z = \mathbf{\Gamma}(z),^*$$

* This function has of course nothing to do with the classical Γ-function.

which yields

$$\left[\bar{w}_1 w_2\right]_{Z_1}^{Z_2} - \int_{Z_1}^{Z_2} |w_2|^2 \frac{\overline{dZ}}{\overline{Q(Z)}} + \int_{Z_1}^{Z_2} |w_1|^2 \, dZ = 0, \tag{2.251}$$

with

$$Q(Z) = G(z)K(z),$$

corresponding to the differential equation

$$\frac{d}{dZ}\left[Q(Z)\frac{dw}{dZ}\right] + w = 0. \tag{2.252}$$

Finally we can get a more symmetric form by putting

$$dZ = \sqrt{\frac{G(z)}{K(z)}}\, dz, \tag{2.26}$$

which leads to

$$\left[\bar{w}_1 w_2\right]_{Z_1}^{Z_2} - \int_{Z_1}^{Z_2} |w_2|^2 \frac{\overline{dZ}}{\overline{S(Z)}} + \int_{Z_1}^{Z_2} |w_1|^2 S(Z)dZ = 0, \tag{2.261}$$

with

$$S(Z) = \sqrt{G(z)K(z)}.$$

The differential equation takes the following form:

$$\frac{d}{dZ}\left[S(Z)\frac{dw}{dZ}\right] + S(Z)w = 0. \tag{2.262}$$

This transformation we shall use in the last section of the paper for the study of the solutions in the neighborhood of an irregular singular point.

Up to this point we have changed only the independent variable. We can of course also make a simultaneous transformation of both independent and dependent variables

$$\begin{aligned} z &= f(Z), \\ w &= g(Z)W. \end{aligned} \tag{2.27}$$

Starting from an equation of the form (2.242), for instance, we can find infinitely many transformations of type (2.27), which furthermore preserve the form of the differential equation. For each set of such variables W, Z we have a Green's transform of type (2.241) of our equation.

The simplest possible of all such transformations is

$$\begin{aligned} w &= W, \\ z &= aZ + b, \end{aligned} \tag{2.28}$$

which carries $J(z)$ into $a^2J(aZ+b)$. We shall return to this transformation in §3.8.

It is evident that w_1 and w_2 play the same rôle in the system (2.12). From this remark it follows that we have also formulas like

$$\text{(2.29)} \qquad \Big[w_1\bar{w}_2\Big]_{z_1}^{z_2} - \int_{z_1}^{z_2} |w_2|^2 \frac{dz}{K(z)} + \int_{z_1}^{z_2} |w_1|^2\overline{G(z)}\,\overline{dz} = 0.$$

This is of course nothing but the conjugate of the expression in formula (2.13). We shall find later on that in all theorems concerning distribution of zeros, w_1 and w_2 enter symmetrically.

2.3. **Two nets of curves.** We can evidently simplify the formulas (2.151) and (2.152) by choosing the path of integration so that one of the following relations is fulfilled, namely

$$\text{(2.31)} \qquad \begin{aligned} d\mathbf{K}_1 &= 0; \qquad d\mathbf{K}_2 = 0; \\ d\mathbf{\Gamma}_1 &= 0; \qquad d\mathbf{\Gamma}_2 = 0. \end{aligned}$$

These paths constitute four families of curves which together form two independent nets of orthogonal trajectories.

In the case of an equation of type (2.242) the **K**-net is simply the net of lines parallel to the axes. The same lines serve as a $\mathbf{\Gamma}$-net of an equation of type (2.252).

On account of the skew symmetry between $G(z)$ and $K(z)$ we need consider only one of the two nets. Our results will hold, mutatis mutandis, for the other net. We choose the $\mathbf{\Gamma}$-net, as the **K**-net is trivial in the most important of all special cases, namely that in which $K(z) = 1$.

Through every point a of the region T in the z-plane where $G(z)$ is regular and furthermore $G(a) \neq 0$, passes one and only one curve of each of the families $\mathbf{\Gamma}_1$ and $\mathbf{\Gamma}_2$, which curves we denote by $\mathbf{\Gamma}_1 a$ and $\mathbf{\Gamma}_2 a$, respectively. These two curves are, of course, orthogonal to each other, with slopes equal to $g_1(a)/g_2(a)$ and $-g_2(a)/g_1(a)$, respectively, if we put

$$\text{(2.32)} \qquad G(z) = g_1(z) + ig_2(z).$$

Hence the $\mathbf{\Gamma}_1$-family has horizontal tangents along the curves $g_1(z) = 0$ and vertical ones along $g_2(z) = 0$. For the $\mathbf{\Gamma}_2$-curves the state of affairs is reversed.

We shall now proceed to consider the behavior of the $\mathbf{\Gamma}$-curves in the neighborhood of an exceptional point of $G(z)$ where this function either vanishes or becomes infinite.* We restrict ourselves to the case where $G(z)$ is analytic in T except for poles.

* Cf. for the theory of curves of this kind F. Lucas, *Géométrie des polynômes*, Journal de l'École Polytechnique, vol. 28 (1879), and F. Klein, *Ueber Riemann's Theorie der Algebraischen Funktionen und ihrer Integrale*, Leipzig, 1882.

First, let $z = a$ be a k-fold zero of $G(z)$. Then

$$\int_a^z G(z)dz = a_k \frac{(z-a)^{k+1}}{k+1} + \dots .$$

If $z - a = re^{i\theta}$ and $a_k = \rho\, e^{i\varphi}$ we have

$$\Re\left\{\int_a^z G(z)dz\right\} = \frac{\rho}{k+1}\, r^{k+1} \cos\left[(k+1)\theta + \varphi\right] + \cdots ;$$
$$\Im\mathrm{m}\left\{\int_a^z G(z)dz\right\} = \frac{\rho}{k+1}\, r^{k+1} \sin\left[(k+1)\theta + \varphi\right] + \cdots .$$

Hence $(k + 1)$ branches of $\mathbf{\Gamma}_1 a$ pass through $z = a$ and the same number of branches of $\mathbf{\Gamma}_2 a$ intersect them there in such a manner that the tangents at a of the one curve make equal angles with each other and bisect the angles formed by the tangents of the other curve.

If $z=a$ is a pole of order $k > 1$ of $G(z)$, we have

$$\int^z G(z)dz = C - \frac{a_k}{k-1}\,\frac{1}{(z-a)^{k-1}} + \cdots .$$

With the same notation as above we obtain

$$\Re\left[\int^z G(z)dz\right] = \gamma - \frac{\rho}{k-1}\, r^{1-k} \cos\left[(k-1)\,\theta - \varphi\right] + \cdots ;$$
$$\Im\mathrm{m}\left[\int^z G(z)dz\right] = \delta + \frac{\rho}{k-1}\, r^{1-k} \sin\left[(k-1)\,\theta - \varphi\right] + \cdots ,$$

where $\gamma + i\,\delta = C$. Let us assume for the sake of simplicity that no logarithmic term appears. Then we can conclude that every curve of the two families which passes in the neighborhood of $z = a$ actually passes through this point. The $\mathbf{\Gamma}_1$-curves are tangent to the lines

$$\arg(z-a) = \left[\varphi + (2\nu+1)\,\frac{\pi}{2}\right]/(k-1) \quad (\nu = 0, 1, \dots, k-2).$$

Replacing $2\nu + 1$ by 2ν we get the tangents to the $\mathbf{\Gamma}_2$-curves.

In the case where $z = a$ is a simple pole of $G(z)$ we have

$$\int^z G(z)dz = C + a_1 \log(z-a) + \cdots$$

and

$$\mathfrak{R}\left[\int^z G(z)dz\right] = \gamma + \alpha_1 \log r - \beta_1 \theta + \cdots;$$
$$\mathfrak{Im}\left[\int^z G(z)dz\right] = \delta + \beta_1 \log r + \alpha_1 \theta + \cdots,$$

where $\gamma + \delta i = C$, and $\alpha_1 + \beta_1 i = a_1$.

If neither α_1 nor β_1 is zero all curves of the net in the vicinity of $z = a$ admit this point as an asymptotic point which they approach in the manner of a logarithmic spiral. When $\beta_1 = 0$ all curves of $\mathbf{\Gamma}_1$ sufficiently near to a are ovals around this point which are approximately circular for great values of γ. In the first approximation, the $\mathbf{\Gamma}_2$-family behaves like a pencil of lines through a. If $\alpha_1 = 0$ the rôles of the two families are interchanged.

When carrying over the results of the discussion of the descriptive properties of the $\mathbf{\Gamma}$-net to the $\mathbf{K}$-net, we have to observe that the rôles of zeros and of poles are interchanged; a zero of $K(z)$ being a pole or logarithmic point of $\mathbf{K}(z)$ and so on.

2.4. $G(z)$ **a polynomial.** Suppose $G(z)$ is a polynomial of degree n

$$G(z) = A_0(z - a_1)^{\nu_1}(z - a_2)^{\nu_2} \ldots (z - a_\mu)^{\nu_\mu} \qquad (\nu_1 + \nu_2 + \ldots + \nu_\mu = n).$$

A non-specialized curve belonging either to $\mathbf{\Gamma}_1$ or $\mathbf{\Gamma}_2$ has no double points at all in the projective plane. Such a curve does not pass through any of the points $a_1, a_2, \cdots, a_\mu$. There is, however, one curve of each family which passes through the point a_k and has a $(\nu_k + 1)$-tuple point there. Thus there are at most μ singular curves in each family.

The intersections with the line at infinity in the projective plane are all distinct for one and the same curve but the same for all curves of the family. The asymptotes are all real and distinct; they all intersect in a fixed point, the center of gravity of the roots $(\nu_1 a_1 + \nu_2 a_2 + \cdots + \nu_\mu a_\mu)/n$, where they furthermore form equal angles with each other. If $\arg A_0 = \varphi_0$ the asymptotic directions of a $\mathbf{\Gamma}_1 a$ are given by

$$\arg z_k = \frac{2k+1}{n+1}\,\frac{\pi}{2} - \frac{\varphi_0}{n+1} \qquad (k = 0, 1, \ldots, n).$$

The asymptotes of the $\mathbf{\Gamma}_2$-curves pass through the same point and bisect the angles between the asymptotes of the $\mathbf{\Gamma}_1$-family. We note that the asymptotes of the curves $g_1(z) = 0$ and $g_2(z) = 0$ also intersect in this point.

A $\mathbf{\Gamma}$-curve can never begin or end in a finite point, nor can it be closed. This implies that every point in the plane can be reached along a suitably chosen path from infinity without crossing the curve in question. This is a consequence of the fact that $\Gamma_1(z)$ and $\Gamma_2(z)$, being harmonic functions throughout the finite plane, can not have maxima or minima.

3. Determination of zero-free regions

3.1. **The real axis.** We shall apply the results of the preceding section to the problem of locating the zeros of a given solution of a linear differential equation of the second order. We start with the real zeros and restrict ourselves to the case when $K(z) = 1$. From formulas (2.151) and (2.152) we obtain

$$\mathfrak{R}\left[\bar{w}_1 w_2\right]_{x_1}^{x_2} - \int_{x_1}^{x_2} |w_2|^2 dx + \int_{x_1}^{x_2} g_1(x)|w_1|^2 dx = 0; \tag{3.11}$$

$$\mathfrak{Im}\left[\bar{w}_1 w_2\right]_{x_1}^{x_2} + \quad * \quad + \int_{x_1}^{x_2} g_2(x)|w_1|^2 dx = 0. \tag{3.12}$$

Now take an arbitrary solution w of the equation (2.242) or a pair of solutions (w_1, w_2) of the corresponding system. Can this solution w, or, more generally, can the product $w_1 w_2$ vanish at two points x_1 and x_2 of a certain interval (a, b) of the real axis? Suppose $g_2(z)$ keeps a constant sign in (a, b). Then by (3.12) w_1, w_2 can vanish once at most in the interval in question, and similarly if $g_1(z) \leqq 0$ in (a,b). We have assumed of course that $G(z) = J(z)$ is analytic in the interval. Thus

Theorem 3.1. *In an interval of the real axis throughout which $J(z)$ is analytic and furthermore $\mathfrak{R}[J(z)] \leqq 0$ or $\mathfrak{Im}[J(z)]$ keeps a constant sign (perhaps vanishing at discrete points) there can be at most one zero of $w\, dw/dz$ where w is an arbitrary solution of $w'' + J(z)w = 0$.*

Thus if w is to oscillate on the axis of reals, supposed free from singular points, we must have $g_1(z) \geqq 0$ and $g_2(z)$ must change sign or vanish identically.

In case there are singular points on the real axis we may be able to get additional information concerning the zeros. Suppose for instance that $z = a$ is a regular singular point of the equation with exponents λ_1 and λ_2 such that $\lambda_1 + \lambda_2 = 1$, $\mathfrak{R}(\lambda_1) > \mathfrak{R}(\lambda_2)$. Then $J(z)$ has a pole of at most the second order at $z = a$. If we assume $\mathfrak{R}(\lambda_1) \gg 1/2$ the expressions in (3.11) and (3.12) will remain finite when we let x_1 or x_2 converge towards a, provided w_1 is the solution corresponding to the exponent λ_1; and $w_1 w_2 \longrightarrow 0$ when $x \longrightarrow a$. Consequently this particular solution will not vanish in an interval (a,b), a being the singular point, where $g_1(z)$ and $g_2(z)$ are subject to the conditions stated in the theorem above. This remark, of course, also applies to more general types of singular points.

It is obvious that the results of this section can be generalized to equations of the type (2.11).

3.2. **Dynamical interpretation.** Theorem 3.1 admits of direct dynamical interpretation.

If we put $w = u + iv$ in equation (2.242), assume $z = x$ to be real and separate reals and imaginaries, we obtain the following system

$$\frac{d^2u}{dx^2} = -g_1u + g_2v; \qquad (3.21)$$

$$\frac{d^2v}{dx^2} = -g_1v - g_2u.$$

These are the equations for the motion of a particle of unit mass, subjected to a force whose components on the radius vector and on a line perpendicular to the same are $g_1\sqrt{u^2+v^2}$ and $-g_2\sqrt{u^2+v^2}$, respectively.

Theorem 3.1 now gives a sufficient condition that a particle starting from the origin in the u,v-plane at the time x_1 with a certain velocity shall continue to move away from the origin during a certain space of time. The condition $g_1(x) \leqq 0$ evidently means that the radial component of the force is always directed from the origin whereas the invariability of sgn $g_2(x)$ means that the rotating component has the same sense during the time interval in question. These conditions are evidently sufficient to prevent the particle from turning towards the origin.

3.3. **Linear paths.** We shall now consider an arbitrary linear segment (z_1, z_2) in the complex plane and an arbitrary solution $w(z)$ of equation (2.242) under the assumption that $J(z)$ is analytic along the segment chosen. Put

$$z = z_1 + re^{i\theta}. \qquad (3.31)$$

From (2.241) we infer

$$\left[\bar{w}\frac{dw}{dr}\right]_0^r - \int_0^r \left|\frac{dw}{dr}\right|^2 dr + e^{2i\theta}\int_0^r |w|^2 J(z)dr = 0. \qquad (3.32)$$

If we put

$$\bar{w}\frac{dw}{dr} = f_1(r) + if_2(r);$$

$$\cos 2\theta\, g_1(z) - \sin 2\theta\, g_2(z) = P(z,\theta); \quad \cos 2\theta\, g_2(z) + \sin 2\theta\, g_1(z) = Q(z,\theta),$$

we obtain

$$f_1(r) - f_1(0) - \int_0^r \left|\frac{dw}{dr}\right|^2 dr + \int_0^r P(z,\theta)|w|^2 dr = 0; \qquad (3.33)$$

$$f_2(r) - f_2(0) + \int_0^r Q(z,\theta)|w|^2 dr = 0. \qquad (3.34)$$

These formulas yield the following

THEOREM 3.31. *There is at most one zero of the product* $w\,dw/dz$, *where* $w(z)$ *is an arbitrary solution of* $w'' + J(z)w = 0$, *on a segment* (z_1, z_2) *in the complex plane where* $J(z)$ *is analytic, provided either*

(i) $P(z, \theta) \leqq 0$ *along the segement, or*

(ii) $Q(z, \theta)$ *keeps a constant sign there.*

If in addition to (i) $f_1(0) \geqq 0$, *or in addition to* (ii) $f_2(0)$ *has opposite sign to that of* $Q(z, \theta)$ *all along the segment, then there is no zero at all of the product* $w\,dw/dz$ *on* (z_1, z_2).

Let us now extend this theorem to a region T in the complex plane where $J(z)$ is analytic. Take a pencil of parallel lines l_{z_0};

$$z = z_0 + re^{i\theta}$$

Each line is characterized by the variable point z_0. Let T be simply connected and of such a shape that every line of the pencil cuts the boundary in two points at most. Then there will be two lines of l_{z_0} each of which meets the boundary in two coincident points, α and β, respectively. These two points α and β divide the boundary in two parts, one of which we shall designate by C and which shall be the locus of the points z_0. Then we have

THEOREM 3.32. *In a region* T, *defined as above, there is at most one zero of the product* $w\,dw/dz$ *on any line* l_{z_0} *if either*

(i) $P(z, \theta) \leqq 0$ *throughout* T, *or*

(ii) $Q(z, \theta) \neq 0$ *in* T.

If in addition to (i) $\Re[e^{i\theta}\overline{w(z)}dw/dz] \geqq 0$ *along* C, *or in addition to* (ii) $\Im\mathrm{m}[e^{i\theta}\overline{w(z)}dw/dz] = 0$ *or has opposite sign to that of* $Q(z, \theta)$ *in* T *all along* C, *then there is no zero of the product in* T.

The first part of the theorem does not lay any serious restrictions on the zeros of w, and the second part seems rather unmanageable. We shall however give some special cases of great interest. Suppose C is a segment of the real axis, $\theta = \pi/2$ and $w(z)$ takes on only real values on C. Then $\Re[e^{i\theta}\overline{w}dw/dz] = 0$ and we obtain

THEOREM 3.321. *If the equation* (2.242) *has a solution* $w(z)$ *which is real on a segment* (a, b) *of the real axis; if further* T *is a region symmetric with reference to* (a,b) *such that every vertical line which cuts the region cuts its boundary twice and meets* (a,b) *in an interior point; and if finally* $\Re[J(z)] \geqq 0$ *throughout* T, *then neither* $w(z)$ *nor* dw/dz *can have any* complex *zeros in* T.

If in the statement of the preceding theorem we replace the real axis by the axis of imaginaries, vertical lines by horizontal and finally suppose $\Re[J(z)] \leqq 0$ in T the theorem still holds, the exceptional zeros now being purely imaginary.

As a simple application we may choose the equation $w'' + w = 0$ with $w = \sin z$. Theorem 3.321 shows that there are no complex zeros of $\sin z$ or $\cos z$.

We can of course also derive a special theorem from formula (3.33), namely

THEOREM 3.322. *If the equation* (2.242) *has a solution* $w(z)$, *real on a segment* (a,b) *of the real axis where* $w\,dw/dz$ *keeps a constant sign, namely the same as that of* $\mathfrak{Im}[J(z)]$ *in a region* T *above the real axis, whose base is* (a,b) *and of which the boundary above the real axis is cut in one point at most by any vertical line; then neither* w *nor* dw/dz *can have a complex zero in* T *or in the symmetric region* $\overline{T}$ *below the real axis.*

A similar theorem can be stated for the imaginary axis.

As a simple illustration take the equation

$$\frac{d^2w}{dz^2} - \frac{1}{z}\,w = 0. \tag{3.35}$$

This equation will be used throughout the remainder of the paper for the purpose of giving concrete applications of our theorems. Thus we need some knowledge of the nature of its solutions. ∞ is an irregular singular point whereas 0 is regular singular with exponents 0 and 1. There is one solution, $E(z) = \sqrt{z}\,J_1(2i\sqrt{z})$, where $J_1(u)$ denotes the ordinary Bessel function of order 1, which is an entire function of z and vanishes at the origin; every other solution is of the form $[c_1 + c_2 \log z]E(z) + c_2H(z)$ where $H(z)$ is an entire function. $E(z)$ is real for all real values of z and has infinitely many real negative zeros; every other solution can be real on a half-axis at most and if it is real on the negative half of the real axis it must oscillate there. A solution which is real for $x > 0$ can have at most one positive zero or extremum. In general w^2 will increase beyond all limit when $x \longrightarrow +\infty$. In fact we have

$$\lim_{x\to+\infty} \frac{w'(x)\sqrt{x}}{w(x)} = +1, \tag{3.351}$$

save for one exceptional integral which yields -1 in the limit.*

As $\mathfrak{R}(-1/z) > 0$ when $R(z) = x < 0$ we infer by theorem 3.321 that no integral of the equation which is real for real negative values can have any complex zeros in the left half-plane. Further $\mathfrak{Im}(-1/z) > 0$ when $x > 0$. Take a solution which is real for real positive values of z and let x_0 be the positive zero of $w(z)\,dw/dz$ if there is any; otherwise positive but arbitrary. Then by theorem 3.322 we infer that such a solution can not vanish in the half-plane $\mathfrak{R}(z) \geqq x_0$.

* Cf. A. Wiman, *Über die reellen Lösungen der linearen Differentialgleichungen zweiter Ordnung*, Arkiv för Matematik, Astronomi och Fysik, vol. 12 (1917), No. 14.

For the exceptional integral we may in fact state a little more, namely that it can not vanish in the right half-plane including the imaginary axis.

For these exceptional integrals we may give a general statement. Let

$$J(z) = -F(z)$$

where $F(z)$ is real and positive for real values of $z > x_1$ and analytic in a region D including the real axis for $\Re(z) > x_1$. Concerning D we make the further assumption that any horizontal line which cuts D shall cut its boundary in one finite point only, and along the part of the line which lies in D one of the two relations

$$\Re[F(z)] > 0 \quad \text{or} \quad \Im\mathrm{m}[F(z)] \neq 0$$

shall hold. Let $W(z)$ be a solution of

$$w'' - F(z)\, w = 0$$

such that $W(z) \longrightarrow 0$ when $z \longrightarrow \infty$ in D along a parallel to the real axis. Such an exceptional integral will under very general assumptions on $F(z)$ tend to zero so rapidly that the integrals in formulas (3.33) and (3.34) converge when the upper limit tends to ∞. This will for instance be the case when $F(z)$ is a polynomial. We conclude that *such an exceptional integral can not admit a finite zero in D nor can its derivative vanish there.*

The results of this section can be considerably extended in various directions. To every theorem we have stated for equations of type (2.242) corresponds a theorem for equations of general self-adjoint form. The reader can easily work out these theorems for himself.

3.4. **The star.** In the preceding section we have considered parallel lines. Let us now take a pencil of lines through a finite point $z = a$ where $J(z)$ is regular and does not vanish. Put

$$z = a + r\, e^{i\theta},$$
$$L(z) = (z - a)^2 J(z) = P(z) + iQ(z),$$

and draw the curves $P(z) = 0$ and $Q(z) = 0$. These curves intersect at $z = a$ where each of them has a double point, the tangents of which are given by

$$\cos 2\theta\, g_1(a) - \sin 2\theta\, g_2(a) = 0,$$
$$\cos 2\theta\, g_2(a) + \sin 2\theta\, g_1(a) = 0,$$

respectively. Let us follow a line in general position from $z = a$. We start with definite signs of $P(z)$ and $Q(z)$, and travel along the line until $Q(z)$ changes sign. If perchance $P(z)$ has been positive or has changed its sign, we break off the line at that point. In case, however, $P(z)$ has been negative all along the segment, we continue to follow the line until finally even $P(z)$ changes sign, at which point we stop. We always stop at a point where $J(z)$ is singular no matter what the signs may be. In the same manner we treat every line of the pencil and determine on each ray a last point p_θ. The assemblage of all segments (a, p_θ) form together a configuration which we shall call *the star belonging to a*.* The boundary of the star is made up of the points p_θ and parts of certain critical rays which are to be found among the tangents that can be drawn from the point $z = a$ to the curves $P(z) = 0$ and $Q(z) = 0$.

The points p_θ are either points of these curves or singular points of $J(z)$.

The appearance of the zero-free star is somewhat different from that of a star of convergence in as much as it in general is composed of two different regions which only touch each other at $z = a$. (See the figure below.) In the

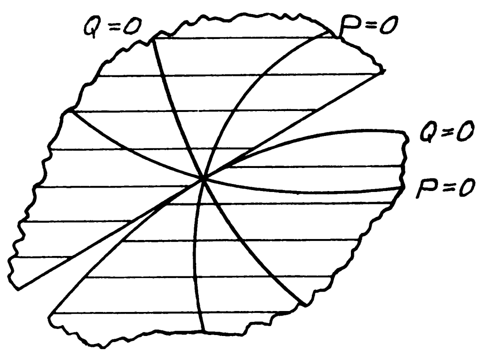

FIG. 1.

neighborhood of this point the boundary of the star is made up of that branch of the curve $Q(z) = 0$ which approaches $z = a$ through a region where $P(z) > 0$, and of the tangent to that branch at $z = a$. The fundamental property of the star is given by the following

THEOREM 3.4. *If $z = a$ is a zero of $w\, dw/dz$ where $w(z)$ is a solution of equation* (2.242), *then this product does not vanish throughout the star belonging to $z = a$, including the regular points of the boundary.*

This theorem is an immediate consequence of Theorem 3.31.†

* It is, of course, the similarity in form and generation to the *star of convergence*, introduced in analysis by Prof. Mittag-Leffler, which suggests this name. The chief property of our star is indicated by the more explicit name *zero-free star of $w(z)$ with respect to $z = a$*.

† For further developments of the properties of the zero-free star cf. the paper quoted in section 3.9 below, also *Convex distribution of the zeros of Sturm-Liouville functions*, Bulletin of the American Mathematical Society, vol. 28 (1922), pp. 261–65 and *A correction*, ibid., vol. 28 (1922), p. 462.

Let us use this theorem for proving that the entire function $E(z)$, considered in the preceding section, as well as its first derivative, has only real and negative zeros. $E(0) = 0$, but the origin is a singular point; we easily find however that the integrals in formulas (3.33) and (3.34) converge when z_1 is allowed to approach 0. Hence we can form the star for the point $z = 0$ and apply Theorem 3.4. We find $L(z) = -z$ and the star consists of the whole plane, except for the negative part of the real axis, which proves our assertion. What we have been proving is, of course, only that $J_1(u)$ has only real zeros, which is a well known fact in the theory of Bessel's functions.

3.5. **The primary domain.** Let us separate reals and imaginaries in formula (2.241). We obtain

$$\Re\left[\bar{w}\frac{dw}{dz}\right]_{z_1}^{z_2} - \int_{z_1}^{z_2}\left|\frac{dw}{dz}\right|^2 dx + \int_{z_1}^{z_2}|w|^2(g_1dx - g_2dy) = 0, \tag{3.51}$$

$$\Im\mathrm{m}\left[\bar{w}\frac{dw}{dz}\right]_{z_1}^{z_2} + \int_{z_1}^{z_2}\left|\frac{dw}{dz}\right|^2 dy + \int_{z_1}^{z_2}|w|^2(g_1dy + g_2dx) = 0. \tag{3.52}$$

Assume $z = z_0$ is a zero of $w\ dw/dz$ lying in a domain T where $g_1(z)$ amd $g_2(z)$ are different from zero and $J(z)$ is regular. In order to have a definite case to deal with we suppose

$$g_1(z) > 0; \qquad g_2(z) < 0$$

in T. Draw parallels to the axes through z_0 and fix the attention on the secona of the resulting quadrants of T. The points of this region are of two different kinds; a point α may be joined with z_0 by a curve which never increases with increasing values of $x = \Re(z)$, or a point may be such that there is no such curve. The points of the first kind form a certain simply connected region D_2 where the subscript 2 denotes the quadrant. Similarly we find a certain region D_4 in the fourth quadrant, the points of which all can be reached from z_0 by a continuous, never-increasing curve. These two regions together form a region $D_{24}(z_0)$ which we agree to call the primary domain of z_0. In view of (3.51) and (3.52) we obtain

THEOREM 3.5. *If there is a solution $w(z)$ of equation* (2.242) *such that $w(z)dw/dz$ vanishes at a point z_0 in a region T where $J(z)$ is analytic and $\Re[J(z)] \neq 0$, $\Im\mathrm{m}[J(z)] \neq 0$, then neither $w(z)$ nor dw/dz can vanish in the primary domain of z_0.*

The primary domain is a region D_{13} if $\Im\mathrm{m}[J(z)] > 0$, but a region D_{24} when $\Im\mathrm{m}[J(z)] < 0$.

The primary domain gives of course only a first rough approximation of the zero-free region in the neighborhood of a given zero, but even that is useful for orientation.

We have assumed that z_0 is not a point of a curve $g_1(z) = 0$ or $g_2(z) = 0$. If z_0 however lies on such a curve, we can still find a similar zero-free domain. Suppose, for example, that a segment (a,b) of the real axis forms a part of $g_2(z) = 0$ and that $a < z_0 < b$; then we can form a primary domain which is of type D_{14} if $g_2(z) > 0$ above the segment and of type D_{23} in the contrary case.

The boundary of a $D_{\mu\nu}(z_0)$ is made up of parts of the curves $g_1(z) = 0$ and $g_2(z) = 0$ and of lines parallel to the axes. It is as a rule possible to cross these boundaries and continue the zero-free region beyond the limits of the primary domain. The formulas (3.5) indicate clearly the way of doing this; we shall take up this question in a more general form in the next section.

3.6. **The standard domain.** Let us take our differential system in the general form

$$\frac{dw_1}{dz} = \frac{1}{K(z)} w_2, \qquad \frac{dw_2}{dz} = -G(z)w_1, \tag{3.61}$$

with the corresponding Green's transform, split up into real and imaginary parts

$$\Re\left[\bar{w}_1 w_2\right]_{z_1}^{z_2} - \int_{z_1}^{z_2} |w_2|^2 d\mathbf{K}_1 + \int_{z_1}^{z_2} |w_1|^2 d\mathbf{\Gamma}_1 = 0, \tag{3.621}$$

$$\Im m\left[\bar{w}_1 w_2\right]_{z_1}^{z_2} + \int_{z_1}^{z_2} |w_2|^2 d\mathbf{K}_2 + \int_{z_1}^{z_2} |w_1|^2 d\mathbf{\Gamma}_2 = 0, \tag{3.622}$$

where as usual

$$d\mathbf{\Gamma} = d\mathbf{\Gamma}_1 + i\, d\mathbf{\Gamma}_2 = G(z)dz, \tag{3.631}$$

$$d\mathbf{K} = d\mathbf{K}_1 + i\, d\mathbf{K}_2 = \frac{dz}{K(z)}. \tag{3.632}$$

We assume $G(z)$ and $K(z)$ to be single-valued analytic functions of z, existing in the whole plane. Then the two nets $\mathbf{\Gamma}$ and $\mathbf{K}$ are uniquely determined; one and the same curve may however correspond to different values of the constant of integration, depending upon which branch of the, in general, many-valued functions $\mathbf{\Gamma}(z)$ or $\mathbf{K}(z)$ we are considering.

A solution of (3.61) is as a rule a many-valued function which undergoes a linear transformation when z describes a closed contour in the plane surrounding singular points of the system. These are the zeros $a_1, a_2, \ldots, a_n, \ldots$ of $K(z)$, the singular points $b_1, b_2, \ldots, b_n, \ldots$ of $G(z)$ and $K(z)$ and, in general, the point at infinity. In order to have to deal with single-valued functions only, we can either join the singular points of the system by properly chosen cuts which z is not allowed to cross, or we can construct a Riemann surface with branch points at the points (a_n), (b_n) and ∞ on which every solution of (3.61) is single-valued. We use the latter scheme and imagine the $\mathbf{\Gamma}$-net and the $\mathbf{K}$-net represented on the Riemann surface.

Now take a point a on a particular leaf of the surface where we know that $W(z) = w_1(z)w_2(z) = 0$. In order to show that another point b on the surface cannot be a zero of $W(z)$ we try to join a and b by a suitable path of integration. Formulas (3.62) suggest the use of the two nets for this purpose. We define a path curve as a *standard path* if it fulfils the following conditions:

(i) *It does not pass through any singular point*;

(ii) *It is composed of a finite number of arcs of curves belonging to the two base nets* $\mathbf{\Gamma}$ *and* $\mathbf{K}$;

(iii) *All along the path one and the same of the following four characteristic inequalities is satisfied, namely*

$$(1)\ \begin{matrix} d\mathbf{\Gamma}_1 \leqq 0, \\ d\mathbf{K}_1 \geqq 0; \end{matrix} \qquad (2)\ \begin{matrix} d\mathbf{\Gamma}_1 \geqq 0, \\ d\mathbf{K}_1 \leqq 0; \end{matrix} \qquad (3)\ \begin{matrix} d\mathbf{\Gamma}_2 \geqq 0, \\ d\mathbf{K}_2 \geqq 0; \end{matrix} \qquad (4)\ \begin{matrix} d\mathbf{\Gamma}_2 \leqq 0, \\ d\mathbf{K}_2 \leqq 0. \end{matrix}$$

With the definition we have chosen the tangent of the path is continuous except at a finite number of points where arcs of different base-curves meet. We make the additional agreement that in a small neighborhood of such a point the two meeting arcs are replaced by an arc which joins continuously and has a continuous tangent. This can always be done in such a manner that the characteristic inequality of the path is preserved.

According as the path in question is characterized by the first, second, third or fourth set of inequalities, we call the path a standard path of the first, second, third or fourth kind, respectively

$$S\mathbf{K}_1^+,\ S\mathbf{K}_1^-,\ S\mathbf{K}_2^+ \text{ and } S\mathbf{K}_2^-.$$

Take a point a on the surface where $G(z)$ and $K(z)$ are analytic and $K(a) \neq 0$ and construct all standard paths which start at a. In this fashion we obtain a certain continuum of curves spread out on the surface. The set of points which belong to at least one of these paths forms a region on the Riemann surface which we call the *standard* domain $D(z_0)$ of z_0. We include the boundary points of $D(z_0)$ in the standard domain except the point z_0 itself and eventual singular points of the differential equation. The fundamental property of $D(z_0)$ is given by

THEOREM 3.6. *If $z = z_0$ is a zero of $W(z) = w_1(z)w_2(z)$ then $W(z)$ cannot vanish in the standard domain of z_0.*

We note that the primary domain serves as a kernel of the standard domain in case $K(z) = 1$.

In our typical example we have

$$\mathbf{K}(z) = z; \qquad \mathbf{\Gamma}(z) = -\log z.$$

The **K**-net consists of the lines parallel to the axes; the $\mathbf{\Gamma}_1$-family is made up of the circles with center at the origin and the $\mathbf{\Gamma}_2$-family of the straight lines through the same point. The solutions of (3.35) are single-valued on the Riemann surface of log z.

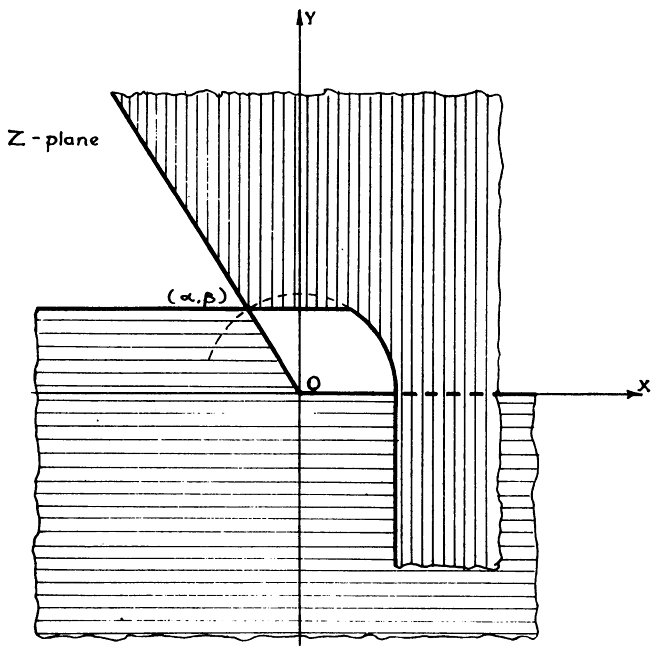

FIG. 2

Take a solution such that $W(x_0) = 0$ where $x_0 < 0$. A simple discussion shows that this function must be real for real negative values. If we construct the standard domain $D(x_0)$ we obtain the whole plane (except the origin) covered once and, in addition, two overlapping regions $\Re(z) \geqq |x_0|$, $\Im m(z) > 0$ and $\Re(z) \geqq |x_0|$, $\Im m(z) < 0$, attached to the lower (upper) side of the positive real axis. Using the method which is explained in the subsequent section, we could even prove a little more, namely: every solution which is real on the negative real axis has only real negative zeros in the sector $-(3\pi/2) < \arg(-z) < (3\pi/2)$.

Now take a solution for which $W(x_0) = 0$ with $x_0 > 0$. Such a solution must be real for real positive values of z. The standard domain of x_0 consists of

(1) the half-plane $\Re(z) \geqq x_0$ in one leaf of the surface;

(2) the region $|z| \leqq x_0$, $\Re(z) \geqq 0$ and $|\Im\mathrm{m}(z)| \leqq x_0$, $\Re(z) < 0$ where the negative part of the real axis is counted as a double line; this all in the same leaf as (1);

(3) the quadrant $\Re(z) < 0$, $\Im\mathrm{m}(z) > 0$ in another leaf, hanging on region (2) along the lower side of the real axis;

(4) the symmetric quadrant in a third leaf, hanging on region (2) along the upper side of the double line.

If the zero is not real we get standard domains which are asymmetric. The figure gives the typical shape of $D(a + ib)$ when $a < 0$, $b > 0$.

3.7. **The standard domain of a curve.** In the preceding sections we have formed various zero-free domains of a point. We can easily extend this conception to an arc of a curve.

Let C be an arc of a continuous curve on the surface, having a continuous tangent except at a finite number of points. We assume further that $G(z)$ is analytic along C and $K(z)$ analytic and different from zero. Let a solution of (3.35) be given along C. By given we understand that there is some means at our disposal by which we can ascertain the signs of the real and of the imaginary parts of $\bar{w}\,(dw/dz)\,K(z)$.

Take a point z_0 on the curve C. From z_0 emanates a set of standard paths, belonging to all four families. Of these families we can as a rule only use two as paths of integration. Put

$$\overline{w(z)}\,\frac{dw}{dz}K(z) = U(z) + i\,V(z). \tag{3.71}$$

If, for example, $U(z_0) > 0$ we can use $S\mathbf{K}_1^+$ but not $S\mathbf{K}_1^-$. Thus we draw from every point on C the standard paths of the kinds which are indicated by the table below.

(3.72)

	negative	zero	positive
$U(z)$	$S\mathbf{K}_1^-$	$S\mathbf{K}_1^+$, $S\mathbf{K}_1^-$	$S\mathbf{K}_1^+$
$V(z)$	$S\mathbf{K}_2^+$	$S\mathbf{K}_2^+$, $S\mathbf{K}_2^-$	$S\mathbf{K}_2^-$

These paths together form a continuum of curves whose points make up the *standard domain of C with respect to* $w(z)$, which we denote by DCw. The points on the boundary of DCw are included in the standard domain by definition except the points on C and the singular points of the differential equation. In view of formulas (3.62) we have

THEOREM 3.7. *If a solution,* $w(z)$, *of* (2.11) *be given, in the sense defined above, along a curve C on the Riemann surface of* $w(z)$, *which curve does not pass through* *any of the singular points of the differential equation, then there is no zero of wdw/dz in the standard domain of C with respect to $w(z)$.*

We have assumed that C does not pass through any of the singular points of the differential equation. This condition can, of course, be abandoned in special cases with great advantage. If, for instance, $z = a$ is a regular singular point with exponents λ_1 and λ_2 where $\Re(\lambda_1) > 1/2$ say, then the expressions in formulas (3.62) remain finite when z_1 is allowed to approach a, if $w_1(z)$ is the solution corresponding to λ_1. Further $w_1(z)w_2(z) \longrightarrow 0$ when $z \longrightarrow a$. Consequently the standard domain of such a point exists and has the same properties as the standard domain of an ordinary regular point.

In case $z = a$ is an irregular singular point we are often able to find certain sectors with vertex at a and corresponding solutions which tend toward 0 when $z \longrightarrow a$ within the sector and approach zero so rapidly that formulas (3.62) are applicable with a as the lower limit of integration. In this case we can not construct a full standard domain around a, as we are allowed to use only such standard paths as, in a small neighborhood of a, lie on the sectors mentioned above. Such sectors can in general be found when, for instance, the irregular singular point is of finite rank.

3.8. **Change of variables.** In section 2.2 we have shown that the Green's transform is invariant under a transformation of the type

$$z = f(Z) \quad \text{or} \quad Z = F(z)$$

where $F(z)$ is, of course, an analytic function of z.

Take a point z_0 in the z-plane and form the corresponding standard domain $D(z_0)$. Suppose $F(z)$ is analytic in D and $F'(z) \neq 0$ there. Then $Z = F(z)$ will map $D(z_0)$ conformally on a region $d(Z_0)$ of the Z-plane. We assert that $d(Z_0)$ is the standard domain of Z_0 in the Z-plane. In fact, every standard path, $S\mathbf{K}$, starting from z_0 in the z-plane, is transformed into a curve, $s\kappa$, that starts at $Z_0 = F(z_0)$ in the Z-plane. In view of the formulas in section 2.2,

$$d\mathbf{\Gamma}(z) \equiv d\gamma(Z); \quad d\mathbf{K}(z) \equiv d\kappa(Z). \tag{3.81}$$

Hence the base nets in the two planes correspond to each other. From this it follows that $s\kappa$ is a standard path in the Z-plane, starting at Z_0 and, furthermore, $s\kappa$ is of the same kind as $S\mathbf{K}$. Thus every point of $d(Z_0)$ belongs to at least one standard path $s\kappa$; consequently $d(Z_0)$ is a standard domain as asserted. We can express this fact also by saying that *the standard domain of a point is an absolute covariant under conformal transformations of the plane.*

All the different forms of the Green's transform, given in section 2.2, are consequently equivalent for our purpose; one choice of independent variable may however lead to simpler expressions for $k(Z)$ and $g(Z)$ or have some other advantage over another choice of variable. In the case of our typical equation

$$w'' - \frac{1}{z} w = 0$$

the transformation in formula (2.25) offers some advantages. By putting

$$Z = -\log z \quad \text{or} \quad z = e^{-Z},$$

the equation becomes

$$\frac{d}{dZ}\left[e^{Z} \frac{dw}{dZ}\right] - w = 0.$$

Every solution is now single-valued, in fact, an entire function of Z. This fact more than compensates the difficulties arising from the slightly more complicated form of the κ-net.

In order to get any new facts concerning the zero-free regions we must change both variables simultaneously. We shall not develop any generalities along this line. A simple example may be sufficient to show what use one can have of a linear transformation.

In section 3.6 we determined the standard domain corresponding to a solution of our typical equation with a positive zero or extremum. Let us indicate the (hypothetical) complex zeros of the principal branch of the solution in question by $z_1, z_2, \ldots, z_n, \ldots$, and $\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_n, \ldots$ where as usual $\bar{z}_n$ stands for the conjugate of z_n. Put $z_n = x_n + iy_n$. Our result showed only that from a certain value of n, $x_n < x_0$ and $y_n > x_0$. We shall prove in section 4.7 that $y_n = O(\sqrt{x_n})$. Thus it is desirable to obtain an upper limit for y_n for all values of n. This can be done as follows. Put

$$z = e^{i\theta} Z.$$

Equation (3.35) becomes

$$\frac{d^2w}{dZ^2} - \frac{e^{i\theta}}{Z} w = 0.$$

Thus we have

$$\gamma(Z) = -e^{i\theta} \log Z.$$

Putting $Z = Re^{i\theta}$ we get

$$\gamma_1(Z) = \sin\theta\,\phi - \cos\theta \log R,$$
$$\gamma_2(Z) = -\sin\theta \log R - \cos\theta\,\phi,$$

which shows that the γ-net is made up of logarithmic spirals. The point x_0 is transformed into $x_0 e^{-i\theta}$; the corresponding γ_1- and γ_2-curves are

$$R = x_0 e^{\tan\theta(\phi+\theta)},$$
$$R = x_0 e^{-\cot\theta(\phi+\theta)},$$

respectively. The first curve has a vertical tangent at

$$\phi_0 = \theta, \quad R_0 = x_0 e^{2\theta \tan\theta}.$$

It is easy to see that this point, as well as the points on the same vector for which $R > R_0$, belongs to a zero-free region. In order to determine the locus of R_0, ϕ_0 we introduce polar coördinates in the old z-plane by putting $z = re^{i\varphi}$. Then $\varphi = 2\theta$, $r = R$ and we obtain

$$r = x_0 e^{\varphi \tan\frac{\varphi}{2}}.$$

Thus there is no zero of $w(z)\, dw/dz$ beyond this curve. Hence

$$y_n < 2\,|z_n| \arg\,(z_n) \,/\, \log |z_n|,$$

or

$$\arg z_n \longrightarrow \pi.$$

3.9. **The domain of influence.*** The zero-free domains considered in the preceding articles have all been obtained by specialization of the path of integration in formula (2.15). We can arrive at a more general type of zero-free region that embraces the previous ones as special cases in the following way.

The formula

$$(3.91) \qquad \Big[\bar{w}_1 w_2\Big]_{z=z_2} - \Big[\bar{w}_1 w_2\Big]_{z=z_1} - \int_{z_1}^{z_2} |w_2|^2 \overline{d\,\mathbf{K}} + \int_{z_1}^{z_2} |w_1|^2 d\,\mathbf{\Gamma} = 0$$

contains four terms. The first term will evidently not vanish if the sum of the other three terms is different from zero. This sum will certainly not be zero if the argument of each of the terms in the sum is known to lie in one and the same angle of opening less than π. The argument of the second term in (3.91) depends essentially upon the initial conditions. We have

$$(3.92) \qquad \arg\,(\bar{w}_1 w_2) = \arg\,(dw/dz)/w + \arg K(z)$$

* This article was added at the revision of the paper in November, 1921.

where $(dw/dz)/w$ at $z = z_1$ is a known number λ which may be 0 or ∞. Put

$$(3.93)\qquad \begin{aligned} \arg\lambda + \arg K(z_1) + \pi &= \iota_1, \\ \pi - \arg d\mathbf{K} &= \kappa_z, \\ \arg d\mathbf{\Gamma} &= \gamma_z. \end{aligned}$$

Given an angle θ such that

$$(3.931)\qquad \theta < \iota_1 < \theta + \pi,$$

we can always choose the path of integration, starting from $z = z_1$, so that

$$(3.932)\qquad \begin{aligned} \theta < \kappa_z < \theta + \pi, \\ \theta < \gamma_z < \theta + \pi, \end{aligned}$$

at all points z on the path C, say. Then it is obvious that the arguments of the third and the fourth terms in (3.91) fulfill the same condition as κ_z and γ_z do. Consequently the sum of the three latter terms in (3.91) is different from zero for this particular choice of the path of integration and

$$(3\ 94)\qquad [\overline{w}_1 w_2]_{z = z_2} \neq 0$$

for any point $z_2(\neq z_1)$ on C. We have of course tacitly assumed that the curve C does not pass through any of the singular points of the differential equation, an assumption that will be made throughout the present section.

To a given angle θ corresponds a set of paths of this nature which we call *the lines of influence with regard to the point* z_1 *and the solution* $w(z|z_1, \lambda)$ where z_1 and λ in an obvious manner denote the initial conditions. Giving θ all possible values, consistent with (3.931), we obtain all lines of influence belonging to $z = z_1$ and the solution in question. These lines we imagine to be traced on the Riemann surface mentioned above in section 3.6. The set of regular points on the surface which can be reached by lines of influence from the point $z = z_1$ forms a zero-free region, *the domain of influence of* z_1 *and* $w(z|z_1, \lambda)$ which we denote by $DI(z_1|\lambda)$.

It is evident that we obtain the most extensive domain for a given point z_1 when $w_1 w_2 = 0$ at $z = z_1$, i.e., when λ is 0 or ∞, there being no restriction (3.931) in that case on θ; and the regions we obtain when $\lambda \neq 0$ or ∞ are interior parts of $DI(z_1|0)$. Further we can verify that the standard domain and the star of the point z_1 are both contained in $DI(z_1|0)$.

In a similar manner we can define the domain of influence with regard to a curve and a solution.*

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* For further developments of these ideas see the author's paper, *On the zeros of Sturm-Liouville functions*, Arkiv för Matematik, Astronomi och Fysik, vol. 16, No. (1922), 17.

4. The asymptotic distribution of the zeros*

4.1. **Reduction to the normal form.** In order to investigate the distribution of the zeros of a solution of a given differential equation in the neighborhood of an irregular singular point we use the transformation in formula (2.26). We have

$$\frac{d}{dz}\left[K(z)\frac{dw}{dz}\right] + G(z)w = 0. \tag{4.11}$$

By putting

$$Z(z|z_0) = \int_{z_0}^{z} \sqrt{\frac{G(z)}{K(z)}}\,dz, \tag{4.12}$$

we obtain

$$\frac{d}{dZ}\left[S(Z)\frac{dw}{dZ}\right] + S(Z)w = 0, \tag{4.13}$$

where

$$S(Z) = \sqrt{G(z)\,K(z)}. \tag{4.131}$$

As a parallel form of (4.13) we also use

$$\frac{d^2w}{dZ^2} + F(Z)\frac{dw}{dZ} + w = 0, \tag{4.14}$$

where

$$F(Z) = \frac{d}{dZ}\,[\log S(Z)]. \tag{4.141}$$

Let us further introduce a new independent variable

$$W = \sqrt{S(Z)}w. \tag{4.15}$$

The differential equation then becomes

$$\frac{d^2W}{dZ^2} + [1 - \phi(Z)]\,W = 0, \tag{4.16}$$

where

$$\phi(Z) = \frac{1}{2}\frac{dF}{dZ} + \frac{1}{4}\,[F(Z)]^2. \tag{4.161}$$

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* This section was rewritten at the revision of the paper. For the results compare with the memoir of P. Boutroux, *Recherches sur les transcendants de M. Painlevé et l'étude asymptotique des équations différentielles du second ordre,* Annales de l'École Normale Supérieure, (3) vol. 30 (1913), pp. 255–375, and (3), vol. 31 (1914), pp. 99–159, especially §§2–6. Cf. further R. Garnier: *Sur les singularités irrégulières des équations différentielles linéaires,* Journal de Liouville, (8) vol. 2 (1919), pp. 99–200, especially §28.

The function $Z(z|z_0)$ depends upon an arbitrary parameter z_0 and, the value of z_0 being fixed, Z is in general infinitely-many-valued. We assume that we can find a determination of Z such that the function $\phi(Z)$ is single-valued and analytic in a region Δ of the Z-plane, extending to infinity and having the following properties:

CONDITION **A**

(1) Δ is simply-connected and smooth;

(2) Every line parallel to the real axis cuts the boundary $\boldsymbol{\Gamma}$ of Δ either (i) in a line-segment, or (ii) in a point, or (iii) not at all;

(3) Δ lies entirely in a sector

$$-\pi + \delta < \arg Z < \pi - \delta; \quad |Z| \geqq R_0 > 0.$$

A region Δ that fulfils a condition **A** is said to be of type **A.** In the case when $\boldsymbol{\Gamma}$ is actually cut by every parallel of the real axis we say the region is of type **Aa**, otherwise of type **Ab**. From the condition **A** it follows that Δ contains a strip Δ_0 given by some equality like

$$\Re(Z) \geqq A > R_0; \quad B_2 \geqq \Im\mathrm{m}(Z) \geqq B_1. \tag{4.17}$$

Furthermore we assume that $\phi(Z)$ in Δ satisfies

CONDITION **B.**

$$|\phi(Z)| < \frac{M}{|Z|^{1+\nu}}$$

where ν and M are positive constants.

4.2. **An integral equation.** The method of successive approximations shows that the general solution of (4.15) is bounded in any strip Δ_0 of *finite width*. Consequently the expression

$$f(Z) = W_0(Z) + \int_Z^\infty \sin(T - Z)\,\phi(T)\,W(T)\,dT \tag{4.21}$$

has a definite meaning and represents an analytic function in Δ_0. Here $W_0(Z)$ is a solution of

$$W_0'' + W_0 = 0, \tag{4.22}$$

$W(Z)$ is a solution of (4.15) and the path of integration is a straight line parallel to the axis of reals. Moreover we find that

$$f''(Z) + f(Z) = \phi(Z)W(Z). \tag{4.23}$$

Consequently if $f(Z)$ is a solution of the integral equation.

$$f(Z) = W_0(Z) + \int_Z^\infty \sin(T - Z)\phi(T) f(T)\, dT, \tag{4.24}$$

then $f(Z)$ is also a solution of (4.15). The equation (4.24) is a singular integral equation of Volterra's type which we can solve in the following way. Putting

$$K_1(Z,T) = \sin(T - Z)\,\phi(T), \tag{4.25}$$

we define the iterated kernals

$$\begin{aligned} K_2(Z, T) &= \int_T^\infty K_1(Z, U)K_1(U,T)\, dU, \\ &\cdots\cdots\cdots \\ K_n(Z, T) &= \int_T^\infty K_1(Z, U)K_{n-1}(U, T)\, dU, \\ &\cdots\cdots\cdots \end{aligned} \tag{4.251}$$

and further

$$E_n(Z) = \int_Z^\infty K_n(Z, T)\, W_0(T)\, dT. \tag{4.252}$$

Then

$$W(Z) = W_0(Z) + \sum_{n=1}^\infty E_n(Z) \tag{4.253}$$

formally satisfies (4.24). If Δ_0 is a strip of finite width and T and Z are points in Δ_0, we have

$$\begin{aligned} &|W_0(Z)| \leqq K, \\ &|\sin(T - Z)| \leqq K, \\ &|K_1(Z, T)| \leqq \frac{KM}{|T|^{\nu+1}}, \end{aligned} \tag{4.26}$$

and consequently

$$|E_n(Z)| < \frac{K}{n!}\left(\frac{KM}{\nu}\right)^n \left|\frac{1}{Z}\right|^{n\nu}. \tag{4.261}$$

Hence the series in (4.253) is absolutely and uniformly convergent in Δ_0 and represents an analytic function of Z in that region, which function is furthermore a solution of the integral equation (4.24) and consequently also of the differential equation (4.15).

We know in advance that $W(Z)$ is bounded in Δ_0 and we find easily that

(4.27) $$|W(Z)| < 2K,$$

when

(4.271) $$|Z| > \left[\frac{2M}{\nu}\right]^{\frac{1}{\nu}} \text{ in } \Delta_0.$$

Using formula (4.24) we can improve this result and obtain a second approximation, namely

(4.272) $$|W(Z) - W_0(Z)| < \frac{2K^2M}{\nu|Z|^\nu},$$

when Z is subject to the condition in formula (4.271).

This shows that $W(Z)$ approaches $W_0(Z)$ indefinitely in Δ_0, or, using the terminology of Boutroux,* the function $W(Z)$ is asymptotic to $W_0(Z)$ in Δ_0.

Similarly we find

(4.273) $$|W'(Z) - W_0'(Z)| < \frac{2K^2M}{\nu|Z|^\nu}$$

when Z satisfies (4.271); this shows that $W'(Z)$ is asymptotic to $W_0'(Z)$ in Δ_0.

4.3. **Oscillatory solutions and truncated ones.** By formula (4.253) we obtain to each solution $W_0(Z)$ of (4.22) a solution $W(Z)$ of the given differential equation (4.15) and $W(Z)$ is asymptotic to $W_0(Z)$. Taking a pair of linearly independent solutions $W_{01}(Z)$ and $W_{02}(Z)$ of (4.22) we obtain a pair of linearly independent solutions $W_1(Z)$ and $W_2(Z)$ of (4.15). In particular we choose

(4.31) $$W_{01}(Z) = e^{iZ}; \quad W_{02}(Z) = e^{-iZ}.$$

The corresponding solutions $W_1(Z)$ and $W_2(Z)$ are asymptotic to e^{iZ} and e^{-iZ}, respectively, and consequently admit of no zeros in Δ_0 beyond a certain limit. The same is true of their derivatives. Even in the full region Δ these solutions do not admit of any zeros outside of a sufficiently large circle. We notice from formula (4.24) that

(4.311) $$e^{-iZ}W_1(Z) = 1 + \frac{\theta_1(Z)}{Z^\nu}, \qquad e^{iZ}\,W_2(Z) = 1 + \frac{\theta_2(Z)}{Z^\nu},$$

where $|\theta_k(Z)| < C$, a certain constant, when $|Z| > R$ in Δ, which proves the assertion. Using a term coined by Boutroux† we call these integrals *truncated in* Δ.

* Loc. cit., pp. 270–273.

† Loc. cit., pp. 261–263.

If the region Δ is of type **Aa** (vide supra) these two integrals are the only truncated ones in Δ. In fact, every solution of (4.15) can be represented in the form

$$W(Z)=c_1W_1(Z)+c_2W_2(Z),\quad c_1c_2\neq 0 \tag{4.32}$$

which is asymptotic to $W_0(Z)=c_1e^{iz}+c_2e^{-iz}$ in Δ. This region being of type **Aa** we can always choose the strip Δ_0, $B_1<\mathfrak{Im}(Z)<B_2$, in such a fashion that it contains the zeros of $W_0(Z)$, $\alpha_n=\alpha+n\pi$, from a certain $n=N_0$ on. As $c_1c_2\neq 0$ we can assume $W_0(Z)=\sin(Z-\alpha)$ without any loss of generality. Let us further denote the parts of Δ which lie above and below Δ_0 respectively by Δ_1 and Δ_{-1}. In these regions

$$W(Z)=\sin(Z-\alpha)+\frac{\theta_0(Z)}{Z^\nu},\ Z \text{ in } \Delta_0; \tag{4.330}$$

$$e^{+iZ}W(Z)=e^{+iZ}\sin(Z-\alpha)+\frac{\theta_1(Z)}{Z^\nu},\ Z \text{ in } \Delta_1; \tag{4.331}$$

$$e^{-iZ}W(Z)=e^{-iZ}\sin(Z-\alpha)+\frac{\theta_{-1}(Z)}{Z^\nu},\ Z \text{ in } \Delta_{-1}; \tag{4.332}$$

Here

$$|\theta_\lambda(Z)|<2K_\lambda^2M/\nu\quad(\lambda=-1,0,+1) \tag{4.333}$$

when

$$|Z|\geqq[2M/\nu]^{\frac{1}{\nu}} \tag{4.334}$$

in Δ_λ, where K_λ denotes the maximum of $|e^{\lambda iZ}\sin(Z-\alpha)|$ in Δ_λ.

Let us mark the points α_n in Δ_0 and surround each of them by a small circle Γ_n, $|Z-\alpha_n|=\epsilon$. On the circle Γ_n we have $|\sin(Z-\alpha)|>\frac{2}{\pi}\epsilon$. If we choose

$$|Z|\geqq R=[\pi K_0^2M/\nu\epsilon]^{\frac{1}{\nu}}, \tag{4.34}$$

we have $|\theta_0(Z)/Z^\nu|\leqq\frac{2}{\pi}\epsilon$. Hence the first term on the right hand side in (4.330) dominates the second one when Z is a point on a circle Γ_n in the part of Δ that lies outside of the circle Γ_R defined by (4.34). We can assume without considerable loss of generality that Γ_R cuts the boundary Γ of Δ in two points only. Then Γ_R divides Δ in two parts, one interior $\Delta-$ and one exterior $\Delta+$. We denote by $\Delta*$ what is left of $\Delta+$ after punching out the interior of the circles Γ_n.

Then, $W(Z)$ *has one zero and only one in each of the circles* Γ_n *in* $\Delta+$ *and no zeros at all in* $\Delta*$. The first part of the statement is a consequence of a well known theorem due to Rouché.* The latter part of the assertion depends upon the fact that $\sin(Z-\alpha)$ is the majorating term in the expression for $W(Z)$ in Δ_*, provided ϵ is small enough. In the same way we find that the zeros of $W'(Z)$ in $\Delta+$ all lie in the circles Γ'_n, $|Z-\alpha'_n|=\epsilon$ where $\alpha'_n=\alpha_n+\frac{\pi}{2}$, namely one in each circle.

Thus in the case of a region Δ of type **Aa** the zeros of $W(Z)$ are approximated by the zeros of a certain sine-function, $C\sin(Z-\alpha)$, to which $W(Z)$ is asymptotic, and the zeros of $W'(Z)$ are approximated by the zeros of $C\cos(Z-\alpha)$. Let the zeros of $W(Z)$ in Δ be denoted by A_n and those of $W'(Z)$ by A'_n where the notation is so chosen that

$$\lim(A_n-\alpha_n)=0 \text{ and } \lim(A'_n-\alpha'_n)=0.$$

Then we say that the points A_n form a *string of zeros in* Δ and similarly the points A'_n.† There are only two linearly independent solutions which do not have a string of zeros in Δ, namely the truncated solutions defined above.

In the case of a region Δ of type **Ab,** however, we have infinitely many independent solutions which are truncated in Δ, namely all for which the zeros of the asymptote-function $W_0(Z)$ lie outside of Δ. On the other hand, given a point set $\alpha_n=\alpha+n\pi$ which lies in Δ from a certain n on, we can always find a solution $W(Z)$ of (4.15) which is asymptotic to $\sin(Z-\alpha)$ in Δ and the zeros of which form a string in Δ, approximated by (α_n). This solution is given by our previous formulas (4.25) with $W_0(Z)=\sin(Z-\alpha)$.

In order to emphasize the dependence of the solution $W(Z)$ on the parameter α we denote it by $W(Z,\alpha)$. The zeros of $W(Z,\alpha)$ in Δ are functions of α which justifies the writing of $A_n(\alpha)$ instead of simply A_n. Let us see how $A_n(\alpha)$ varies with α. We put $\alpha=\sigma+i\tau$ and let σ increase from σ_0 to $\sigma_0+\pi$, keeping τ constant $=\tau_0$. Then $A_n(\alpha)$ describes a certain curve from $A_n(\alpha_0)$ to $A_n(\alpha_0+\pi)=A_{n+1}(\alpha_0)$, which shows the relationship between the functions $A_n(\alpha)$. The path followed by $A_n(\alpha)$ is almost a straight line from a point close to $\alpha_n=\alpha_0+n\pi$ to a point close to α_{n+1}. If σ continues to increase, $A_n(\alpha)$ will describe a certain curve $\Sigma(\tau_0)$ that joins the different zeros in the string, belonging to $W(Z,\alpha_0)$. We call this curve a *zero-curve* (of solutions) of the differential equation (4.15). The curve is evidently asymptotic to the line $\mathfrak{Im}(Z)=\tau_0$ and is uniquely characterized by its asymptote. Through every point in $\Delta+$ there passes one and only one zero-curve.

4.4. **Varied conditions.** In the preceding articles we have assumed $\phi(Z)$ to

* Rouché, *Mémoire sur la série de Lagrange,* Journal de l'École Polytechnique, cahier 39 (1862), pp. 217–218.

† Boutroux and Garnier both use the term *ligne de zéros* for what we denote by *string of zeros.*

be analytic in an infinite region in the right half-plane. The case when $\phi(Z)$ is analytic and fulfils a condition **B** in a region Λ in the left half-plane of type **A** (with suitable change of condition **A**(3)) can be treated in a similar manner. We find that the general solution of (4.15) is asymptotic to some sine-function $W_0(Z)$ in Λ and there are solutions the zeros of which form a string asymptotic to a given point set $\alpha_{-n} = \alpha - n\pi$ in Λ.

A special case of large importance is when $\phi(Z)$ is analytic and fulfils a condition **B** in an upper half-plane Υ, defined by $\mathfrak{Im}(Z) > B_2$. In this case every solution is asymptotic to some sine-function $W_{0\alpha}(Z)$ in Υ_+, the extreme right part of Υ, and asymptotic to some other sine-function $W_{0\beta}(Z)$ in Υ_-, the extreme left part of Υ. Thus the solution may be truncated in Υ_+ or in Υ_- or in both of these regions. If the solution in question is oscillatory in Υ_+ then the zeros form one and only one string which is approximated by a point set (α_n). And if the solution is oscillatory in Υ_- then the zeros form a single string that approaches a point set (β_{-n}) indefinitely.

On the other hand given a sine-function $W_{0\alpha}(Z)$ there exists a solution $W_+(Z)$ of (4.15) that is asymptotic to $W_{0\alpha}$ in Υ_+, and there is another solution $W_-(Z)$ that is asymptotic to $W_{0\alpha}$ in Υ_-. But if $\tau = \mathfrak{Im}(\alpha)$ is a very large number then there exists a solution $W_\alpha(Z)$ that is asymptotic to $\sin(Z - \alpha)$ throughout Υ. The zeros of $W_\alpha(Z)$ are approximated by $\alpha \pm n\pi$ and consequently the two strings in Υ_+ and in Υ_- join into one doubly unlimited string. There are evidently no zeros above this string and only a finite number below it in Υ.

We can of course treat a lower half-plane in a similar manner.

4.5. **The distribution in the z-plane.** Let us now try to pass back to the z-plane. We will be chiefly concerned with the case of a region Δ in the right half-plane. The function

$$Z = \int_{z_0}^{z} \sqrt{\frac{G(z)}{K(z)}}\, dz \tag{4.51}$$

establishes a one-to-one and conformal correspondence between the points of the interior of the region Δ in the Z-plane and the interior points of a certain region D of the z-plane. This latter region is of course simply connected but may overlap itself infinitely often. In the case of overlapping we consider D as part of a Riemann surface $\mathfrak{D}$ over the z-plane on which we proceed to study the integrals $w(z)$. These solutions are of course single-valued and analytic in the interior of D (on $\mathfrak{D}$) which does not contain any of the singular points of the differential equation (4.11). There is however at least one singular point on the boundary of D corresponding to $Z = \infty$.

By this conformal mapping, the zero-curves $\Sigma(\tau)$ of the differential equation (4.15) are transformed into a set of curves $S(t)$ in D. The points α_n are carried over into points a_n in D; similarly the points A_n into points A_n. The circles Γ_n go over into closed contours C_n around a_n. Finally the circular arc Γ_R in Δ is carried over into a curve C_R in D which separates this region into two parts D_+ and D_- corresponding to Δ_+ and Δ_- respectively. The points of D_+ that remain after leaving out the interior of the closed contours C_n we denote by D_*.

To the solution $W(Z,\alpha)$ of (4.15) corresponds a solution $w(z,a)$ of (4.11) by the relation

$$w(z, a) = [S(Z)]^{-\frac{1}{2}}W(Z, \alpha). \tag{4.52}$$

If α_n lies in Δ from a certain $n = N_0$ on, then the corresponding points a_n will finally lie in D. The solution $w(z,a)$ will have precisely one zero within each of the curves C_n in D_+, namely at A_n, and there will be no zeros at all in D_*. If D corresponds to a region Δ of type **Aa** there are two and only two linearly independent solutions which are truncated in D, namely those corresponding to $W_1(Z)$ and $W_2(Z)$, and every other solution oscillates infinitely often in D_+ When the corresponding region Δ is of type **Ab,** however, there are infinitely many solutions which are truncated in D.

The zero-curves of the equation (4.11) as far as the region D is concerned are precisely the curves S mentioned above. These curves are asymptotic to the curves $\mathfrak{S}$ satisfying the differential equation

$$\mathfrak{Im}\left\{\sqrt{\frac{G(z)}{K(z)}}dz\right\} = 0. \tag{4.53}$$

The integral curves of this equation we shall call the *asymptotic zero-curves* of (4.11).

Through every point a in D (on $\mathfrak{D}$) passes one curve $\mathfrak{S}$ and only one. Follow this curve $\mathfrak{S}(a)$ in the direction of increasing values of $\mathfrak{R}(Z)$ and mark the points $a_1, a_2, \ldots, a_n, \ldots$ defined by the equalities

$$n\pi = \mathfrak{R}\left\{\int_a^{a_n}\sqrt{\frac{G(z)}{K(z)}}\,dz\right\} \qquad (n=1, 2, \ldots), \tag{4.54}$$

the integral taken along $\mathfrak{S}(a)$.

By our previous investigation we know that there is a definite solution $w(z,a)$ of (4.11) having the points a_n as limiting points of its zeros in the sense that the actual zeros A_n can be so numbered that $\lim (A_n - a_n) = 0$. Let us draw two circles Γ_{R_1} and Γ_{R_2} in the Z-plane with radii $R_2 > R_1 > R$. Suppose for the sake of simplicity that these circles cut Γ, the boundary of Δ, each in two points only. The maps in the z-plane of these circular arcs are two curves C_{R_1} and C_{R_2} in D which together with the boundary of D enclose a curvilinear quadrilateral

$[D]$ on $\mathfrak{D}$. This region is cut in two points only by $\mathfrak{S}$ (a), z_1 on C_{R_1} and z_2 on C_{R_2} say. Then the number of zeros of $w(z, a)$ in $[D]$ is given by

$$(4.55) \qquad N[D] = \frac{1}{\pi}\Re\left\{\int_{z_1}^{z_2}\sqrt{\frac{G(z)}{K(z)}}\,dz\right\} + \theta,^{*}$$

where the integral is taken along $\mathfrak{S}$ (a) and $-1 \leqq \theta \leqq +1$.

Let us now consider the zeros of $w'(z,a)$. We have

$$(4.56) \qquad \frac{dw}{dz} = \left[\frac{dW}{dZ} - \frac{1}{2}\,F(Z)W(Z)\right]\left[S(Z)\right]^{-\frac{1}{2}}\frac{dZ}{dz}.$$

The second and the third factors do not concern us. We note that $W(Z)$ is bounded in Δ_0 and assume that

$$(4.561) \qquad |F(Z)| < \frac{M}{|Z|^{\mu}} \qquad (\mu > 0).^{\dagger}$$

Further, the function dW/dZ vanishes once in each of the circles Γ'_n and not at all in Δ'_*. Using the theorem of Rouché, quoted above, we conclude that the function $dW/dZ - \frac{1}{2}\,F(Z)W(Z)$ vanishes once and only once in each of these circles and is different from zero in every finite point of Δ_*, provided the quantity ϵ which enters in formula (4.34) is small enough. Hence dW/dZ vanishes once and only once in each of the corresponding closed contours C'_n in $D+$ and not at all in D'_*.

Just a few words about regions Λ and Υ considered in section 4.4. A region Λ of type **A** in the Z-plane corresponds to a region L in the z-plane. The zeros form a simple string just as in the case of a region D. A region Υ corresponds to a region U in the z-plane; this region consists of three parts $U+$, $U-$ and U° corresponding to $\Upsilon+$, $\Upsilon-$ and the intermediate portion of Υ respectively. The zeros form simple strings in $U+$ and $U-$ (one string in each at most); in certain cases these strings may join to one doubly unlimited string.

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* Compare the result in the real case given by Wiman, loc. cit., p. 7. Professor Wiman has informed me that he has used the more or less classical transformation in formula (4.12) in his lectures at the University of Upsala for studying the properties of solutions of linear differential equations of the second order in the complex plane. The general character of the distribution of the zeros in the neighbourhood of an irregular singular point has been known to him for some years.

† The function $F(Z)$ satisfies a Riccati equation (4.161). Using this equation it can be proved that $F(Z)$ will satisfy (4.561) in Δ with $\mu = 1$ or ν where ν is the exponent entering in condition **B,** provided $\nu \geqq 1$. When $0 < \nu < 1$ the general solution of (4.161) may admit strings of poles in Δ. Leaving out properly chosen regions around these poles, condition (4.561) is still satisfied.

4.6. **Polynomial coefficients.*** Let us assume that $K(z)$ and $G(z)$ are polynomials

(4.61) $$K(z) = z^k + \cdots; \quad G(z) = g_0 z^g + \cdots .$$

Infinity is an irregular singular point if $g - k \geqq -1$, which is the only interesting case in this connection.

In this case $Z(z|z_0)$ is in general an abelian integral of the third kind. In the neighborhood of infinity we have, putting $g - k + 2 = m$,

(4.62) $$Z = \frac{2}{m}\sqrt{g_0}\, z^{\frac{m}{2}}\left\{1 + \sum_1^{\infty}{}' \frac{c_n}{z^n} + c_{m/2}\frac{\log z}{z^{m/2}}\right\},$$

where the logarithmic term occurs only when m is an even number. Conversely we find

(4.621) $$z = \left[\frac{m}{2\sqrt{g_0}} Z\right]^{\frac{2}{m}}\left\{1 + \sum_{1,1}^{\infty} C_{nr} Z^{-\frac{2}{m}(n+r)} (\log Z)^r\right\},$$

the series being convergent in any sector in the neighborhood of infinity. Further we have

(4.63) $$F(Z) = \frac{G(z)K'(z) + G'(z)K(z)}{[G(z)]^{\frac{3}{2}}\,[K(z)]^{\frac{1}{2}}}.$$

Expanding in ascending powers of z we obtain

(4.631) $$F(Z) = \frac{g+k}{2\sqrt{g_0}}\, z^{-\frac{m}{2}}\left\{1 + \sum_1^{\infty}\frac{a_n}{z^n}\right\}.$$

Hence, substituting the expansion for z from formula (4.631)

(4.632) $$F(Z) = \frac{g+k}{m}\frac{1}{Z}\left\{1 + \begin{matrix}\text{double series of the same}\\ \text{type as in formula (4.621)}\end{matrix}\right\}.$$

The expansion for $\phi(Z)$ is of similar form but starts with a factor

$$[(3k - g + 4)/4m^2](1/Z^2).$$

Thus $\phi(Z)$ satisfies a condition **B** with $\nu = 1$ in any region outside of a sufficiently large circle in which the argument of Z is bounded. Let us take as our region Δ the part of the Z-plane in which $|Z| > R$ and $\Re(Z) \geqq 0$. If

* Cf. J. Horn, *Über die irregulären Integrale der linearen Differentialgleichungen zweiter Ordnung*, Acta Mathematica, vol. 23 (1900), pp. 171–201, especially p. 198.

R is sufficiently large this region is conformally mapped by $z(Z|Z_0)$ on a smooth sectorial region $D^{(\mu)}$ in the z-plane in which

$$(4.64)\qquad \frac{1}{m}\Big[(2\mu-1)\pi-\theta_0\Big]-\delta < \arg z < \frac{1}{m}\Big[(2\mu+1)\pi-\theta_0\Big]+\delta$$

where δ is a small number, $\theta_0 = \arg g_0$ and $\mu = 0, 1, \ldots, m-1$ depending upon which determination of $\sqrt[m]{Z}$ we use in formula (4.621). The determination of the logarithm does not affect the form of the region appreciably.

The asymptotic zero-curves in the z-plane are given by

$$(4.65)\qquad r^{\frac{m}{2}} \sin \frac{1}{2}(m\theta + \theta_0) + \cdots = \text{const.} \qquad (z = r e^{i\theta})$$

showing the leading term. They form m pencils in the neighborhood of infinity which point they approach in the directions

$$(4.651)\qquad \theta^{(\mu)} = \frac{1}{m}\Big[2\mu\pi-\theta_0\Big] \qquad (\mu=0, 1, \ldots, m-1).$$

In general $w(z|a)$ is not single-valued in the neighborhood of infinity. Let $\mathfrak{D}$ be the part of the Riemann surface of $\log z$ which lies outside of a large circle. Then the solution is single-valued on $\mathfrak{D}$. The zeros of $w(z|a)$ are seen to form m strings asymptotic to the directions $\theta^{(\mu)}$ in *each* leaf of $\mathfrak{D}$. In particular if the solution is uniform in the neighborhood of infinity we find m strings in all.

If $N(r)$ denotes the number of zeros in a string within the circle $|z| = r$ and we put

$$(4.66)\qquad n(r) = \frac{2}{m\pi}\sqrt{|g_0|}\, r^{\frac{m}{2}}$$

then $N(r)/n(r) \longrightarrow 1$ when $r \longrightarrow \infty$.

Let us take up the question of truncated solutions. By the results of section 4.3 we know that there are exactly two solutions which are truncated in one of the directions $\theta^{(\mu)}$. Hence the total number of truncated solutions is at most equal to $2m$. This number, however, actually reduces to m as we shall see.

Let us take a modified region $\widetilde{\Delta}$ bounded by a large circular arc $|Z| = R$, $\frac{\pi}{2} - \delta \geqq \arg Z \geqq -\frac{\pi}{2} + \delta$ (δ a small fixed positive quantity) and the tangents at the end-points of this arc, extended to infinity in the left half-plane. This is a region of type **A** in which $\phi(Z)$ satisfies a condition **B**. The two solutions, $W_1(Z)$ and $W_2(Z)$, truncated in $\widetilde{\Delta}$ are asymptotic to e^{iZ} and e^{-iZ}, respectively, in this region. Moreover they keep their asymptotic form in a wider region, $W_1(Z)$ being asymptotic to e^{iZ} when $-\pi < \arg Z < 2\pi$ and $W_2(Z)$ being asymptotic to e^{-iZ} when $-2\pi < \arg Z < \pi$, as can be proved by considering a region $\overset{\frown}{\Lambda}$ symmetric to $\overset{\frown}{\Delta}$. Hence $W_1(Z) \longrightarrow 0$ in the upper half of $\overset{\frown}{\Delta}$ and $W_2(Z) \longrightarrow 0$ in the lower half of $\overset{\frown}{\Delta}$ provided $|\mathfrak{Im}(Z)| \longrightarrow \infty$. These solutions are furthermore uniquely determined by this property, as the solution of the integral equation (4.24) is unique.

The region $\overset{\frown}{\Delta}$ corresponds to m different regions $\overset{\frown}{D}^{(\mu)}$ in the z-plane in which

$$\theta^{(\mu-1)} + \epsilon \leqq \arg z \leqq \theta^{(\mu+1)} - \epsilon \qquad (\mu = 0, 1, \ldots, m-1). \tag{4.67}$$

These regions $\overset{\frown}{D}^{(\mu)}$ of course have common parts; in fact, $\overset{\frown}{D}^{(\mu)}$ has a part $U^{(\mu)}$, $\theta^{(\mu)} + \epsilon \leqq \arg z \leqq \theta^{(\mu+1)} - \epsilon$, in common with $\overset{\frown}{D}^{(\mu+1)}$ and so on. By formula (4.52) we conclude that there is one solution truncated in $\overset{\frown}{D}^{(\mu)}$ which tends towards zero in $U^{(\mu)}$ and a second solution truncated in $\overset{\frown}{D}^{(\mu+1)}$ which approaches zero in the same region $U^{(\mu)}$. There is, however, only one solution which tends towards 0 in $U^{(\mu)}$. Thus we find the number of truncated solutions reduces to m; these solutions are uniquely characterized by their asymptotic properties: the solution $w_\mu(z)$ which approaches zero when $z \longrightarrow \infty$ in $U^{(\mu)}$ is truncated in the adjacent directions $\theta^{(\mu)}$ and $\theta^{(\mu+1)}$. It is worth while observing that the truncated solutions preserve the same asymptotic representation in three adjacent regions $U^{(\kappa)}$, namely $U^{(\mu-1)}$, $U^{(\mu)}$ and $U^{(\mu+1)}$ in the case of $w_\mu(z)$. This number m of truncated solutions is actually reached as is shown by considering the differential equations $w'' + z^g w = 0$.

Let us take the case when $K(z) \equiv 1$. Then the general solution of the differential equation which is an entire function of z can be represented in the form

$$w_\lambda(z) = w_1(z) - \lambda w_2(z), \tag{4.68}$$

where $w_1(z)$ and $w_2(z)$ are linearly independent solutions and λ is a complex parameter. When is this solution truncated? The answer is: λ *has to be one of the asymptotic values of the meromorphic function* $\lambda(z) = w_1(z)/w_2(z)$, or, in other words, λ *has to be a transcendental singular point of the inverse function* $z(\lambda)$. A quantity a is said to be an asymptotic value of an entire or meromorphic function $f(z)$ if there is a path L (a Jordan curve) tending to infinity along which $f(z)$ tends towards a. The condition is necessary because, if $w_\lambda(z)$ is a truncated solution, then we can find paths L along which $w_\lambda(z)/w_2(z) \longrightarrow 0$. Hence $w_1(z)/w_2(z) \longrightarrow \lambda$ along L; thus λ is an asymptotic value of $\lambda(z)$. The condition is also sufficient. If λ is an asymptotic value, $w_\lambda(z)/w_2(z) \longrightarrow 0$ along the corresponding path L. But

$$w_\lambda(z)/w_2(z) = W_\lambda(Z)/W_2(Z) \sim W_{0\lambda}(Z)/W_{02}(Z), \tag{4.69}$$

with obvious notation. This shows that the path can be taken outside of the sectors $\theta^{(\mu)} - \delta < \arg z < \theta^{(\mu)} + \delta$ and, moreover, that the limit is zero only if $W_{0\lambda}(Z)$ is either e^{iZ} or e^{-iZ}. But then $W_\lambda(Z)$ is truncated in $\tilde{\Delta}$ and consequently $w_\lambda(z)$ is truncated in a corresponding region $\tilde{D}^{(\mu)}$.

This result shows further that every function of the meromorphic family $[c_1w_1(z) + c_2w_2(z)]/[c_3w_1(z) + c_4w_2(z)]$ has at most m asymptotic values.

4.7. **Concluding remarks.** The discussion in this section has given us information on the distribution of the zeros in the neighborhood of an irregular singular point, in as much as we have means by which to determine the number of strings of zeros and their directions, and furthermore to find the restrictions imposed on the absolute value of the difference of two consecutive zeros in a string. This is often enough to give a fairly good picture of the general character of the distribution. The results in the third section, on the other hand, give information about zero-free regions around a given point or curve, and thus about the restrictions on the argument of the difference between two zeros. Hence the two methods are complementary.

The reader may find it instructive to work out for himself the distribution problems for the differential equation

$$w'' - \frac{1}{z}w = 0, \tag{4.71}$$

by combining the two methods. We find $Z = 2i\sqrt{z}$ and the transformed normal form is

$$W'' + \left(1 - \frac{3}{4Z^2}\right)W = 0. \tag{4.72}$$

The asymptotic zero-curves of (4.71) are parabolas with focus at the origin and the negative real axis as axis. The distribution of the zeros in the vicinity of the asymptotic parabola is very regular.

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