MACLAURIN EXPANSION OF THE INTERPOLATION POLYNOMIAL DETERMINED BY $2n+1$ EVENLY SPACED POINTS*

BY

GEORGE RUTLEDGE

1. Introduction. In a recent paper† by the writer, the polynomial determined by the $2n+1$ points

$$(1)\qquad (-nh, y_{-n}), \ldots, (-h, y_{-1}),\ (0, y_0),\ (h, y_1), \ldots, (nh, y_n)$$

is written down in terms of the determinant

$$(2)\qquad D = \begin{vmatrix} 1 & 2^2 & 3^2 & \cdots & n^2 \\ 1 & 2^4 & 3^4 & \cdots & n^4 \\ 1 & 2^6 & 3^6 & \cdots & n^6 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 & 2^{2n} & 3^{2n} & \cdots & n^{2n} \end{vmatrix} = n!\,3!\,5!\cdots(2n-1)!$$

---

* Presented to the Society, February 24, 1923.

† Journal of Mathematics and Physics of the Massachusetts Institute of Technology, vol. 2 (1922), p. 47. In this paper the cofactors A_{ij} are explicitly expressed as follows:

$$\frac{A_{ij}}{2D} = (-1)^{i+j}\frac{\binom{2n}{n-j}}{(2n)!}\sum^{(j)} r_1^2 r_2^2 \cdots r_{n-i}^2,$$

where $\sum^{(j)} r_1^2 r_2^2 \cdots r_{n-i}^2$ represents the sum of the squares of the $\binom{n-1}{n-i}$ products of the first n integers excepting j, taken $n-i$ at a time, $i \neq n$. For $i = n$, $\sum^{(j)} r_1^2 r_2^2 \cdots r_{n-i}^2$ is unity by definition.

and the cofactors A_{ij} of its elements j^{2i}, as follows:

$$
\begin{aligned}
P^{[2n]}(x) = {} & y_0 \\
& + \left\{\frac{A_{11}}{2D}(y_1 - y_{-1}) + \frac{2A_{12}}{2D}(y_2 - y_{-2}) + \cdots + \frac{nA_{1n}}{2D}(y_n - y_{-n})\right\}\frac{x}{h} \\
& + \left\{\frac{A_{11}}{2D}(y_1 + y_{-1} - 2y_0) + \frac{A_{12}}{2D}(y_2 + y_{-2} - 2y_0) + \cdots \right. \\
& \qquad\qquad \left. \cdots + \frac{A_{1n}}{2D}(y_n + y_{-n} - 2y_0)\right\}\frac{x^2}{h^2} \\
(3) \qquad & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& + \left\{\frac{A_{n1}}{2D}(y_1 - y_{-1}) + \frac{2A_{n2}}{2D}(y_2 - y_{-2}) + \cdots + \frac{nA_{nn}}{2D}(y_n - y_{-n})\right\}\frac{x^{2n-1}}{h^{2n-1}} \\
& + \left\{\frac{A_{n1}}{2D}(y_1 + y_{-1} - 2y_0) + \frac{A_{n2}}{2D}(y_2 + y_{-2} - 2y_0) + \cdots \right. \\
& \qquad\qquad \left. \cdots + \frac{A_{nn}}{2D}(y_n + y_{-n} - 2y_0)\right\}\frac{x^{2n}}{h^{2n}}.
\end{aligned}
$$

This formula is comparable with the Lagrange formula in ease of verification (independently of evaluation of the determinant D and its cofactors).

The object of the present paper is to express the coefficient of x^m in (3) in terms of mth difference quotients, thus obtaining a form of (3) which suggests Maclaurin's series as a limiting case under proper conditions.

2. **Transformation of the coefficients.** The scheme of differences to be employed is the following:

$$
(4) \qquad
\begin{array}{cccccccc}
\cdot & & & & & & & \\
y_{-3} & \cdot & & & & & & \\
 & \Delta'_{-3} & \cdot & & & & & \\
y_{-2} & & \Delta''_{-2} & \cdot & & & & \\
 & \Delta'_{-2} & & \Delta'''_{-2} & \cdot & & & \\
y_{-1} & & \Delta''_{-1} & & \Delta^{IV}_{-1} & \cdot & & \\
 & \Delta'_{-1} & & \Delta'''_{-1} & & \Delta^{V}_{-1} & \cdot & \\
y_0 & & \Delta''_0 & & \Delta^{IV}_0 & & \Delta^{VI}_0 & \cdot\,, \\
 & \Delta'_1 & & \Delta'''_1 & & \Delta^{V}_1 & \cdot & \\
y_1 & & \Delta''_1 & & \Delta^{IV}_1 & \cdot & & \\
 & \Delta'_2 & & \Delta'''_2 & \cdot & & & \\
y_2 & & \Delta''_2 & \cdot & & & & \\
 & \Delta'_3 & \cdot & & & & & \\
y_3 & \cdot & & & & & & \\
\cdot & & & & & & &
\end{array}
$$

where $y_3 - y_2 = \Delta'_3$, $\Delta'_3 - \Delta'_2 = \Delta''_2$, etc.

Between the differences of orders one and three, and between the differences of orders two and four, we have, respectively, the following relations:

$$
\begin{aligned}
&\Delta'_{-1}+\Delta'_1 = (\Delta'_{-1}+\Delta'_1),\\
&\Delta'_{-2}+\Delta'_2 = (\Delta'_{-1}+\Delta'_1)+\ (\Delta'''_{-1}+\Delta'''_1),\\
(5)\quad &\Delta'_{-3}+\Delta'_3 = (\Delta'_{-1}+\Delta'_1)+2(\Delta'''_{-1}+\Delta'''_1)+\ (\Delta'''_{-2}+\Delta'''_2),\\
&\Delta'_{-4}+\Delta'_4 = (\Delta'_{-1}+\Delta'_1)+3(\Delta'''_{-1}+\Delta'''_1)+2(\Delta'''_{-2}+\Delta'''_2)+(\Delta'''_{-3}+\Delta'''_3),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdot;
\end{aligned}
$$

$$
\begin{aligned}
&\Delta''_0 = (\Delta''_0),\\
&\Delta''_{-1}+\Delta''_1 = (2\Delta''_0)+\ (\Delta^{\mathrm{IV}}_0),\\
(6)\quad &\Delta''_{-2}+\Delta''_2 = (2\Delta''_0)+2(\Delta^{\mathrm{IV}}_0)+\ (\Delta^{\mathrm{IV}}_{-1}+\Delta^{\mathrm{IV}}_1),\\
&\Delta''_{-3}+\Delta''_3 = (2\Delta''_0)+3(\Delta^{\mathrm{IV}}_0)+2(\Delta^{\mathrm{IV}}_{-1}+\Delta^{\mathrm{IV}}_1)+(\Delta^{\mathrm{IV}}_{-2}+\Delta^{\mathrm{IV}}_2),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdot.
\end{aligned}
$$

Exactly the same relations, of course, hold between the differences of order $2i-1$ and those of order $2i+1$; and between the differences of order $2i$ and those of order $2i+2$.

We shall find it advantageous, as will at once appear, to make formal use of certain binomial coefficients, and of the notation (4), to write (3) thus:

$$
\begin{aligned}
P^{[2n]}(x) &= y_0\\
&+\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{jA_{ij}}{2D}\left[\binom{j-1}{0}(\Delta'_{-1}+\Delta'_1)+\binom{j-2}{0}(\Delta'_{-2}+\Delta'_2)+\cdots\right.\\
(7)\qquad &\qquad\qquad \left.\cdots+\binom{0}{0}(\Delta'_{-j}+\Delta'_j)\right]\frac{x^{2i-1}}{h^{2i-1}}\\
&+\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{A_{ij}}{2D}\left[\binom{j}{1}(\Delta''_0)+\binom{j-1}{1}(\Delta''_{-1}+\Delta''_1)+\cdots\right.\\
&\qquad\qquad \left.\cdots+\binom{1}{1}(\Delta''_{-j+1}+\Delta''_{j-1})\right]\frac{x^{2i}}{h^{2i}}.
\end{aligned}
$$

The coefficients of (7) are now transformed by successive application of (5) and (6) (and the generalization of these relations) until the coefficient of x^{2i-1} contains differences of order $2i-1$ and the coefficient of x^{2i} contains differences of order $2i$. *It will then appear that the differences of lower order, though formally involved, have actually been eliminated.*

By use of (5) and the identity

$$(8)\quad \binom{n}{k}\cdot 1+\binom{n-1}{k}\cdot 2+\binom{n-2}{k}\cdot 3+\cdots+\binom{k}{k}\cdot(n-k+1)=\binom{n+2}{k+2},$$

we may reduce the expression

$$(9)\quad \left[\binom{j-1}{0}(\Delta'_{-1}+\Delta'_1)+\binom{j-2}{0}(\Delta'_{-2}+\Delta'_2)+\cdots+\binom{0}{0}(\Delta'_{-j}+\Delta'_j)\right]$$

to the form

$$(10)\quad \left[\binom{j}{2}(\Delta'''_{-1}+\Delta'''_1)+\binom{j-1}{2}(\Delta'''_{-2}+\Delta'''_2)+\cdots+\binom{2}{2}(\Delta'''_{-j+1}+\Delta'''_{j-1})\right],$$

except for a term in $(\Delta'_{-1}+\Delta'_1)$ which by combination with similar terms for $j=1,2,3,\ldots,n$ is eliminated from the coefficients of $x^3, x^5, \ldots,$ as stated above, and as will presently appear.

Repetition of this process yields from (10)

$$(11)\quad \left[\binom{j+1}{4}(\Delta^{\mathrm{v}}_{-1}+\Delta^{\mathrm{v}}_1)+\binom{j}{4}(\Delta^{\mathrm{v}}_{-2}+\Delta^{\mathrm{v}}_2)+\cdots+\binom{4}{4}(\Delta^{\mathrm{v}}_{-j+2}+\Delta^{\mathrm{v}}_{j-2})\right]$$

and so on, the final expression in the sequence being

$$(12)\quad \left[\binom{2j-2}{2j-2}(\Delta^{[2j-1]}_{-1}+\Delta^{[2j-1]}_1)\right].$$

In like manner by use of (6) and the identity (8) we may reduce the expression

$$(13)\quad \left[\binom{j}{1}(\Delta''_0)+\binom{j-1}{1}(\Delta''_{-1}+\Delta''_1)+\cdots+\binom{1}{1}(\Delta''_{-j+1}+\Delta''_{j-1})\right]$$

to the forms

$$(14)\quad \left[\binom{j+1}{3}(\Delta_0^{\mathrm{IV}})+\binom{j}{3}(\Delta_{-1}^{\mathrm{IV}}+\Delta_1^{\mathrm{IV}})+\cdots+\binom{3}{3}(\Delta_{-j+2}^{\mathrm{IV}}+\Delta_{j-2}^{\mathrm{IV}})\right],$$

$$(15)\quad \left[\binom{j+2}{5}(\Delta_0^{\mathrm{VI}})+\binom{j+1}{5}(\Delta_{-1}^{\mathrm{VI}}+\Delta_1^{\mathrm{VI}})+\cdots+\binom{5}{5}(\Delta_{-j+3}^{\mathrm{VI}}+\Delta_{j-3}^{\mathrm{VI}})\right],$$

and so on, with exceptions of the same nature as in the case of the expression (9). The final expression in the sequence is

$$(16)\qquad \left[\binom{2j-1}{2j-1}(\Delta_0^{[2j]})\right].$$

3. Enumeration of the differences. By use of the identities

$$(17)\qquad \binom{n}{k}+\binom{n-1}{k}+\binom{n-2}{k}+\cdots+\binom{k}{k}=\binom{n+1}{k+1}$$

and

$$(18)\quad \binom{n}{k}+2\binom{n-1}{k}+2\binom{n-2}{k}+\cdots+2\binom{k}{k}=\frac{2n-k+1}{k+1}\binom{n}{k},$$

we find that the expressions (9), (10), (11), ... contain, respectively,

$$(19)\qquad 2\binom{j}{1},\quad 2\binom{j+1}{3},\quad 2\binom{j+2}{5},\ \ldots$$

differences, and that the expressions (13), (14), 15), ... contain, respectively,

$$(20)\qquad \frac{j}{1}\binom{j}{1},\quad \frac{j}{2}\binom{j+1}{3},\quad \frac{j}{3}\binom{j+2}{5},\ldots$$

differences.

Consequently, if we replace the expressions in the brackets in (7) by the reduced expressions in the higher differences, (7) becomes (apart from the omitted differences of lower order) *with respect only to the number of differences involved*

$$
(21) \qquad \begin{aligned} y_0 &+ \sum_{i=1}^{n} \sum_{j=i}^{n} \frac{A_{ij}}{D}\left[j\binom{j+i-1}{2i-1} \Delta^{[2i-1]}\text{'s}\right]\frac{x^{2i-1}}{h^{2i-1}} \\ &+ \sum_{i=1}^{n} \sum_{j=i}^{n} \frac{A_{ij}}{D}\left[\frac{j}{2i}\binom{j+i-1}{2i-1} \Delta^{[2i]}\text{'s}\right]\frac{x^{2i}}{h^{2i}}, \end{aligned}
$$

where, because of the termination, for $i=j$, of the sequences of expressions (9), (10), (11), ..., and (13), (14), (15), ..., we have $j \geqq i$.

We shall now show that

$$
(22) \qquad \sum_{j=i}^{n} \frac{A_{ij}}{D} j \binom{j+i-1}{2i-1} = \frac{1}{(2i-1)!}
$$

and hence that

$$
(23) \qquad \sum_{j=i}^{n} \frac{A_{ij}}{D} \frac{j}{2i} \binom{j+i-1}{2i-1} = \frac{1}{(2i)!};
$$

also that

$$
(24) \qquad \sum_{j=k}^{n} \frac{A_{ij}}{D} j \binom{j+k-1}{2k-1} = 0, \ k<i,
$$

and hence that

$$
(25) \qquad \sum_{j=k}^{n} \frac{A_{ij}}{D} \frac{j}{2k} \binom{j+k-1}{2k-1} = 0, \ k<i.
$$

We shall find it convenient to introduce the notation ($j \geqq i$)

$$
(26) \quad a_{ij} = j(2i-1)!\binom{j+i-1}{2i-1} = j^2(j^2-1)(j^2-2^2)\cdots(j^2-(i-1)^2).
$$

In order to establish (22), (23), (24), (25), we have then to prove

$$
(27) \qquad \sum_{j=i}^{n} a_{ij} A_{ij} = D,
$$

and

$$
(28) \qquad \sum_{j=k}^{n} a_{kj} A_{ij} = 0, \ k<i.
$$

But the determinant D is readily written in the form

$$(29)\quad D=\begin{vmatrix} 1 & 2^2 & 3^2 & 4^2 & \cdots \\ 0 & 2^2(2^2-1) & 3^2(3^2-1) & 4^2(4^2-1) & \cdots \\ 0 & 0 & 3^2(3^2-1)(3^2-2^2) & 4^2(4^2-1)(4^2-2^2) & \cdots \\ 0 & 0 & 0 & 4^2(4^2-1)(4^2-2^2)(4^2-3^2) & \cdots \\ \cdot\ \cdot & & \cdot & \cdot & \cdots \\ \cdot\ \cdot & & \cdot & \cdot & \cdots \end{vmatrix},$$

the nth row of this form being obtained by multiplying the rows of (2) in order by the coefficients of $x^2, x^4, x^6, \ldots, x^{2n}$ in the polynomial $x^2(x^2-1)(x^2-2^2)\cdots(x^2-(n-1)^2)$ and adding. The remaining rows are obtained in like manner. If only the first i rows of (2) are so transformed the cofactors of elements in rows i to n inclusive are the same in (29) as in (2). From this the truth of (27) and (28) is at once evident. From (24) and (25), which are now established, it is clear that differences of lower order are eliminated by the transformations of Section 2, as there predicted, and from this fact in combination with (22) and (23) we have the following result:

THEOREM. *The coefficient of x^m in the polynomial determined by the points* (1) *is the product of the reciprocal of $m!$ and a weighted average of difference quotients of order m.*

4. **Explicit expansion of $P^{[2n]}(x)$ in terms of difference quotients.** In the light of the foregoing theorem the explicit expansion of the polynomial $P^{[2n]}(x)$ in the manner specified by the theorem becomes important. This expansion results from the transformations of Section 2, and is as follows:

$$(30)\quad \begin{aligned} P^{[2n]}(x) = y_0 &+ \sum_{i=1}^{n}\sum_{j=i}^{n}\frac{jA_{ij}}{2D}\left[\binom{j+i-2}{2i-2}(\Delta_{-1}^{[2i-1]}+\Delta_{1}^{[2i-1]})\right. \\ &+\binom{j+i-3}{2i-2}(\Delta_{-2}^{[2i-1]}+\Delta_{2}^{[2i-1]})+\cdots+\left.\binom{2i-2}{2i-2}(\Delta_{-j+i-1}^{[2i-1]}+\Delta_{j-i+1}^{[2i-1]})\right]\frac{x^{2i-1}}{h^{2i-1}} \\ &+\sum_{i=1}^{n}\sum_{j=i}^{n}\frac{A_{ij}}{2D}\left[\binom{j+i-1}{2i-1}(\Delta_0^{[2i]})+\binom{j+i-2}{2i-1}(\Delta_{-1}^{[2i]}+\Delta_{1}^{[2i]})\right. \\ &+\cdots+\left.\binom{2i-1}{2i-1}(\Delta_{-j+i}^{[2i]}+\Delta_{j-i}^{[2i]})\right]\frac{x^{2i}}{h^{2i}}. \end{aligned}$$

Combination of (26) and (30) now enables us to write $P^{[2n]}(x)$ in the form*

$$
\begin{aligned}
P^{[2n]}(x) &= y_0 \\
&+ \sum_{i=1}^{n} \frac{1}{(2i-1)!} \sum_{k=}^{n-i+1} \frac{\Delta_{-k}^{[2i-1]} + \Delta_{k}^{[2i-1]}}{2h^{2i-1}} \cdot \frac{\sum_{j=i-1+k}^{n} j \binom{j+i-1-k}{2i-2} A_{ij}}{\sum_{j=i}^{n} j \binom{j+i-1}{2i-1} A_{ij}} x^{2i-1} \\
&+ \sum_{i=1}^{n} \frac{1}{(2i)!} \sum_{k=}^{n-i} \frac{\Delta_{-k}^{[2i]} + \Delta_{k}^{[2i]}}{2h^{2i}} \cdot \frac{\sum_{j=i+k}^{n} 2i \binom{j+i-1-k}{2i-1} A_{ij}}{\sum_{j=i}^{n} j \binom{j+i-1}{2i-1} A_{ij}} x^{2i}.
\end{aligned}
\tag{31}
$$

Formula (31) presents explicitly the weightings of the various difference quotients in the weighted averages mentioned in the theorem of Section 3.

For $n = 1, 2, 3, 4$ we have the following numerical results:

The polynomial determined by three points $(n = 1)$:

$$
P^{[2]}(x) = y_0 + \frac{\Delta'_{-1} + \Delta'_1}{2h} x + \frac{1}{2!} \frac{\Delta''_0}{h^2} x^2 \tag{32}
$$

The polynomial determined by five points $(n = 2)$:

$$
\begin{aligned}
P^{[4]}(x) = y_0 &+ \left\{ \frac{7}{6} \frac{\Delta'_{-1} + \Delta'_1}{2h} - \frac{1}{6} \frac{\Delta'_{-2} + \Delta'_2}{2h} \right\} x \\
&+ \frac{1}{2!} \left\{ \frac{7}{6} \frac{\Delta''_0}{h^2} - \frac{1}{6} \frac{\Delta''_{-1} + \Delta''_1}{2h^2} \right\} x^2 \\
&+ \frac{1}{3!} \left\{ \frac{\Delta'''_{-1} + \Delta'''_1}{2h^3} \right\} x^3 + \frac{1}{4!} \left\{ \frac{\Delta^{\mathrm{IV}}_0}{h^4} \right\} x^4.
\end{aligned}
\tag{33}
$$

The polynomial determined by seven points $(n = 3)$:

$$
\begin{aligned}
P^{[6]}(x) = y_0 &+ \left\{ \frac{111}{90} \frac{\Delta'_{-1} + \Delta'_1}{2h} - \frac{24}{90} \frac{\Delta'_{-2} + \Delta'_2}{2h} + \frac{3}{90} \frac{\Delta'_{-3} + \Delta'_3}{2h} \right\} x \\
&+ \frac{1}{2!} \left\{ \frac{111}{90} \frac{\Delta''_0}{h^2} - \frac{23}{90} \frac{\Delta''_{-1} + \Delta''_1}{2h^2} + \frac{2}{90} \frac{\Delta''_{-2} + \Delta''_2}{2h^2} \right\} x^2
\end{aligned}
$$

---

* For $k = 0$, $\Delta_{-k}^{[2i]} = 0$, by definition.

$$
\begin{aligned}
&+\frac{1}{3!}\left\{\frac{5}{4}\,\frac{\Delta_{-1}'''+\Delta_{1}'''}{2h^3}-\frac{1}{4}\,\frac{\Delta_{-2}'''+\Delta_{2}'''}{2h^3}\right\}x^3\\
(34)\qquad&+\frac{1}{4!}\left\{\frac{4}{3}\,\frac{\Delta_{0}^{IV}}{h^4}-\frac{1}{3}\,\frac{\Delta_{-1}^{IV}+\Delta_{1}^{IV}}{2h^4}\right\}x^4\\
&+\frac{1}{5!}\left\{\frac{\Delta_{-1}^{V}+\Delta_{1}^{V}}{2h^5}\right\}x^5+\frac{1}{6!}\left\{\frac{\Delta_{0}^{VI}}{h^6}\right\}x^6.
\end{aligned}
$$

The polynomial determined by nine points ($n=4$):

$$
\begin{aligned}
P^{[8]}(x)=y_0&+\left\{\frac{3198}{2520}\,\frac{\Delta_{-1}'+\Delta_{1}'}{2h}-\frac{834}{2520}\,\frac{\Delta_{-2}'+\Delta_{2}'}{2h}\right.\\
&\qquad\left.+\frac{174}{2520}\,\frac{\Delta_{-3}'+\Delta_{3}'}{2h}-\frac{18}{2520}\,\frac{\Delta_{-4}'+\Delta_{4}'}{2h}\right\}x\\
&+\frac{1}{2!}\left\{\frac{3198}{2520}\,\frac{\Delta_{0}''}{h^2}-\frac{779}{2520}\,\frac{\Delta_{-1}''+\Delta_{1}''}{2h^2}\right.\\
&\qquad\left.+\frac{110}{2520}\,\frac{\Delta_{-2}''+\Delta_{2}''}{2h^2}-\frac{9}{2520}\,\frac{\Delta_{-3}''+\Delta_{3}''}{2h^2}\right\}x^2\\
(35)\qquad&+\frac{1}{3!}\left\{\frac{164}{120}\,\frac{\Delta_{-1}'''+\Delta_{1}'''}{2h^3}-\frac{51}{120}\,\frac{\Delta_{-2}'''+\Delta_{2}'''}{2h^3}+\frac{7}{120}\,\frac{\Delta_{-3}'''+\Delta_{3}'''}{2h^3}\right\}x^3\\
&+\frac{1}{4!}\left\{\frac{181}{120}\,\frac{\Delta_{0}^{IV}}{h^4}-\frac{68}{120}\,\frac{\Delta_{-1}^{IV}+\Delta_{1}^{IV}}{2h^4}+\frac{7}{120}\,\frac{\Delta_{-2}^{IV}+\Delta_{2}^{IV}}{2h^4}\right\}x^4\\
&+\frac{1}{5!}\left\{\frac{4}{3}\,\frac{\Delta_{-1}^{V}+\Delta_{1}^{V}}{2h^5}-\frac{1}{3}\,\frac{\Delta_{-2}^{V}+\Delta_{2}^{V}}{2h^5}\right\}x^5\\
&+\frac{1}{6!}\left\{\frac{3}{2}\,\frac{\Delta_{0}^{VI}}{h^6}-\frac{1}{2}\,\frac{\Delta_{-1}^{VI}+\Delta_{1}^{VI}}{2h^6}\right\}x^6\\
&+\frac{1}{7!}\left\{\frac{\Delta_{-1}^{VII}+\Delta_{1}^{VII}}{2h^7}\right\}x^7+\frac{1}{8!}\left\{\frac{\Delta_{0}^{VIII}}{h^8}\right\}x^8.
\end{aligned}
$$

The weightings of the mean difference quotients of the first and second orders in the weighted averages involved in the coefficients of x and x^2 are of particular interest, and for low degrees are of practical importance. We therefore tabulate these weightings for degrees 2, 4, 6, 8, 10, the weightings in any given column being in order of numerically ascending subscripts of the mean difference quotients:

Weightings of first Difference Quotients

(36)

Degree 2	Degree 4	Degree 6	Degree 8	Degree 10
1.000 000	1.166 666	1.233 333	$1.26\dot{9}\,047\,6\dot{1}$	$1.29\dot{1}\,269\,8\dot{4}$
	$-0.166\,666$	$-0.266\,666$	$-0.33\dot{0}\,952\,3\dot{8}$	$-0.37\dot{5}\,396\,8\dot{2}$
		0.033 333	$0.06\dot{9}\,047\,6\dot{1}$	$0.10\dot{0}\,793\,6\dot{5}$
			$-0.00\dot{7}\,142\,8\dot{5}$	$-0.01\dot{8}\,253\,9\dot{6}$
				$0.00\dot{1}\,587\,3\dot{0}$

Weightings of second Difference Quotients

(37)

Degree 2	Degree 4	Degree 6	Degree 8	Degree 10
1.000 000	1.166 666	1.233 333	$1.26\dot{9}\,047\,6\dot{1}$	$1.29\dot{1}\,269\,8\dot{4}$
	$-0.166\,666$	$-0.255\,555$	$-0.309\,\dot{1}26\,98\dot{4}$	$-0.344\,\dot{6}82\,53\dot{9}$
		0.022 222	$0.04\dot{3}\,650\,7\dot{9}$	$0.06\dot{1}\,428\,5\dot{7}$
			$-0.003\,\dot{5}71\,42\dot{8}$	$-0.008\,\dot{6}50\,79\dot{3}$
				$0.000\,\dot{6}34\,92\dot{0}$

By means of (36) and (37) the first and second derivatives of $P^{[2n]}(x)$, $n = 1, 2, 3, 4, 5$, at the point $(0, y_0)$ are quickly determined without computing differences of higher order than that of the desired derivative. The values given in (36) and (37) are readily checked by taking for $y_{-n}, \ldots, y_{-1}, y_0, y_1, \ldots, y_n$ the values, for $x = -n, \ldots, -1, 0, 1, \ldots, n$, respectively, of any polynomial (for example, x^m) of degree less than or equal to $2n$.

5. **Derivation of Stirling's form of** $P^{[2n]}(x)$. The polynomial $P^{[2n]}(x)$ is derived in the foregoing by methods in no way dependent on the classic formulas of Lagrange and Stirling. Stirling's formula, on the other hand, is very readily derived from (31).

We shall use the customary notation

$$\frac{\Delta'_{-1} + \Delta'_1}{2} = \mu \Delta'_0, \qquad \frac{\Delta'''_{-1} + \Delta'''_1}{2} = \mu \Delta'''_0, \cdots. \tag{38}$$

From (32) we have

$$P^{[2]}(x) = y_0 + \mu \Delta'_0 \frac{x}{h} + \frac{1}{2!} \Delta''_0 \frac{x^2}{h^2}. \tag{39}$$

If to (39) we add a polynomial of degree 4 which has for its terms of the third and fourth degree the terms of the third and fourth degree of $P^{[4]}(x)$, as specified by (33), and which vanishes for $x/h = -1, 0, 1$, the difference between the polynomial thus obtained and $P^{[4]}(x)$ will be of degree 2 at most, and will vanish at three points. The new polynomial

$$y_0 + \mu\Delta_0' \frac{x}{h} \frac{1}{2!} \Delta_0'' \frac{x^2}{h^2} + \frac{1}{3!} \mu \Delta_0''' \frac{x}{h}\left(\frac{x^2}{h^2} - 1\right) + \frac{1}{4!} \Delta_0^{\text{IV}} \frac{x^2}{h^2}\left(\frac{x^2}{h^2} - 1\right) \tag{40}$$

is therefore identical with $P^{[4]}(x)$.

Since, for $i = j = n$, formulas (30) and (31) yield

$$\begin{aligned} &\frac{2nA_{nn}}{2D} \mu \Delta_0^{[2n-1]} \frac{x^{2n-1}}{h^{2n-1}} + \frac{A_{nn}}{2D} \Delta_0^{[2n]} \frac{x^{2n}}{h^{2n}} \\ &\qquad = \frac{1}{(2n-1)!} \mu \Delta_0^{[2n-1]} \frac{x^{2n-1}}{h^{2n-1}} + \frac{1}{(2n)!} \Delta_0^{[2n]} \frac{x^{2n}}{h^{2n}}, \end{aligned} \tag{41}$$

the argument by which (40) is obtained from (39) may be extended indefinitely, yielding the well known Stirling interpolation formula.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
CAMBRIDGE, MASS.