ON RIESZ AND CESÀRO METHODS OF SUMMABILITY*

BY

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1. Introduction. Marcel Riesz‡ formulated the following method of summability: Let r be any complex constant and, given a series $u_0+u_1+u_2+\cdots$, let

$$(1.1)\qquad A_r:\qquad \alpha_n = \sum_{k=0}^{n-1}\left(1-\frac{k}{n}\right)^r u_k \qquad (n = 1, 2, 3, \cdots);$$

if $\lim_{n\to\infty}\alpha_n = L$, then $\sum u_n$ is said to be summable A_r to L.

In a second note, Riesz§ gave a method which is, when $r>0$, equivalent|| (vide Theorem 4.4) to the following: Let r be a complex constant and let¶

$$(1.2)\qquad B_r:\qquad \beta(t) = \sum_{k=0}^{[t]-1}\left(1-\frac{k}{t}\right)^r u_k, \qquad 1 \leqq t < \infty;$$

if $\beta(t)$ approaches a limit L as t becomes infinite over the real set $t\geqq 1$, then $\sum u_n$ is summable B_r to L. The second method of Riesz is the following: Let $\Re(r)>0$, and let

$$(1.3)\qquad D_r:\qquad \delta(t) = \sum_{k=0}^{[t^-]}\left(1-\frac{k}{t}\right)^r u_k, \qquad 1 \leqq t < \infty;$$

if $\delta(t)$ converges to L as t becomes infinite continuously, then $\sum u_n$ is summable D_r to L. The method D_r is known as the Riesz method of order r and type $\lambda_n = n$**, and has proved to be one of the most useful of all methods of summability.

In his second note, Riesz outlined a proof that D_r is equivalent to C_r,

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‡ Comptes Rendus, vol. 149 (1909), pp. 18–22. In this note Riesz considered only real positive orders r.

§ Comptes Rendus, vol. 152 (1911), pp. 1651–1654. Here again Riesz considered only the case $r>0$.

|| The terminology used in this paper is that given by W. A. Hurwitz, Bulletin of the American Mathematical Society, vol. 28 (1922), pp. 17–36.

¶ We use the symbols $[t]$ and $[t^-]$ to denote respectively the greatest integer $\leqq t$ and the greatest integer $<t$.

** Hardy-Riesz, *General Theory of Dirichlet's series*, Cambridge Tracts in Mathematics and Mathematical Physics, No. 18.

the Cesàro method of order r, when $r>0$. Chapman* has stated that Riesz's proof of equivalence of D_r and C_r holds when $r>-1$; but this statement is incorrect as Theorem 2.1 shows. Hobson† has given a more detailed proof of equivalence of D_r and C_r when $r>0$.

In a third note, Riesz‡ outlined a proof that A_r and C_r are equivalent when $-1<r<1$, and showed that this equivalence does not hold for certain values of $r>1$.

It is the object of this paper to discuss A_r, B_r, D_r, C_r, and closely related methods of summability. We shall be especially interested in orders r with real part $\Re(r)<0$.

In §2 we show that D_r does not constitute a useful method of summability when $\Re(r)<0$; and in §§2–3 we discuss modifications of D_r which may be expected to be useful when $\Re(r)<0$. For each complex r, these modifications are found to be equivalent to B_r. In §4 we show that B_r and D_r are equivalent when $r\geqq 0$. In §§5–7 we obtain auxilliary results from which it follows in §8 that B_r and C_r are equivalent when $-1<\Re(r)<0$. The theorems of §8 give a complete solution of the problem which furnished the point of departure of this investigation. In §9 we give relations between methods B_r of different orders. We show in §§10–11 that A_r does not possess certain properties of B_r when $\Re(r)<-1$; in particular when $\Re(r)$ is less than a certain constant between -2 and -1, A_r is not consistent with convergence. Finally, in §12 we point out that when $\Re(r)<0$, the methods A_r, B_r, and C_r are equivalent over a certain class of series.

2. Ineffectiveness of D_r when $\Re(r)<0$. It is well known that D_r is regular when r is real and $\geqq 0$. It can be shown further that D_r is regular when $\Re(r)>0$; and that D_r is not regular when $\Re(r)=0$ but $r\neq 0$. To show that D_r does not constitute a useful method of summability when $\Re(r)<0$, we will prove the following Theorem.

THEOREM 2.1. *In order that $\sum u_n$ may be summable D_r when $\Re(r)<0$, it is necessary and sufficient that $\sum u_n$ have at most a finite number of terms different from zero.*

Sufficiency is easily established. To prove necessity, let us suppose that $u_p\neq 0$ for a certain index $p>0$; then $\lim_{h\to 0+}|\delta(p+h)|=+\infty$. Hence if $u_k\neq 0$ for an infinite set of values of k, then $\delta(t)$ is unbounded over every interval

* Proceedings of the London Mathematical Society, (2), vol. 9 (1910–11), p. 374, second footnote.

† *Theory of Functions of a Real Variable.* vol. II, 1926, pp. 90–98.

‡ Proceedings of the London Mathematical Society, (2), vol. 22 (1923–24), p. 418.

(N, ∞) so that $\delta(t)$ cannot converge as $t\to\infty$ and the theorem is proved.*

Theorem 2.1 and its proof make it clear that if a useful generalization to orders with real part $\Re(r)<0$ of the Riesz method D_r is to be obtained, the upper index of summation with respect to k must be a function of t which is definitely less than $[t^-]$. The two methods defined by the transformations

$$\pi(t) = \sum_{k=0}^{[t-\theta]}\left(1-\frac{k}{t}\right)^r u_k, \qquad \theta<t<\infty, \tag{2.2}$$

$$\rho(t) = \sum_{k=0}^{[(t-\theta)^-]}\left(1-\frac{k}{t}\right)^r u_k, \qquad \theta<t<\infty, \tag{2.3}$$

where θ is a positive constant, suggest themselves at once as modifications of D_r which may be useful for every complex order.

Let r be any complex number. Then, corresponding to any given series $\sum u_n$, the functions $\pi(t)$ and $\rho(t)$ are equal except when t is of the form $t=n+\theta$ and $u_n\neq 0$, in which case $\pi(n+\theta)\neq\rho(n+\theta)$. Furthermore the transforms $\pi(t)$ and $\rho(t)$ are continuous except when $t=n+\theta$ and $u_n\neq 0$; here $\pi(t)$ and $\rho(t)$ have finite jumps, $\pi(t)$ having right-hand continuity and $\rho(t)$ having left-hand continuity. It follows that if either $\pi(t)$ or $\rho(t)$ converges as $t\to\infty$, then the other must also converge to the same value as $t\to\infty$. Hence the methods (2.2) and (2.3) are equivalent. We elect to consider the first rather than the second of these.

3. **Consideration of (2.2) for different values of θ.** In this section we will establish a theorem which will be of fundamental importance in the sequel; and will show that, for any fixed complex r, the methods (2.2) obtained by selecting different positive values of θ are equivalent to B_r.

THEOREM 3.1. *If $\sum u_n$ is summable (2.2) with r a fixed complex constant and θ a fixed positive constant, then*

$$\lim_{n\to\infty} u_n/n^r = 0. \tag{3.11}$$

Suppose $\sum u_n$ is summable (2.2) to L. Then

$$\lim_{t\to\infty}\sum_{k=0}^{[t-\theta]}\left(1-\frac{k}{t}\right)^r v_k = 0$$

* There is an apparent inconsistency between Theorem 2.1 and Chapman's statement (loc. cit., p. 401) that the Dirichlet series $1^{-s}-2^{-s}+3^{-s}-4^{-s}+\cdots$, $s>0$, is summable $(R, n, -r)$, i.e. D_{-r}, when $r<s$. The last equation of p. 399 shows that Chapman has used the transformation B_r rather than D_r; and furthermore the second equation of p. 400 is correct only when $[n]=n$. Therefore, as a matter of fact, Chapman has not shown that $\sum(-1)^n(n+1)^{-s}$ is summable D_{-r} when $r<s$; what he has shown is that the series is summable A_{-r} when $r<s$. However, it follows from this result and Theorem 12.1 that the series $\sum(-1)^n(n+1)^{-s}$ is summable B_{-r} and C_{-r} as well as A_{-r} when $r<s$.

where $v_0 = u_0 - L$ and $v_n = u_n$ when $n > 0$. Given $\epsilon > 0$, choose $T > \theta$ such that

$$\left| \sum_{k=0}^{[t-\theta]} \left(1 - \frac{k}{t}\right)^r v_k \right| < \frac{\epsilon}{2}, \qquad t > T. \tag{3.12}$$

Let $n > T+1$, let $0 < h < 1$, and set $t = n + \theta - h$ in (3.12) to obtain

$$\left| \sum_{k=0}^{n-1} \left(1 - \frac{k}{n+\theta-h}\right)^r v_k \right| < \frac{\epsilon}{2}, \qquad 0 < h < 1. \tag{3.13}$$

The left member of (3.13) is a continuous function of h over the closed interval $0 \leqq h \leqq 1$; hence we may take the limit as $h \to 0$ to obtain

$$\left| \sum_{k=0}^{n-1} \left(1 - \frac{k}{n+\theta}\right)^r v_k \right| \leqq \frac{\epsilon}{2}, \qquad n > T+1. \tag{3.14}$$

Again we may set $t = n + \theta$ in (3.12) and write the last term of the sum as a separate term to obtain

$$\left| \sum_{k=0}^{n-1} \left(1 - \frac{k}{n+\theta}\right)^r v_k + \left(\frac{\theta}{n+\theta}\right)^r v_n \right| < \frac{\epsilon}{2}, \qquad n > T+1. \tag{3.15}$$

Combining (3.14) and (3.15), we find that $|\theta^r v_n/(n+\theta)^r| < \epsilon$ when $n > T+1$. Hence $\lim_{n\to\infty} \theta^r v_n/(n+\theta)^r = 0$ so that $\lim_{n\to\infty} v_n/n^r = 0$ and, since $v_n = u_n$ when $n > 0$, (3.11) follows. Thus Theorem 3.1 is proved.

A slight modification of the preceding argument shows that if $\sum u_n$ is bounded (2.2), then u_n/n^r is bounded for all $n > 0$.

THEOREM 3.2. *If r is a complex constant and*

$$\pi^{(\theta)}(t) = \sum_{k=0}^{[t-\theta]} \left(1 - \frac{k}{t}\right)^r u_k, \quad \pi^{(\theta')}(t) = \sum_{k=0}^{[t-\theta']} \left(1 - \frac{k}{t}\right)^r u_k$$

represent two different methods of the form (2.2) *with $\theta > 0$, $\theta' > 0$, and if furthermore* $\lim_{n\to\infty} u_n/n^r = 0$, *then*

$$\lim_{t\to\infty} \left\{ \pi^{(\theta)}(t) - \pi^{(\theta')}(t) \right\} = 0. \tag{3.21}$$

To establish this result, we may assume that $\theta > \theta'$ and show that the difference in the left member of (3.21) consists of a finite number of terms each of which approaches zero as $t \to \infty$.

From the two preceding theorems we obtain at once

THEOREM 3.3. *When r is any complex constant, the methods obtained by assigning different positive values to θ in* (2.2) *are equivalent.*

For if $\sum u_n$ is summable (2.2) for a positive value of θ, then $\lim u_n/n^r=0$ by Theorem 3.1; hence the hypotheses of Theorem 3.2 are satisfied and the conclusion (3.21) completes the proof of Theorem 3.3.

The only representative of the set of methods (2.2) which we will consider in the sequel is that for which $\theta=1$; in this case (2.2) becomes B_r.

4. **Relations between B_r and D_r when $\Re(r)\geqq 0$.** Before passing to a study of B_r when $\Re(r)<0$, we wish to point out that B_r is closely related to the familiar Riesz method D_r when $\Re(r)\geqq 0$.

THEOREM 4.1. *If $\Re(r)\geqq 0$ and $\lim u_n/n^r=0$, then*

$$\lim_{t\to\infty}\left\{\delta(t)-\beta(t)\right\}=0. \tag{4.11}$$

We find from (1.2) and (1.3) that $|\delta(t)-\beta(t)|\leqq|u_{[t]}/[t]^r|$ when $\Re(r)\geqq 0$ and $t>1$, and Theorem 4.1 follows.

THEOREM 4.2. *If $\Re(r)\geqq 0$, then D_r includes B_r.*

If $\sum u_n$ is summable B_r to L so that $\lim \beta(t)=L$, then $\lim u_n/n^r=0$ by Theorem 3.1 with $\theta=1$; hence the hypotheses of Theorem 4.1 are satisfied, the conclusion (4.11) shows that $\lim \delta(t)=L$, and Theorem 4.2 is proved.

THEOREM 4.3. *If $r\geqq 0$, then B_r includes D_r.*

The proposition being evident when $r=0$, we suppose $r>0$. Let $\sum u_n$ be summable D_r to L so that $\lim \delta(t)=L$. Then, using the fact (§1) that C_r includes D_r when $r>0$, we see that $\sum u_n$ must be summable C_r and hence, as is well known, that $\lim u_n/n^r=0$. Hence Theorem 4.1 shows that $\lim\beta(t)=L$ and Theorem 4.3 is proved.

Combining Theorems 4.2 and 4.3 with the fact that D_r and C_r are equivalent when $r\geqq 0$, we obtain

THEOREM 4.4. *If $r\geqq 0$, then B_r, D_r, and C_r are equivalent.*

5. **A relation between the A_r and B_r transforms when $\Re(r)<0$.** We proceed to establish some preliminary propositions, interesting in themselves, which will enable us to obtain relations between B_r and C_r.

THEOREM 5.1. *When $\Re(r)<0$, the assumption that*

$$\lim_{t\to\infty}\beta(t)=\lim_{t\to\infty}\sum_{k=0}^{[t]-1}\left(1-\frac{k}{t}\right)^r u_k=L \tag{5.11}$$

is equivalent to the two assumptions

$$\lim_{n\to\infty} u_n/n^r=0 \tag{5.12}$$

and

$$\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \sum_{k=0}^{n-1} \left(1 - \frac{k}{n}\right)^r u_k = L.^* \tag{5.13}$$

That (5.11) implies (5.12) follows from Theorem 3.1 with $\theta = 1$; and that (5.11) implies (5.13) follows from the fact that $\alpha_n = \beta(n)$. Hence our problem here is to show that (5.12) and (5.13) together imply (5.11).

A consideration of the sequence v_n defined by $v_0 = u_0 - L$ and $v_n = u_n$, $n > 0$, shows that it is sufficient to prove (5.12) and (5.13) imply (5.11) when $L = 0$. We suppose therefore that $\Re(r) < 0$, that (5.12) holds, and that (5.13) holds with $L = 0$; we will show that (5.11) holds with $L = 0$.

Given $\epsilon > 0$, choose an index $N > 0$ so great that

$$\left| \sum_{k=0}^{n-1} \left(1 - \frac{k}{n}\right)^r u_k \right| < \frac{\epsilon}{2}, \qquad n \geqq N, \tag{5.14}$$

and

$$| u_n/n^r | < \epsilon/(4B|r|), \qquad n \geqq N, \tag{5.15}$$

where $B = 2^{-r'}(1 + \sum_{p=1}^{\infty} p^{r'-1})$ and $r' = \Re(r)$. Next, choose an index $P > N$ so great that

$$\frac{|r|}{t^2} \left| \sum_{k=0}^{N-1} k \left(1 - \frac{k}{t}\right)^{r-1} u_k \right| < \frac{\epsilon}{4}, \qquad t > P. \tag{5.16}$$

Let $n > P$ and consider the function

$$\beta(t) = \sum_{k=0}^{n-1} \left(1 - \frac{k}{t}\right)^r u_k, \qquad n \leqq t < n + 1. \tag{5.17}$$

Using (5.14), we see that

$$|\beta(n)| < \epsilon/2. \tag{5.18}$$

Differentiating (5.17) we find

$$\beta'(t) = \frac{r}{t^2} \sum_{k=0}^{n-1} k \left(1 - \frac{k}{t}\right)^{r-1} u_k, \qquad n \leqq t < n + 1,$$

where the derivative for $t = n$ is a right-hand derivative. Hence

$$\beta'(t) \leqq \frac{|r|}{t^2} \left| \sum_{k=0}^{N-1} k \left(1 - \frac{k}{t}\right)^{r-1} u_k \right| + \frac{|r|}{t^2} \sum_{k=N}^{n-1} k^{1+r'} \left(1 - \frac{k}{t}\right)^{r'-1} | u_k/k^r |,$$

and using (5.16) and (5.15) we obtain

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* Consideration of independence of (5.12) and (5.13) is relegated to §10 where we study A_r.

$$|\beta'(t)| < \epsilon/4 + \epsilon\Phi_n(t)/(4B), \tag{5.19}$$

where

$$\Phi_n(t) = \sum_{k=1}^{n-1} \frac{k^{r'}(t-k)^{r'-1}}{t^{r'}}, \qquad n \leqq t < n+1. \tag{5.20}$$

But since $r' < 0$,

$$\sum_{k=1}^{[t/2]} \frac{k^{r'}(t-k)^{r'-1}}{t^{r'}} \leqq \frac{1}{t} \sum_{k=1}^{[t/2]} \left(1-\frac{k}{t}\right)^{r'-1} \leqq \frac{1}{t} \sum_{k=1}^{[t/2]} \left(\frac{1}{2}\right)^{r'-1} \leqq 2^{-r'},$$

and

$$\sum_{k=[t/2]+1}^{n-1} \frac{k^{r'}(t-k)^{r'-1}}{t^{r'}} \leqq \left(\frac{[t/2]+1}{t}\right)^{r'} \sum_{k=[t/2]+1}^{n-1} (t-k)^{r'-1} \leqq 2^{-r'} \sum_{p=1}^{\infty} p^{r'-1},$$

so that

$$\Phi_n(t) \leqq 2^{-r'}\left(1 + \sum_{p=1}^{\infty} p^{r'-1}\right) = B, \qquad n \leqq t < n+1. \tag{5.21}$$

From (5.19) and (5.21) we obtain

$$|\beta'(t)| < \epsilon/2, \qquad n \leqq t < n+1. \tag{5.22}$$

Using (5.18), (5.22), and the formula

$$|\beta(t)| \leqq |\beta(n)| + \int_n^t |\beta'(t)|\, dt, \qquad n \leqq t < n+1,$$

we find that $|\beta(t)| < \epsilon$, $n \leqq t < n+1$.

We have shown that if $n > P$, then $|\beta(t)| < \epsilon$, $n \leqq t < n+1$. It follows that if $t > P+1$, then $|\beta(t)| < \epsilon$. Hence $\lim \beta(t) = 0$ and Theorem 5.1 is proved.

6. Lemmas involving C_r. The Cesàro method C_r of order r (r not a negative integer) is defined by the transformation

$$C_r: \qquad \gamma_n = \sum_{k=0}^{n} a_{nk} u_k \qquad (n = 0, 1, 2, \cdots), \tag{6.01}$$

where

$$a_{nk} = \frac{\Gamma(n+1)\Gamma(n-k+1+r)}{\Gamma(n+1+r)\Gamma(n-k+1)}, \qquad 0 \leqq k \leqq n. \tag{6.02}$$

The following two lemmas will be used in the next section.

LEMMA 6.1. *Corresponding to each complex constant r (not a negative integer) there is a bounded sequence C_{nk} of constants such that for each positive index n and each index $k<n$*

$$a_{nk} = \left(1 - \frac{k}{n}\right)^r \left(1 + \frac{C_{nk}}{n-k}\right). \tag{6.11}$$

Using the familiar asymptotic expansion of the logarithm of the gamma function of a complex argument,* we find

$$\log \{\Gamma(n+1+r)/\Gamma(n+1)\} = r \log n + H_n/n, \qquad n > 0, \tag{6.12}$$

where H_n is a bounded sequence of constants. Subtracting (6.12) from the relation obtained by replacing n by $n-k$ in it, we obtain

$$\log a_{nk} = r \log \{(n-k)/n\} - H_n/n + H_{n-k}/(n-k) \tag{6.13}$$

when $n>0$ and $k<n$. The lemma results from (6.13). The following lemma is easily deduced from (6.12).

LEMMA 6.2. *When r is not a negative integer*

$$\lim_{n\to\infty} n^r a_{nn} = \Gamma(1+r). \tag{6.21}$$

In §8 we shall need

LEMMA 6.3. *When $\Re(r) \leqq -1$, r not a negative integer, the condition* lim $u_n/n^r = 0$ *is not necessary in order that $\sum u_n$ may be summable C_r.*

The inverse of C_r is, when r is not an integer, given by

$$u_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{k+r}{r} \binom{r+1}{n-k} \gamma_k \tag{6.31}$$

or

$$u_n = \sum_{k=0}^{n} \frac{\sin \pi r}{\pi} \frac{\Gamma(2+r)}{\Gamma(k+1)} \frac{\Gamma(k+1+r)}{\Gamma(1+r)} \frac{\Gamma(n-k-1-r)}{\Gamma(n-k+1)} \gamma_k. \tag{6.32}$$

Corresponding to each complex r which is not an integer, let the sequence $\gamma_n^{(r)}$ be defined by $\gamma_0^{(r)} = \pi/\{\sin \pi r \Gamma(2+r)\}$ and $\gamma_n^{(r)} = 0$ when $n>0$; and let $\sum u_n^{(r)}$ be the series whose C_r transform is $\gamma_n^{(r)}$. Substituting in (6.32) we find

$$u_n^{(r)} = \Gamma(n-1-r)/\Gamma(n+1) = n^{-2-r}(1+o(1)) \tag{6.33}$$

so that

* See, for example, J. L. W. V. Jensen (translation by T. H. Gronwall), Annals of Mathematics, (2), vol. 17 (1916), p. 136.

$$(6.34)\qquad u_n^{(r)}/n^r = n^{-2(1+r)}(1 + o(1)).$$

The series $\sum u_n^{(r)}$ is summable C_r to 0 and, when $\mathfrak{R}(r) \leqq -1$, the right member of (6.34) fails to converge to 0 as $n \to \infty$; thus Lemma 6.3 is established.

7. A relation between the A_r and C_r transforms when $\mathfrak{R}(r) < 0$. With the lemmas of §6 at our disposal, we are in a position to prove the following theorem.

THEOREM 7.1. *If $\mathfrak{R}(r) < 0$ (r not a negative integer) and the terms of $\sum u_n$ satisfy the condition*

$$(7.11)\qquad \lim_{n\to\infty} u_n/n^r = 0,$$

then

$$(7.12)\qquad \lim_{n\to\infty} (\gamma_n - \alpha_n) = 0,$$

*where γ_n and α_n represent respectively the C_r and A_r transform of $\sum u_n$.**

Letting $\sum u_n$ be any series for which (7.11) holds, we have for each $n > 1$

$$(7.13)\qquad \gamma_n - \alpha_n = a_{nn}u_n + \sum_{k=0}^{n-1}\left\{a_{nk} - \left(1 - \frac{k}{n}\right)^r\right\}u_k.$$

Writing $a_{nn}u_n$ in the form $(n^r a_{nn})(u_n/n^r)$, we see from Lemma 6.2 and (7.11) that it approaches zero as n becomes infinite. Furthermore the coefficient of u_0 is zero for each n. Hence it follows from (7.13) that

$$(7.14)\qquad \gamma_n - \alpha_n = o(1) + \sum_{k=1}^{n-1}\left\{a_{nk} - \left(1 - \frac{k}{n}\right)^r\right\}u_k,$$

and we may use Lemma 6.1 to obtain

$$(7.15)\qquad \gamma_n - \alpha_n = o(1) + \sum_{k=1}^{n-1}\left(1 - \frac{k}{n}\right)^r \frac{C_{nk}}{n-k}u_k.$$

Choosing a constant C such that $|C_{nk}| < C$ when $0 < k < n$, we obtain

$$(7.16)\qquad |\gamma_n - \alpha_n| \leqq o(1) + C\sum_{k=1}^{n-1}\frac{k^{r'}(n-k)^{r'-1}}{n^{r'}}|u_k/k^r|,$$

where $r' = \mathfrak{R}(r)$.

* It should be noted that the hypotheses of this theorem are not sufficient to ensure that either of the sequences γ_n or α_n is convergent, and hence that this theorem gives an especially important relation between Cesàro and Riesz transforms.

Now (7.16) shows that (7.12) will follow if $\lim v_n = 0$ implies $\lim V_n = 0$ when V_n is defined by

$$(7.17) \qquad V_n = \sum_{k=1}^{n-1} \frac{k^{r'}(n-k)^{r'-1}}{n^{r'}} v_k.$$

Thus we can establish Theorem 7.1 by proving that the transformation defined by (7.17) is regular over the set of sequences which converge to zero. To prove the latter result, it is necessary as well as sufficient* to prove that

$$(7.18) \qquad \lim_{n\to\infty} k^{r'}(n-k)^{r'-1}/n^{r'} = 0 \qquad (k = 1, 2, 3, \cdots),$$

and that

$$(7.19) \qquad W_n = \sum_{k=1}^{n-1} k^{r'}(n-k)^{r'-1}/n^{r'} < M \quad (n = 2, 3, 4, \cdots),$$

for some constant M which may depend on r but must be independent of n. It is clear that (7.18) holds for any value of r. That (7.19) holds when $\Re(r) < 0$ follows from (5.20) and (5.21) since $W_n = \Phi_n(n)$. Thus Theorem 7.1 is proved.

8. **Relations between B_r and C_r.** The preceding results enable us to establish the following two theorems.

THEOREM 8.1. *If $\Re(r) < 0$, r not a negative integer, then C_r includes B_r.*

Suppose $\sum u_n$ is summable B_r to L so that $\lim \beta(t) = L$. Then by Theorem 5.1, (5.12) and (5.13) hold and we may use Theorem 7.1 to show that $\lim \gamma_n = L$. Thus Theorem 8.1 is proved.

THEOREM 8.2. *If $-1 < \Re(r) < 0$, then B_r includes C_r; if $\Re(r) \leqq -1$, B_r does not include C_r.*

Suppose $-1 < \Re(r) < 0$ and $\sum u_n$ is summable C_r to L. Then $\lim \gamma_n = L$. Since, as is well known, (7.11) is a necessary condition for summability C_r when $\Re(r) > -1$, we can apply Theorem 7.1 to obtain $\lim \alpha_n = L$; an application of Theorem 5.1 completes the proof of the first part of Theorem 8.2. To prove the second part suppose $\Re(r) \leqq -1$, and, of course, that r is not a negative integer. By Lemma 6.3, there is a series $\sum u_n$ summable C_r for which (5.12) fails; hence by Theorem (5.1), $\sum u_n$ is not summable B_r and the second part of Theorem 8.2 is proved.

Theorems 8.1 and 8.2 yield

* Kojima, Tôhoku Mathematical Journal, vol. 12 (1917), pp. 291–326; p. 300.

THEOREM 8.3. *If* $-1<\mathfrak{R}(r)<0$, *then* B_r *and* C_r *are equivalent; if* $\mathfrak{R}(r)\leqq -1$, B_r *and* C_r *are not equivalent.*

THEOREM 8.4. *If* r *is real and* >-1, *then* B_r *and* C_r *are equivalent.*

When $-1<r<0$, this is included in Theorem 8.3. When $r>0$, the result is included in Theorem 4.4.

Cesàro's method C_r of summability is, as is well known, not regular when $\mathfrak{R}(r)<0$. When $-1<\mathfrak{R}(r)<0$, C_r will evaluate only a subset of the set of all convergent series, and will evaluate no divergent series; hence, as might be expected, C_r occupies, for this range of values of r, a prominent place in the theory of series. On the other hand when r is real and <-1, C_r can evaluate to zero certain divergent series of positive terms (see, for example, §6). Owing to this fact, and also to the fact that many useful properties which hold when $\mathfrak{R}(r)>-1$ fail when $\mathfrak{R}(r)\leqq -1$, the method C_r has received little attention when $\mathfrak{R}(r)\leqq -1$.

It is of interest to note that Theorems 8.1 and 8.2 show that B_r is equivalent to C_r over precisely the range of values of r with negative real parts over which C_r has been useful, namely the range $-1<\mathfrak{R}(r)<0$.

In the next section, we will show that summability B_r is significant even when $\mathfrak{R}(r)<-1$.

9. **Relations between methods B_r of different orders.** In this section we prove six theorems on relations between methods B_r of different orders.

THEOREM 9.1. *If* $\mathfrak{R}(r)<-1$ *and* $\mathfrak{R}(r)\leqq\mathfrak{R}(s)$, *then* B_s *includes* B_r.

Let $\sum u_n$ be summable B_r to L so that $\lim \beta^{(r)}(t)=L$. Then, by Theorem 5.1, $\lim u_n/n^r=0$. We may write

$$\beta^{(r)}(t)-\beta^{(s)}(t)=\sum_{k=0}^{[t]-1}\left\{\left(1-\frac{k}{t}\right)^r-\left(1-\frac{k}{t}\right)^s\right\}u_k$$
$$=\sum_{k=1}^{[t]-1}k^r\left\{\left(1-\frac{k}{t}\right)^r-\left(1-\frac{k}{t}\right)^s\right\}u_k/k^r.$$

We see that Theorem 9.1 will follow if the transformation

$$W(t)=\sum_{k=1}^{[t]-1}k^r\left\{\left(1-\frac{k}{t}\right)^r-\left(1-\frac{k}{t}\right)^s\right\}w_k \tag{9.11}$$

is regular over the set of all sequences w_n which converge to zero.

Letting $d_k(t)$ represent the coefficient of w_k in (9.11), we have evidently

$$\lim_{t\to\infty} d_k(t)=0 \qquad (k=1,2,3,\cdots). \tag{9.12}$$

Also

$$\sum_{k=1}^{[t]-1} | d_k(t) | \leqq \sum_{k=1}^{[t-1]} k^{r'}\left\{\left(1 - \frac{k}{t}\right)^{r'} + \left(1 - \frac{k}{t}\right)^{s'}\right\} \leqq 2\sum_{k=1}^{[t]-1} k^{r'}\left(1 - \frac{k}{t}\right)^{r'}$$

where $r' = \Re(r)$ and $s' = \Re(s)$. Since $r' < -1$,

$$\sum_{k=1}^{[t/2]} k^{r'}\left(1 - \frac{k}{t}\right)^{r'} \leqq 2^{-r'} \sum_{k=1}^{[t/2]} k^{r'} < 2^{-r'} \sum_{k=1}^{\infty} k^{r'},$$

and

$$\sum_{k=[t/2]+1}^{[t]-1} k^{r'}\left(1 - \frac{k}{t}\right)^{r'} \leqq 2^{-r'} \sum_{k=[t/2]+1}^{[t]-1} (t - k)^{r'} < 2^{-r'} \sum_{k=1}^{\infty} k^{r'}.$$

Hence

$$\sum_{k=1}^{[t]-1} | d_k(t) | < 2^{-r'+2} \sum_{k=1}^{\infty} k^{r'}. \tag{9.13}$$

The conditions (9.12) and (9.13) ensure that (9.11) has the desired property and Theorem 9.1 is proved.

From Theorem 9.1, we obtain at once

THEOREM 9.2. *If* $\Re(r) = \Re(s) < -1$, *then* B_r *and* B_s *are equivalent.*

From Theorems 9.1 and 5.1 we obtain

THEOREM 9.3. *If* $\Re(r) < -1$ *and* $\sum u_n$ *is summable* B_r *to* L, *then* $\sum u_n$ *converges to* L, *the convergence being absolute.*

That $\sum u_n$ must converge to L follows from the fact that B_0, which includes B_r by Theorem 9.1, represents convergence. Again, by Theorem 5.1, $\lim u_n/n^r = 0$; hence $|u_n| < n^{r'}$, $r' = \Re(r) < -1$, for all sufficiently great n, and absolute convergence of $\sum u_n$ follows. Thus Theorem 9.3 is proved.

THEOREM 9.4. *If* $-1 < \Re(r) < \Re(s) < 0$, *then* B_s *includes* B_r. *If* $-1 < r < s$, *then* B_s *includes* B_r.

The first part of the Theorem follows from the fact (Theorem 8.3) that B_r and C_r are equivalent when $-1 < \Re(r) < 0$ and the fact that C_s includes C_r when $-1 < \Re(r) < \Re(s)$. The second part follows from a similar application of Theorem 8.4.

To complete Theorems 9.1 and 9.4, it would be desirable to determine whether B_s includes B_r when $-1 = \Re(r) < \Re(s) < 0$. Neither the method of proof of Theorem 9.1 nor that of Theorem 9.4 throws light on this question. A partial answer to this question is given by the following theorem.

THEOREM 9.5. *If* $-1=r<\Re(s)<0$, *then* B_s *includes* B_r.

We shall give a proof of Theorem 9.5 after having proved Theorem 10.1 below. After having proved Theorem 9.5, we can use Theorems 9.1, 9.4, and 9.5 to give a relation of inclusion between any two methods B_r of real orders, namely

THEOREM 9.6. *If* $r<s$, *then* B_s *includes* B_r.

10. **Consideration of** A_r. Since it is sometimes convenient to use transformations involving a continuous parameter, and at other times a discontinuous parameter, it is important to know whether A_r and B_r are equivalent, and whether the results which we have established for B_r hold also for A_r.

Using Theorem 8.4 and the result of Riesz that A_r and C_r are equivalent when $-1<r<1$, we see that A_r and B_r are equivalent when $-1<r<1$. We proceed to show that A_r and B_r have very different properties when $\Re(r)<-1$.

A series $\sum u_k$ is said to be summable by the Abel method P to L if $\sum u_k x^k$ converges for $|x|<1$ and $\lim_{x\to 1-}\sum u_k x^k=L$. We shall say that $\sum u_k$ is summable P^* to L if $\sum u_k x^k$ converges for all sufficiently small $|x|$ and generates an analytic function $u(x)$ such that $\lim_{x\to 1-} u(x)=L$.† It is evident that P^* includes P and that P does not include P^*.

THEOREM 10.1. *If* $\Re(r)\leqq -1$ *and* $r\neq -1$, *then* P^* *does not include* A_r; *if* r *is real and* $\geqq -1$, *then* P^* *includes* A_r.

Let $\sum u_k$ be summable A_r to L; then $\alpha_n\to L$ where

$$(10.11)\qquad (n+1)^r\alpha_{n+1}=\sum_{k=0}^{n}(n+1-k)^r u_k.$$

From (10.11) we obtain when $|x|<1$,

$$(10.12)\qquad \sum_{n=0}^{\infty}(n+1)^r\alpha_{n+1}x^n=\sum_{n=0}^{\infty}\sum_{k=0}^{n}(n+1-k)^r u_k x^n.$$

Letting $u(x)$ be the analytic function determined by the equation

$$(10.13)\qquad \sum_{n=0}^{\infty}(n+1)^r\alpha_{n+1}x^n=u(x)\sum_{n=0}^{\infty}(n+1)^r x^n,$$

we see that when $|x|$ is sufficiently small, say $|x|<\delta$, $u(x)$ is a convergent power series in x. A comparison of (10.12) and (10.13) suffices to show that

† This modification of Abel's method was introduced by Silverman-Tamarkin, Mathematische Zeitschrift, vol. 29 (1928), pp. 161–170; p. 169.

$$u(x) = \sum_{k=0}^{\infty} u_k x^k, \qquad |x| < \delta; \tag{10.14}$$

hence $u(x)$ is the analytic function generated by $\sum u_k x^k$. That P^* includes A_r when r is real and $\geqq -1$ follows at once from the conditions for regularity† of the transformation defined by (10.13); this also follows from a result of Silverman-Tamarkin, loc.cit.

We shall prove the first part of Theorem 10.1 by a method which shows that P^* and A_r are inconsistent when $\mathfrak{R}(r) < -1$. Corresponding to each complex r, let $\sum u_n^{(r)}$ be the series having for its A_r transform the sequence $\alpha_1 = 1$, $\alpha_n = 0$ when $n > 1$. Then $\sum u_n^{(r)}$ is summable A_r to 0. Using (10.14) and (10.13), we see that the analytic function $u^{(r)}(x)$ generated by $\sum u_k^{(r)} x^k$ is given by

$$u^{(r)}(x) \sum_{n=0}^{\infty} (n+1)^r x^n = 1. \tag{10.15}$$

But when $r = -1 - ih$, h real and $\neq 0$, we have as $x \to 1^-$‡

$$\sum_{n=0}^{\infty} (n+1)^{-1-ih} x^n \sim \Gamma(ih) \{\log (1/x)\}^{ih}$$

so that $\lim_{x\to 1^-} u^{(r)}(x)$ does not exist and $\sum u_n^{(r)}$ is non-summable P^*. On the other hand, if $\mathfrak{R}(r) < -1$, then $\sum (n+1)^r$ converges to $\zeta(-r)$ which is finite and different from zero; hence $\sum u_n^{(r)}$ is summable P^* to $1/\zeta(-r)$ which is finite and different from the A_r value of $\sum u_n^{(r)}$. Thus Theorem 10.1 is proved.

We pass now to a proof of Theorem 9.5. Let $\sum u_k$ be summable B_{-1} to L. Then by Theorem 5.1, $nu_n \to 0$ and $\sum u_k$ is summable A_{-1} to L. Then by Theorem 10.1, $\sum u_k$ is summable P^* to L. But $\sum u_k x^k$ must converge when $|x| < 1$ since $nu_n \to 0$; hence $\sum u_k$ is summable P to L. Therefore, by Tauber's Theorem§ $\sum u_k$ must converge to L. Since $nu_n \to 0$ and $\sum u_k$ converges to L, it follows‖ that $\sum u_k$ is summable C_s for every s with $\mathfrak{R}(s) > -1$. Finally summability B_s for every s with $-1 < \mathfrak{R}(s) < 0$ follows from Theorem 8.3 and Theorem 9.5 is proved.

We have shown in the proof of Theorem 10.1 that when $\mathfrak{R}(r) < -1$, the transformation A_r can evaluate to 0 a series which is not summable P to 0 and which is therefore not convergent to 0. Using this result and Theorem 9.3, we obtain

† W. A. Hurwitz, loc. cit., p. 20.

‡ Lindelöf, *Le Calcul des Résidus*, p. 139.

§ A. Tauber, Monatshefte für Mathematik und Physik, vol. 8 (1897), pp. 273–277.

‖ Hardy and Littlewood, Proceedings of the London Mathematical Society, (2), vol. 11 (1912), p. 462.

THEOREM 10.2. *If* $\Re(r) < -1$, *then* A_r *and* B_r *are not equivalent.*

Theorem 10.1 also shows that the methods A_r do not, in contrast to the methods B_r, form for real values of r a set of consistent methods of summability whose effectiveness increases steadily as r increases.

We can now see that (5.12) is not a consequence of (5.13) when $\Re(r) < -1$ by proving

THEOREM 10.3. *When* $\Re(r) < -1$, *the condition* $u_n/n^r \to 0$ *is not necessary in order that* $\sum u_n$ *may be summable* A_r.

If the condition were necessary, it would follow from Theorem 5.1 that A_r and B_r would be equivalent and Theorem 10.2 would be contradicted.

In the next section, we give a theorem which is interesting in connection with Theorem 10.3, and give further properties of A_r.

11. Consideration of A_r when $\Re(r) < \zeta$. Let ζ, $-2 < \zeta < -1$, be the real negative root of the equation.

$$2^r + 3^r + 4^r + \cdots = 1. \tag{11.01}$$

We shall now prove

THEOREM 11.1. *If* $\Re(r) < \zeta$ *and* $\sum u_n$ *is bounded* A_r, *then* u_n/n^r *is bounded for all* $n > 0$.†

Let $\sum u_n$ be bounded A_r, $\Re(r) < \zeta$, so that α_n, being defined by

$$\alpha_n = \sum_{k=0}^{n-1} \left(1 - \frac{k}{n}\right)^r u_k, \tag{11.11}$$

is a bounded sequence. Since $r' = \Re(r) < \zeta$, it follows that

$$0 < \theta_r = 2^{r'} + 3^{r'} + 4^{r'} + \cdots < 1. \tag{11.12}$$

Choose an index $p > 1$ so great that

$$2^{-r'+1} \sum_{k=p}^{\infty} k^{r'} < (1 - \theta_r)/2 \tag{11.13}$$

and let a sequence v_n be defined by the formulas $v_n = 0$, $n < p$; $v_p = u_0 + u_1 + \cdots + u_p$; and $v_n = u_n$, $n > p$. Then

$$\alpha_n = o(1) + \sum_{k=p}^{n-1} \left(1 - \frac{k}{n}\right)^r v_k = o(1) + \sum_{k=p}^{n-1} k^r \left(1 - \frac{k}{n}\right)^r (v_k/k^r).$$

Hence we can prove Theorem 10.1 by showing that boundedness of W_n

† $\sum u_n$ is said to be bounded A_r when its A_r transform is a bounded sequence.

implies boundedness of w_n whenever

$$W_n = \sum_{k=p}^{n} k^r \left(1 - \frac{k}{n+1}\right)^r w_k, \qquad n > p. \tag{11.14}$$

Let d_{nk} represent the coefficient of w_k in (11.14). Then when $n > 2p$,

$$\sum_{k=p}^{n-p-1} |d_{nk}| \leqq 2^{-r'+1} \sum_{k=p}^{\infty} k^{r'} < (1 - \theta_r)/2; \tag{11.15}$$

the first inequality being obtained by considering separately the sums when k ranges from p to $[(n+1)/2]$ and from $[(n+1)/2]+1$ to $n-p-1$. Also

$$\lim_{n\to\infty} \sum_{k=n-p}^{n-1} |d_{nk}| = \sum_{k=2}^{p} k^{r'} < \theta_r. \tag{11.16}$$

Combining (11.15) and (11.16), we obtain

$$\limsup_{n\to\infty} \sum_{k=p}^{n-1} |d_{nk}| \leqq (1 + \theta_r)/2 < 1. \tag{11.17}$$

Since

$$\lim_{n\to\infty} d_{n,n} = 1, \tag{11.18}$$

we may use (11.17) and the fact that $d_{nk} = 0$ when $k < p$ to obtain

$$\liminf_{n\to\infty} \left\{ |d_{n,n}| - \sum_{k=0}^{n-1} |d_{nk}| \right\} > 0. \tag{11.19}$$

Owing to (11.19), the fact that the transformation (11.14) has the desired property results from the following lemma.

LEMMA 11.2. *If the coefficients in the transformation*

$$W_n = \sum_{k=0}^{n} d_{nk} w_k \tag{11.21}$$

satisfy (11.19) *and if W_n is a bounded sequence, then w_n is a bounded sequence.*

To prove this lemma, let w_n be an unbounded sequence; we shall show that W_n is an unbounded sequence. Since w_n is unbounded, we can choose an increasing sequence n_j of indices such that $|w_{n_j}| \geqq |w_k|$ when $0 \leqq k < n_j$. Then

$$\begin{aligned} |W_{n_j}| &\geqq - \sum_{k=0}^{n_j - 1} |d_{n_j k}| \, |w_k| + |d_{n_j n_j}| \, |w_{n_j}| \\ &\geqq \left\{ |d_{n_j n_j}| - \sum_{k=0}^{n_j-1} |d_{n_j k}| \right\} |w_{n_j}| . \end{aligned}$$

But lim $|w_{n_j}| = +\infty$ and using (10.19) we see that lim $|W_{n_j}| = +\infty$. Hence W_n is an unbounded sequence, Lemma 11.2 is proved, and Theorem 11.1 follows.

THEOREM 11.2. *If $\mathfrak{R}(r) < \zeta$, every series summable A_r is convergent, but not necessarily to the value to which it is summable.*

That a series summable A_r must be convergent follows from Theorem 11.1; in fact boundedness A_r is sufficient to ensure absolute convergence of $\sum u_n$. That the A_r and convergence values need not be equal is shown by the series $\sum u_n^{(r)}$ used in the proof of Theorem 10.1.

Since C_r can evaluate certain divergent series when $\mathfrak{R}(r) < -1$, r not a negative integer, and A_r can evaluate only absolutely convergent series when $\mathfrak{R}(r) < \zeta$, it follows that C_r and A_r are not equivalent when $\mathfrak{R}(r) < \zeta$.

The methods A_r, $\mathfrak{R}(r) < \zeta$, may be of use for classification of convergent series; but use of such methods for evaluation of series is open to the objection that they are, by Theorem 11.2, inconsistent with convergence.

12. **Conclusion.** In conclusion we point out that while A_r, B_r, and C_r are not mutually equivalent when $\mathfrak{R}(r) < -1$, there is an important class of series over which these methods are equivalent. In fact, a combination of Theorems 5.1 and 7.1 yields the following theorem.

THEOREM 12.1. *If $\mathfrak{R}(r) < 0$, being $\neq -1, -2, \cdots$ when C_r is involved, and* lim $u_n/n^r = 0$, *and if $\sum u_n$ is summable by one of the methods A_r, B_r, and C_r, then it is summable to the same value by the other two methods.*

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