CONVERGENCE IN VARIATION AND RELATED TOPICS†

BY

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1. **Introduction.** In recent papers‡ by Adams and Clarkson and by Adams and Lewy the notions of convergence in variation and convergence in length have been examined. In AC it has been shown that if a sequence converges in variation and satisfies certain further restrictions which are clearly needed, the sequence of reciprocals converges in variation. The central purpose of the present paper is to determine so far as we are able the transformations which when applied to sequences of functions, preserve various types of convergence, such as convergence in variation or length and other types which we shall introduce. This paper also leads us to certain generalizations of results in AC and AL, such as Theorem 5.4 wherein convergence in length is seen to be invariant under addition and multiplication when *only one* of the limit functions is absolutely continuous.

In §2 we assemble certain preliminary definitions, notations, and conventions. §3 is devoted to preliminary theorems and lemmas, among them being Theorems 3.1 and 3.2 which might be of interest in themselves, their full power, in fact, not being used in this paper. Theorem 3.2 is a substitution theorem for Lebesgue integrals which is more general than other theorems of this type known to us in literature. Certain results in §3 are, however, obvious analogues of results in AC. Transforms of sequences are discussed in §4, wherein Theorems 4.1 and 4.2 form the kernel of the paper. The remainder of the paper consists largely of various applications of these two theorems, convergence in length being discussed in §5 together with convergence almost in the mean, uniform convergence in length in §6, and strong convergence in §7. Certain miscellaneous applications are made in §8; these include Theorem 8.1 which points out a necessary and sufficient condition for convergence in the mean, and Theorems 8.3 and 8.4 which are generalizations of a theorem of Plessner.

2. **Notation; preliminary definitions and conventions.** In this paper we shall consistently use x, y, t to denote real numbers or variables, and use the

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‡ Adams and Clarkson, *On convergence in variation*, Bulletin of the American Mathematical Society, vol. 40 (1934), pp. 413–417. Adams and Lewy, *On convergence in length*, Duke Mathematical Journal, vol. 1 (1935), pp. 19–26. Hereinafter these papers will be referred to as AC and AL, respectively.

letters X, Y, ξ, ψ, Ψ, η, u, v, U, V with or without subscripts to denote real-valued functions. All other functions are to be regarded as complex functions of a real variable unless the contrary is expressly stated. We shall also employ a and b with or without subscripts to designate real numbers with $a<b$.

If Q is a condition involving x, then $\underset{x}{E}[Q]$ is a set defined as follows: A point x belongs to $\underset{x}{E}[Q]$ if x satisfies the condition Q. We use the notation $[x_1, x_2]$ to denote a closed interval.

If f is a function and if $[a, b]$ is included in its domain, then the symbol $T_a^b(f)$ (read the total variation from a to b of f) will be used to denote the least upper bound (finite or infinite) of numbers of the form

$$\sum_{j=1}^{k} | f(t_j) - f(t_{j-1}) |,$$

where $a=t_0<t_1<t_2< \cdots <t_k=b$. We define $T_b^a(f)=-T_a^b(f)$ and $T_a^a(f)=0$. If $T_a^b(f)<\infty$, then f is said to be of b.v. (bounded variation) on $[a, b]$. Since it will sometimes be necessary to display the variable with respect to which the total variation is taken, we employ the notation $T_{t=a}^{b}f(t)$ as an alternate for $T_a^b(f)$.

We shall use a.c. as an abbreviation for absolute continuity and employ p.p. to denote almost everywhere (presque partout), and designate the outer measure of a set R by $|R|$. It will also be convenient to refer to Euclidean space of n dimensions as simply n-space. Furthermore, a function will be said to be increasing on a set if it is strictly increasing there, a function will be said to be monotone on a set if it is either non-increasing or non-decreasing there.

The following convention, will be used throughout the paper. If f is defined on $[a, b]$, then the function f' is defined on $[a, b]$ by the following relations:

$f'(t)=$ the derivative of f at t wherever it exists finite,

$f'(t)=0$ for all other t on $[a, b]$.

We also agree: If $[a, b]$ is the domain of f and if $\lim_{h\to 0+}f(t+h)$ exists for $a\leqq t<b$ and if $\lim_{h\to 0+}f(t-h)$ exists for $a<t\leqq b$, then $f(t+)$ and $f(t-)$ are defined for t on $[a, b]$ as follows:

$$f(a-) = f(a); \quad f(b+) = f(b); \quad f(t+) = \lim_{h\to 0+} f(t+h), \quad a \leqq t < b;$$
$$f(t-) = \lim_{h\to 0+} f(t-h), \quad a < t \leqq b.$$

We shall denote by CR the set of all finite-valued complex functions whose domain is $[0, 1]$. If f and g are any points in CR and if α is any complex number, then by $f+g$ is meant that point G in CR such that $G(t)=f(t)+g(t)$ for t on $[0, 1]$, by $f\cdot g$ is meant that point G in CR such that $G(t)=f(t)\cdot g(t)$ for t on

$[0, 1]$, by αf is meant that point G in CR such that $G(t)=(\alpha)\cdot f(t)$ for t on $[0, 1]$; and, provided f does not vanish on $[0, 1]$, by α/f is meant that point G in CR such that $G(t)=\alpha/f(t)$ for t on $[0, 1]$. We denote by I and θ the elements of CR defined respectively by

$$I(t) = t, \qquad \theta(t) = 0; \qquad 0 \leqq t \leqq 1.$$

If f is in CR, then $\|f\|$, read norm of f, is defined as $|f(0)|+T_0^1(f)$. The space BV is a subspace of CR defined by $BV=(CR)\cdot \underset{f}{E}[\|f\|<\infty]$ and the subspace RBV is defined by $RBV=(BV)\cdot \underset{f}{E}$ [f is real].

Totally distinct from these spaces and used merely for convenience is the space CC defined as the space of continuous functions on the finite complex plane to the finite complex plane.

In concluding this section we lay down the following definitions.

Definition 2.1. If f is a point in CR and ϕ is a function whose domain includes the range of f and whose range is included in the set of finite complex numbers (finite complex plane), then $\phi:f$ is defined to be that point G in CR for which $G(t)=\phi\{f(t)\}$, $0\leqq t\leqq 1$.

Definition 2.2. If Y is a real point in CR and u is a function on a part of two-space to one-space and if $u(t, Y(t))$ is defined for t on $[0, 1]$, then by $(u|Y)$ is meant the real point Ψ in CR such that $\Psi(t)=u(t, Y(t))$, $0\leqq t\leqq 1$.

Definition 2.3. *Convergence in variation*. By $f_n-v\rightarrow f_0$, read f_n converges in variation to f_0, is meant this: f_n is in BV for $n=0, 1, 2, \cdots$; $f_n(t)\rightarrow f_0(t)$ for t on $[0, 1]$; $\|f_n\|\rightarrow\|f_0\|$.

Definition 2.4. *Uniform convergence in variation*. By $f_n-uv\rightarrow f_0$ is meant this: $f_n-v\rightarrow f_0$ with $f_n(t)\rightarrow f_0(t)$ uniformly for t on $[0, 1]$.

Definition 2.5. *Convergence in length*. By $Y_n-l\rightarrow Y_0$ is meant this: Y_n is in RBV for $n=0, 1, 2, \cdots$; $(I+iY_n)-v\rightarrow(I+iY_0)$.

Definition 2.6. *Uniform convergence in length*. By $Y_n-ul\rightarrow Y$ is meant this: $Y_n-l\rightarrow Y_0$; $Y_n(t)\rightarrow Y_0(t)$ uniformly for t on $[0, 1]$.

Definition 2.7. *Strong convergence*. By $f_n-s\rightarrow f_0$ is meant this: f_n is in BV for $n=0, 1, 2, \cdots$; $\|f_n-f_0\|\rightarrow 0$.

Definition 2.8. If α is a point in n_1-space with $\alpha=(\alpha_1, \alpha_2, \cdots, \alpha_{n_1})$ and β a point in n_2-space with $\beta=(\beta_1, \beta_2, \cdots, \beta_{n_2})$, then $\alpha\circ\beta$ is the point $(\alpha_1, \alpha_2, \cdots, \alpha_{n_1}, \beta_1, \beta_2, \cdots, \beta_{n_2})$ in (n_1+n_2)-space.

3. **Preliminary results.** In this section certain preliminary results will be actually proved while others which are quite simple or well known will simply be stated both for completeness and for use later.

From the definition of total variation it is clear that if $a\leqq c\leqq b$ with f defined on $[a, b]$, then $T_a^b(f)=T_a^c(f)+T_c^b(f)$. Another corollary is the following semi-continuity property: the relation $f_n(t) \to f_0(t)$ for t on $[a, b]$ implies $\liminf_{n\to\infty} T_a^b(f_n) \geqq T_a^b(f_0)$. It is likewise easily verified that the relation $f_n - v \to f_0$ with f_0 continuous implies $f_n - uv \to f_0$ (see AC, Theorem 2 and corollary to Theorem 4).

Of considerable use is the following

LEMMA 3.1. *The relations $f_n - v \to f_0$ and $f_n - uv \to f_0$ imply respectively the relations*† $S_n(t) \to S_0(t)$ *for t on* $[0, 1]$; $S_n(t) \to S_0(t)$, *uniformly for t on* $[0, 1]$ *where $S_n(t) = T_0^t(f_n)$ for t on* $[0, 1]$.

Several properties of the norm in CR are now set forth in the following lemma.

LEMMA 3.2. *If f and g are in CR and α is a complex number, then*

$$\|f + g\| \leqq \|f\| + \|g\|, \quad \|\alpha f\| = |\alpha| \cdot \|f\|, \quad \|fg\| \leqq \|f\| \cdot \|g\|.$$

The first two relations are quite simple. We sketch a proof of the last which is seen to reduce to proving

$$\left(\sum_{n=0}^{N} |a_n - a_{n-1}|\right) \cdot \left(\sum_{n=0}^{N} |b_n - b_{n-1}|\right) \geqq \sum_{n=0}^{N} |a_n b_n - a_{n-1} b_{n-1}|,$$

where a_n and b_n are complex numbers ($n = 0, 1, \cdots, N$) and $|a_{-1}| + |b_{-1}| = 0$. Suppose the above relation, which is obviously true if N is replaced by 0, to be true for N replaced by an integer k with $0 \leqq k < N$. Then it follows that

$$\left(|a_{k+1} - a_k| + \sum_{n=0}^{k} |a_n - a_{n-1}|\right) \cdot \left(|b_{k+1} - b_k| + \sum_{n=0}^{k} |b_n - b_{n-1}|\right)$$

$$= |a_{k+1} - a_k| \cdot |b_{k+1} - b_k| + |a_{k+1} - a_k| \cdot \sum_{n=0}^{k} |b_n - b_{n-1}|$$

$$+ |b_{k+1} - b_k| \cdot \sum_{n=0}^{k} |a_n - a_{n-1}| + \left(\sum_{n=0}^{k} |a_n - a_{n-1}|\right) \cdot \left(\sum_{n=0}^{k} |b_n - b_{n-1}|\right)$$

$$\geqq |(a_{k+1} - a_k)(b_{k+1} - b_k) + (a_{k+1} - a_k) b_k + (b_{k+1} - b_k) a_k|$$

$$+ \sum_{n=0}^{k} |a_n b_n - a_{n-1} b_{n-1}| = \sum_{n=0}^{k+1} |a_n b_n - a_{n-1} b_{n-1}|.$$

This induction completes the proof.

The following two lemmas are particular cases of Minkowski's inequality.

† AC, corollaries to Theorems 2 and 5. The proof given there holds equally well for the functions considered here.

LEMMA 3.3. *If* $0 \leqq a_j \leqq b_j$ (*for* $j=1, 2, \cdots, k$), *then*

$$\sum_{j=1}^{k} (b_j^2 - a_j^2)^{1/2} \leqq \left[\left(\sum_{j=1}^{k} b_j \right)^2 - \left(\sum_{j=1}^{k} a_j \right)^2 \right]^{1/2}.$$

LEMMA 3.4. *If X and Y are summable on* $[a, b]$, *then*

$$\int_a^b \{ | X(t) | + | Y(t) | \} dt \geqq \int_a^b (\{ X(t)\}^2 + \{Y(t)\}^2)^{1/2} dt$$
$$\geqq \left\{ \left(\int_a^b | X(t) | \, dt \right)^2 + \left(\int_a^b | Y(t) | \, dt \right)^2 \right\}^{1/2}.$$

LEMMA 3.5. *If X is a function defined on* $[a, b]$, *then* X' *is measurable on* $[a, b]$.

This is a corollary of a theorem found in Saks, *Théorie de l'Intégrale* (Chapter 3, p. 47, Theorem 1).

LEMMA 3.6. *Let* $[a, b]$ *be the domain of the function* X *and denote* $\underset{t}{E}[|X'(t)| > 0]$ *by* P. *If* D *is a set of measure* 0, *then the set* $P \cdot \underset{t}{E}[X(t) \varepsilon D]$ *is likewise of measure* 0.

Let $Q = \underset{t}{E}[X(t)\varepsilon D]$ and suppose $|PQ| > 0$. Define $P_1 = \underset{t}{E}[X'(t) > 0]$. There is no loss of generality in assuming $|P_1 Q| > 0$. P_1 being measurable, it is easily established that there exist positive numbers ϵ_1, ϵ_2, and a closed set C such that

$$C \subset P_1, \quad |P_1 - C| < |P_1 Q|, \quad \frac{X(t_2) - X(t_1)}{t_2 - t_1} \geqq \epsilon_1$$

if $t_1 \varepsilon C$, $t_2 \varepsilon C$, and $0 < |t_2 - t_1| < \epsilon_2$. Thus $|CQ| > 0$, so that there exists an interval $[a_0, b_0]$ of length $< \epsilon_2$ for which $|CQ[a_0, b_0]| > 0$. Defining $C^* = C \cdot [a_0, b_0]$ and $Q^* = C^* \cdot Q$ we observe $|Q^*| > 0$ and C^* is closed, so that if X_1 is defined on C^* so as to coincide with X and defined on the remainder of $[a_0, b_0]$, which is made up of a set of non-overlapping intervals, by linear interpolation then it appears that X_1 is continuous and increasing on $[a_0, b_0]$ with

$$|X_1(t_2) - X(t_1)| \geqq \epsilon_1 |t_2 - t_1|, \ (a_0 \leqq t_1, t_2 \leqq b_0); \ X_1(t) = X(t), \ (t \varepsilon C^*).$$

The continuity of X_1 follows from the existence of the derivative of X at all points of C^*. Let τ be defined on the interval $[X_1(a_0), X_1(b_0)]$ by the relation $\tau\{X_1(t)\} = t$ $(a_0 \leqq t \leqq b_0)$. Clearly τ satisfies a Lipschitz condition on $[X_1(a_0), X_1(b_0)]$, and thus transforms the set $D \cdot [X_1(a_0), X_1(b_0)]$ which is of measure 0, into a set D^* of measure 0. But

$$Q^* = C^*Q = C^*\underset{t}{E}[X(t)\varepsilon D] = C^*\underset{t}{E}[X_1(t)\varepsilon D] = C^*D^*.$$

Hence the contradiction $|Q^*| = 0$.

COROLLARY 3.1. *Let $[a, b]$ be the domain of X and let $[\alpha, \beta]$ include its range. If ψ_1 and ψ_2 are finite-valued functions defined on $[\alpha, \beta]$ and equal p.p. there, then for almost all t on $[a, b]$, $\psi_1\{X(t)\}X'(t) = \psi_2\{X(t)\}X'(t)$.*

THEOREM 3.1. *Let $[a, b]$ be the domain of the function X; let $[\alpha, \beta]$ include its range; let Ψ be a.c. on $[\alpha, \beta]$ and denote $\Psi\{X(t)\}$ by $G(t)$ for t on $[a, b]$. If X has a derivative p.p. on $[a, b]$, then*

$$G'(t) = \Psi'\{X(t)\}X'(t)$$

for almost all t on $[a, b]$.

Let it first be noted that the theorem does not answer the question as to whether or not G is differentiable p.p.† We now proceed with the proof of the theorem.

Let $R = \underset{t}{E}[X'(t) = 0]$. If $|R| = 0$ the next paragraph by itself furnishes a proof. However, in case $|R| > 0$, it may be noted first that corresponding to $\epsilon(>0)$ there exist positive numbers M, ϵ_1 and a closed subset C, of R, such that $|R - C| < \epsilon$ with $|X(t_2) - X(t_1)| \leqq M|t_2 - t_1|$ if $t_1 \varepsilon C$, $t_2 \varepsilon C$, and $|t_2 - t_1| < \epsilon_1$. Likewise readily verified is the existence of a function X_1 satisfying a Lipschitz condition on $[a, b]$ (the constant involved may be $> M$), and in addition fulfilling: $X_1(t) = X(t)$ for $t \varepsilon C$. Now X_1 and X both transform C into a measurable (closed) set C' with

$$|C'| \leqq \int_C |X_1'(t)|\, dt = \int_C |X'(t)|\, dt = 0.$$

But Ψ being a.c. on $[\alpha, \beta]$, transforms C' into a set C'' likewise of measure 0. Hence G transforms C into C'', a set of measure 0, so that $G'(t) = 0$ for almost all t in C; for supposing the contrary leads immediately to a contradiction of Lemma 3.6. Hence, since ϵ was arbitrary, we conclude $G'(t) = 0$ for almost all t in R. Thus $G'(t) = \Psi'\{X(t)\}X'(t)$ for almost all t in R.

However, upon denoting $\underset{t}{E}[|X'(t)| > 0]$ by P, it is seen from Lemma 3.6 that

$$\lim_{h \to 0} \frac{\Psi\{X(t+h)\} - \Psi\{X(t)\}}{X(t+h) - X(t)} = \Psi'\{X(t)\}$$

† In fact, from the work of N. Bary, Mathematische Annalen, vol. 103 (1930), p. 611, a definite answer to this question can be given. Let F be a continuous function nowhere differentiable on $[0, 1]$. There then exist functions G_1, G_2, G_3 and a.c. functions Ψ_1, Ψ_2, Ψ_3, ϕ_1, ϕ_2, ϕ_3 such that

$$F(t) = G_1(t) + G_2(t) + G_3(t), \quad G_j(t) = \Psi_j\{\phi_j(t)\} \quad (j = 1, 2, 3;\ 0 \leqq t \leqq 1).$$

Thus at least one of the functions G_1, G_2, G_3 fails to have a derivative on a set of measure >0.

for almost all t in P. Hence $G'(t)=\Psi'\{X(t)\}X'(t)$ for almost all t on $[a, b]$.†

COROLLARY 3.2. *Let $[a, b]$ be the domain of X whereon it is a.c. and let $[\alpha, \beta]$ include its range. If Ψ satisfies a Lipschitz condition on $[\alpha, \beta]$ or if Ψ is a.c. on $[\alpha, \beta]$ with X monotone on $[a, b]$, then the function G defined by $G(t)=\Psi\{X(t)\}$ $(a\leqq t\leqq b)$, is a.c. on $[a, b]$ with $G'(t)=\Psi'\{X(t)\}$ $X'(t)$ for almost all t on $[a, b]$.*

COROLLARY 3.3. *Let $[a, b]$ be the domain of X; let $[\alpha, \beta]$ include its range; let Ψ be a.c. on $[\alpha, \beta]$; and denote $\Psi\{X(t)\}$ by $G(t)$ for t on $[a, b]$. If X has a vanishing derivative p.p. on $[a, b]$, then a necessary and sufficient condition that G is a.c. on $[a, b]$ is that Ψ is constant on the range of X.*

Now combining Theorem 3.1 with Corollary 3.1 we obtain

THEOREM 3.2. *Let X be defined on $[a, b]$ and differentiable p.p. there; let $[\alpha, \beta]$ include the range of X; let ψ be defined, finite-valued, and summable on $[\alpha, \beta]$ and denote $\int_\alpha^x\psi(s)ds$ by $\Psi(x)$ for x on $[\alpha, \beta]$; finally let $G(t)=\Psi\{X(t)\}$ for t on $[a, b]$. Under these circumstances*

$$\int_{X(a)}^{X(t)} \psi(x)dx = \int_a^t \psi\{X(s)\}X'(s)ds \qquad (a \leqq t \leqq b)$$

if and only if G is a.c. on $[a, b]$.‡

LEMMA 3.7. *If f is defined on $[a, b]$ then*

$$T_a^b(f) \geqq \int_a^b |f'(t)|\, dt,$$

the sign of equality holding if f is a.c. on $[a, b]$.

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† A slight modification in proof establishes:

If Ψ satisfies Lusin's condition N on $[\alpha, \beta]$ (i.e., transforms sets of measure 0 into sets of measure 0) and if X is continuous on $[a, b]$ in addition to being differentiable p.p. there, then $G'(t)=\Psi'\{X(t)\}X'(t)$ for almost all t on $[a, b]$.

It may also be noted that a slight simplification in proof may be achieved by use of a theorem of N. Bary, loc. cit., p. 190.

‡ This theorem is a generalization of results previously obtained by de la Vallée Poussin, these Transactions, vol. 16 (1915), p. 466, and by Fichtenholz, Bulletin de l'Académie Royale de Belgique, Classe des Sciences, ser. 5, vol. 8 (1922), p. 441. In the notation of the present theorem it was shown by de la Vallée Poussin that absolute continuity of X implies the equivalence of the following three statements:

(i) $\int_{X(a)}^{X(b)}\psi(x)dx=\int_a^b\psi\{X(s)\}X'(s)ds$;

(ii) G is a.c.;

(iii) $\int_a^b\psi\{X(s)\}X'(s)ds$ exists.

On the other hand Fichtenholz showed that absolute continuity of G together with monotonicity and continuity of X implies (i).

LEMMA 3.8. *Let σ be monotone and continuous on $[a, b]$. If f is defined on $[\sigma(a), \sigma(b)]$, then*

$$T_{t=a}^{b} f\{\sigma(t)\} = \left| T_{\sigma(a)}^{\sigma(b)}(f) \right|.$$

LEMMA 3.9. *If f is a.c. on $[a, b]$ with $f'(t)=\mu$ (constant) for almost all t on $[a, b]$, then*

$$\int_a^b |f'(t) - \Delta\delta^{-1}|\, dt \leqq (\mu^2\delta^2 - |\Delta|^2)^{1/2} + 2(\mu\delta - |\Delta|),$$

where $\delta=b-a$ and $\Delta=f(b)-f(a)$.

The lemma is clearly true in case $\Delta=0$. In the alternative case we define

$$F(t) = |\Delta|\,(f(t) - f(a))\Delta^{-1} = \xi(t) + i\eta(t) \qquad (a \leqq t \leqq b).$$

Now (Lemma 3.4)

$$\begin{aligned}\mu\delta &= \int_a^b |f'(t)|\, dt = \int_a^b |F'(t)|\, dt = \int_a^b (\{\xi'(t)\}^2 + \{\eta'(t)\}^2)^{1/2} dt \\ &\geqq \left(\left\{\int_a^b |\xi'(t)|\, dt\right\}^2 + \left\{\int_a^b |\eta'(t)|\, dt\right\}^2\right)^{1/2} \\ &\geqq \left(\left\{\int_a^b \xi'(t) dt\right\}^2 + \left\{\int_a^b |\eta'(t)|\, dt\right\}^2\right)^{1/2} \\ &= \left(|\Delta|^2 - \left\{\int_a^b |\eta'(t)|\, dt\right\}^2\right)^{1/2}.\end{aligned}$$

whence $(\mu^2\delta^2 - |\Delta|^2) \geqq \{\int_a^b |\eta'(t)|\, dt\}^2$, so that from the relations

$$\int_a^b \left| |\Delta|\, f'(t)\Delta^{-1} - \xi'(t) \right| dt = \int_a^b |\eta'(t)|\, dt,$$

$$\int_a^b |\xi'(t) - \mu|\, dt = \int_a^b \{\mu - \xi'(t)\} dt = \mu\delta - |\Delta|,$$

$$\int_a^b \left| \mu - |\Delta|\, \delta^{-1} \right| dt = \mu\delta - |\Delta|,$$

follows the relation

$$\begin{aligned}\int_a^b |f'(t) - \Delta\delta^{-1}|\, dt &= \int_a^b \left| |\Delta|\, f'(t)\Delta^{-1} - |\Delta|\, \delta^{-1} \right| dt \\ &\leqq (\mu^2\delta^2 - |\Delta|^2)^{1/2} + 2(\mu\delta - |\Delta|).\end{aligned}$$

THEOREM 3.3. *If f_n is an a.c. function in BV with $|f_n'(t)| = \mu_n$ for almost all t on* [0, 1] *($n=0, 1, 2, \cdots$), then the relation $f_n - v \to f_0$ implies $\|f_n - f_0\| \to 0$.*

From

$$\mu_n = \int_0^1 |f_n'(t)|\, dt = T_0^1(f_n) \qquad (n = 0, 1, 2, \cdots)$$

we conclude $\mu_n \to \mu_0$. Let ϵ be any positive number. There exists a partition $(0 = t_0 < t_1 < t_2 < \cdots < t_k = 1)$ such that $0 \leqq \mu_0 - \sum_{j=1}^{k} |f_0(t_j) - f_0(t_{j-1})| < \epsilon$. Letting $\Delta_{n,j} = f_n(t_j) - f_n(t_{j-1})$ and $\delta_j = t_j - t_{j-1}$, ($n = 0, 1, 2, \cdots; j = 1, 2, \cdots, k$), we have the obvious relation

$$\sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} |\Delta_{n,j}\delta_j^{-1} - \Delta_{0,j}\delta_j^{-1}|\, dt = \sum_{j=1}^{k} |\Delta_{n,j} - \Delta_{0,j}| \qquad (n = 0, 1, 2, \cdots).$$

Use of Lemma 3.9 and Lemma 3.3 yields (for $n = 0, 1, 2, \cdots$)

$$\sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} |f_n'(t) - \Delta_{n,j}\delta_j^{-1}|\, dt \leqq \sum_{j=1}^{k} (\mu_n^2\delta_j^2 - |\Delta_{n,j}|^2)^{1/2} + \sum_{j=1}^{k} 2(\mu_n\delta_j - |\Delta_{n,j}|) \leqq K_n,$$

where

$$K_n = \left\{\mu_n^2 - \left(\sum_{j=1}^{k} |\Delta_{n,j}|\right)^2\right\}^{1/2} + 2\left\{\mu_n - \sum_{j=1}^{k} |\Delta_{n,j}|\right\}.$$

Combining these last two relations with the relation obtained by setting $n = 0$ in the last relation we conclude

$$\int_0^1 |f_n'(t) - f_0'(t)|\, dt = \sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} |f_n'(t) - f_0'(t)|\, dt$$

$$\leqq K_n + K_0 + \sum_{j=1}^{k} |\Delta_{n,j} - \Delta_{0,j}| \qquad (n = 0, 1, 2, \cdots).$$

Hence

$$\limsup_{n\to\infty} \int_0^1 |f_n'(t) - f_0'(t)|\, dt$$

$$\leqq 2K_0 = 2\left\{\left(\mu_0 - \sum_{j=1}^{k} |\Delta_{0,j}|\right)\left(\mu_0 + \sum_{j=1}^{k} |\Delta_{0,j}|\right)\right\}^{1/2} + 4\left(\mu_0 - \sum_{j=1}^{k} |\Delta_{0,j}|\right)$$

$$\leqq 2(\epsilon \cdot 2\mu_0)^{1/2} + 4\epsilon,$$

the theorem following from the arbitrariness of ϵ.

4. **Transforms of sequences in** BV. We prove the following lemma.

LEMMA 4.1. *If $\{X_n\}$ is a sequence of functions in RBV satisfying the condition*

$$|X_n(t_2) - X_n(t_1)| \leqq M|t_2 - t_1| \qquad (0 \leqq t_1, t_2 \leqq 1; n = 0, 1, 2, \cdots),$$

and if Ψ is a real function which satisfies a Lipschitz condition on every finite interval, then the relation $\|X_n - X_0\| \to 0$ implies the relation $\|\Psi:X_n - \Psi:X_0\| \to 0$.

Let $[a, b]$ be such that $X_n(t)$ is in $[a, b]$ for $0 \leqq t \leqq 1$; $n = 0, 1, 2, \cdots$. Let $M_1(>0)$ dominate $\Psi'(x)$ for $a \leqq x \leqq b$; let $P_n = \underset{t}{E}[X_n'(t) > 0]$, $R_n = \underset{t}{E}[X_n'(t) = 0]$, $N_n = \underset{t}{E}[X_n'(t) < 0]$ for $n = 0, 1, 2, \cdots$ and denote by P^*, R^*, N^* those points on $[0, 1]$ at which the metric density of P_0, R_0, N_0 respectively is 1; let $\{\psi_p\}$ be a sequence of functions continuous on $[a, b]$ and dominated there by M_1, such that $\int_a^b H_p(x)dx \to 0$, where $H_p(x) = |\Psi'(x) - \psi_p(x)|$ for x on $[a, b]$. Since $\Psi:X_n$ satisfies a Lipschitz condition, it becomes clear in the light of Corollary 3.2 that the truth of the theorem is equivalent to showing $F_n(t) \to 0$ for t on $[0, 1]$, where F_n is defined by

$$F_n(t) = \int_0^t |\Psi'\{X_n(s)\}X_n'(s) - \Psi'\{X_0(s)\}X_0'(s)|\,ds.$$

To establish this relation we shall prove the following: If $\{F_n^*\}$ is any subsequence of $\{F_n\}$, then a subsequence $\{G_n\}$ of $\{F_n^*\}$ exists such that $G_n(t) \to 0$ for t on $[0, 1]$.

Since the sequence $\{F_n\}$ is comprised of non-decreasing functions which uniformly satisfy a Lipschitz condition, it appears as a corollary of Helly's theorem that there exists a subsequence $\{G_n\}$ of $\{F_n^*\}$ and a function G_0 satisfying a Lipschitz condition such that $G_n(t) \to G_0(t)$ for t on $[0, 1]$.

Now let t_0 be any point of P^* and denote $[t_0, s]$ by Q_s for $0 \leqq s \leqq 1$. From Theorem 3.2 follows the relation

$$\left|\int_{t_0}^s |\Psi'\{X_n(t)\}X_n'(t) - \psi_p\{X_n(t)\}X_n'(t)|\,dt\right| = \left|\int_{t_0}^s H_p\{X_n(t)\}\,|X_n'(t)|\,dt\right|$$

$$= \left|\int_{t_0}^s H_p\{X_n(t)\}X_n'(t)dt\right| + 2\int_{Q_sN_n} H_p\{X_n(t)\}\,|X_n'(t)|\,dt$$

$$\leqq \int_a^b H_p(x)dx + 4M_1M|Q_sN_n|$$

$$\leqq \int_a^b H_p(x)dx + 4M_1M\{|Q_sP_0N_n| + |Q_s(N_0 + R_0)|\}$$

$$(n = 0, 1, 2, \cdots; p = 1, 2, 3, \cdots; 0 \leqq s \leqq 1).$$

Thus

$$\left|\int_{t_0}^{s} \left| \psi_p\{X_0(t)\}X_0'(t) - \Psi'\{X_0(t)\}X_0'(t) \right| dt\right|$$
$$\leqq \int_a^b H_p(x)dx + 4M_1M\left| Q_s(N_0 + R_0) \right|$$

for $p=1, 2, 3, \cdots; 0 \leqq s \leqq 1$; furthermore, since X_n' converges in measure to X_0', it may easily be seen that $|Q_sP_0N_n| \to 0$ as $n \to \infty$ and hence that

$$\limsup_{n\to\infty}\left|\int_{t_0}^{s} \left| \Psi'\{X_n(t)\}X_n'(t) - \psi_p\{X_n(t)\}X_n'(t) \right| dt\right|$$
$$\leqq \int_a^b H_p(x)dx + 4M_1M\left| Q_s(N_0 + R_0) \right|$$

for $p=1, 2, 3, \cdots; 0 \leqq s \leqq 1$. By combining these two relations with the obvious relation

$$\limsup_{n\to\infty}\left|\int_{t_0}^{s} \left| \psi_p\{X_n(t)\}X_n'(t) - \psi_p\{X_0(t)\}X_0'(t) \right| dt\right| = 0$$
$$(p = 1, 2, 3, \cdots; 0 \leqq s \leqq 1),$$

we obtain

$$\limsup_{n\to\infty} \left| F_n(s) - F_n(t_0) \right|$$
$$= \limsup_{n\to\infty}\left|\int_{t_0}^{s} \Psi'\{X_n(t)\}X_n'(t) - \Psi\{X_0'(t)\}X_0'(t)\, dt\right|$$
$$\leqq 2\int_a^b H_p(x)dx + 8M_1M\left|Q_s(N_0 + R_0)\right| \quad (p = 1, 2, 3, \cdots; 0 \leqq s \leqq 1)$$

so that upon letting $p \to \infty$ we conclude

$$\left| G_0(s) - G_0(t_0) \right| = \lim_{n\to\infty} \left| G_n(s) - G_n(t_0) \right|$$
$$\leqq \limsup_{n\to\infty} \left| F_n(s) - F_n(t_0) \right| \leqq 8M_1M\left| Q_s(N_0 + R_0) \right|$$

for s on $[0, 1]$. Thus, since t_0 is a point at which the metric density of P_0 is 1,

$$\lim_{s\to t_0} \frac{\left| Q_s(N_0 + R_0) \right|}{\left| s - t_0 \right|} = 0$$

which implies $G_0'(t_0) = 0$.

If t_0 is a point of N^*, a similar proof establishes $G_0'(t_0) = 0$; if $t_0 \varepsilon R^*$, then the relation $G_0'(t_0) = 0$ is a consequence of the easily proved inequality

$$\limsup_{n\to\infty} \left| F_n(s) - F_n(t_0) \right| \leqq 2M_1M\left| Q_s(N_0 + P_0) \right| \quad (0 \leqq s \leqq 1).$$

Thus $G_0'(t)=0$ for almost all t on $[0, 1]$, and hence $G_0(t)=0$ for t on $[0, 1]$. This completes the proof.

The following is now readily established.

LEMMA 4.2. *If $\{X_n\}$ is a sequence of monotone functions in RBV satisfying the condition*

$$|X_n(t_2) - X_n(t_1)| \leqq M|t_2 - t_1| \qquad (0 \leqq t_1, t_2 \leqq 1; n = 0, 1, 2, \cdots),$$

and if Ψ is a real function which is a.c. on every finite interval, then the relation $\|X_n - X_0\| \to 0$ implies $\|\Psi:X_n - \Psi:X_0\| \to 0$.

To show this, approximate Ψ' in the mean with bounded measurable functions and apply Lemma 4.1 and Theorem 3.2. It is of some interest to note that some of the functions which comprise $\{X_n\}$ may be increasing while others are decreasing.

DEFINITION 4.1. Let u be a function on two-space to one-space. If there is a function A on one-space to n_1-space, a function B on one-space to n_2-space, and a function U on (n_1+n_2)-space to one-space such that

(i) $A(x) = (A_1(x), A_2(x), \cdots, A_{n_1}(x))$, $B(y) = (B_1(y), B_2(y), \cdots, B_{n_2}(y))$ $(-\infty < x, y < \infty)$;

(ii) all first partial derivatives of $U(\alpha_1, \alpha_2, \cdots, \alpha_{n_1}, \beta_1, \beta_2, \cdots, \beta_{n_2})$ with respect to these arguments are continuous functions on (n_1+n_2)-space;

(iii) $u(x, y) = U(A(x) \circ B(y)) = U(A_1(x), \cdots, A_{n_1}(x), B_1(y), \cdots, B_{n_2}(y))$ for $-\infty < x, y < \infty$;

(iv) on every finite interval either

A and B are a.c., or

A satisfies a Lipschitz† condition and B is a.c., or

A is a.c. and B satisfies a Lipschitz condition, or

A and B satisfy a Lipschitz condition;

then u is said to be respectively either $\mathfrak{R}$ or $\mathfrak{R}_1$ or $\mathfrak{R}_2$ or $\mathfrak{R}_{12}$.

DEFINITION 4.2. Let ϕ be a function in CC with $\phi(x+iy) = u(x, y) + iv(x, y)$ for $-\infty < x, y < \infty$; let f be a point in BV with $f = X + iY$. If the functions u and v are $\mathfrak{R}$ with X and Y monotone, or $\mathfrak{R}_1$ with Y monotone, or $\mathfrak{R}_2$ with X monotone, or $\mathfrak{R}_{12}$, then ϕ is said to be *applicable* to f.

DEFINITION 4.3. Let Y be a point in RBV and let u be $\mathfrak{R}$. If Y is monotone or if u is $\mathfrak{R}_2$, then u is said to be *applicable* (R) to Y.

DEFINITION 4.4. If ϕ is applicable to f and if

$$|\phi:f(t\pm) - \phi:f(t)| = T_{\lambda=0}^{1}[\phi\{(1-\lambda)f(t) + \lambda f(t\pm)\}] \qquad (0 \leqq t \leqq 1),$$

then ϕ is *strictly applicable* to f.

† Throughout the paper we shall consider definitions which involve purely metric properties of a function on $[a, b]$ to one-space to be generalized in the customary manner to functions on $[a, b]$ to n-space.

DEFINITION 4.5. If u is applicable (R) to Y and if for each t on $[0, 1]$ it is true that $u\{t, (1-\lambda)Y(t)+\lambda Y(t\pm)\}$ is monotone in λ for t on $[0, 1]$, then u is said to be *strictly applicable* (R) to Y.

Definitions 4.2 and 4.4 are formulated to facilitate the discussion of the following problem: Suppose $f_n=(X_n+iY_n)-v\rightarrow(X_0+iY_0)=f_0$ and suppose ϕ in CC with $\phi(x+iy)=u(x, y)+iv(x, y)$ for $-\infty<x, y<\infty$. What conditions on ϕ will imply $\phi:f_n-v\rightarrow\phi:f_0$? This is equivalent to asking what condition on ϕ will imply

$$T_{t=0}^{1}[u\{X_n(t), Y_n(t)\} + iv\{X_n(t), Y_n(t)\}] \rightarrow T_{t=0}^{1}[u\{X_0(t), Y_0(t)\} + iv\{X_0(t), Y_0(t)\}].$$

Definitions 4.3 and 4.5 will be used in connection with convergence in length.

LEMMA 4.3. *Let σ and g be in CR with σ real and non-decreasing and $\sigma(0)=0$, $\sigma(1)=1$. If $f=g:\sigma$ with $f(t\pm)=g\{\sigma(t\pm)\}$ for t on $[0, 1]$, then*

$$\|g\| = \|f\| + T_0^1(\Lambda),$$

where

$$\Lambda(t) = 2^{\operatorname{sgn} t(t-1)}\{T_{\sigma(t-)}^{\sigma(t+)}(g) - |f(t)-f(t-)| - |f(t+)-f(t)|\} \quad (0\leqq t\leqq 1).$$

Let f_0 and σ_0 be defined on $[-2, 2]$ as follows:

$$\begin{aligned} f_0(t) &= f(t), & \sigma_0(t) &= \sigma(t) & &(0\leqq t\leqq 1),\\ f_0(t) &= f(1), & \sigma_0(t) &= \sigma(1) & &(1< t\leqq 2),\\ f_0(t) &= f(0), & \sigma_0(t) &= \sigma(0) & &(-2\leqq t< 0),\end{aligned}$$

and let

$$\Lambda_0(t) = 2^{-1}\{T_{\sigma_0(t-)}^{\sigma_0(t+)}(g) - |f_0(t)-f_0(t-)| - |f_0(t+)-f_0(t)|\} \quad (-2\leqq t\leqq 2).$$

The truth of the lemma is clearly equivalent to showing

$$T_0^1(g) = T_{-2}^2(f_0) + T_{-2}^2(\Lambda_0).$$

To do this let $S\equiv(0=s_0<s_1<s_2<\cdots<s_k=1)$ be any partition of $[0, 1]$ and let $H(s', s'')$ be defined for $0\leqq s'\leqq s''\leqq 1$ by

$$H(s', s'') = \sum_{j=1}^{q}|g(p_j) - g(p_{j-1})|,$$

where $p_0=s'$, $p_q=s''$, and q is one more than the number of points of S on $\underset{s}{E}[s'<s<s'']$ and $p_1<p_2<\cdots<p_{q-1}$ are these points (if any). Consider the set

$$E = \underset{t}{E}[H(0, \sigma_0(t)) \leqq T_{-2}^{t}(f_0) + T_{-2}^{t}(\Lambda_0)]$$

and denote by E' the set of those points which are left-hand limit points† of E. Noting that $-1\varepsilon E'$ let $P=\sup E'$ and suppose $P<2$ in an attempt to show that $P=2$. Since $P\varepsilon E'$ there is a point P_1 (>-2) of E which is less than P but sufficiently close to P so that $\underset{s}{E}[\sigma_0(P_1)<s<\sigma_0(P-)]$ contains no point of S. It is likewise clear that there exists a point P_2 $(>P$ and $<2)$ such that $\underset{s}{E}[\sigma_0(P+)<s<\sigma_0(P_2)]$ contains no point of S. Now $P_1\varepsilon E$ implies

$$H(0, \sigma_0(P_1)) \leqq T_{-2}^{P_1}(f_0) + T_{-2}^{P_1}(\Lambda_0) \tag{1}$$

and from the fact that Λ_0 vanishes everywhere on $[-2, 2]$ except for a denumerable set it follows that

$$T_{\sigma(P-)}^{\sigma(P+)}(g) - | f_0(P) - f_0(P-) | - | f_0(P+) - f_0(P) | \leqq T_{P_1}^{R}(\Lambda_0) \quad (P<R<P_2)$$

which implies

$$\begin{aligned} H(\sigma_0(P-), \sigma_0(P+)) &\leqq T_{\sigma_0(P-)}^{\sigma_0(P+)}(g) \\ &\leqq | f_0(P) - f_0(P-) | + | f_0(P+) - f_0(P) | + T_{P_1}^{R}(\Lambda_0) \quad (P<R<P_2). \end{aligned} \tag{2}$$

Furthermore the fact that $\underset{s}{E}[\sigma_0(P_1)<s<\sigma_0(P-)]+\underset{s}{E}[\sigma_0(P+)<s<\sigma_0(P_2)]$ contains no point of S combines with the hypothesis of the lemma to yield

$$\begin{aligned} H(\sigma_0(P_1), \sigma_0(P-)) &+ H(\sigma_0(P+), \sigma_0(R)) \\ &= | f_0(P-) - f_0(P_1) | + | f_0(R) - f_0(P+) | \quad (P<R<P_2), \end{aligned} \tag{3}$$

so that upon adding (1), (2), and (3) we obtain, for $P<R<P_2$,

$$\begin{aligned} H(0, \sigma_0(R)) &\leqq H(0, \sigma_0(P_1)) + H(\sigma_0(P_1), \sigma_0(P-)) \\ &\quad + H(\sigma_0(P-), \sigma_0(P+)) + H(\sigma_0(P+), \sigma_0(R)) \\ &\leqq T_{-2}^{P_1}(f_0) + | f_0(P-) - f_0(P_1) | + | f_0(P) - f_0(P-) | \\ &\quad + | f_0(P+) - f_0(P) | + | f_0(R) - f_0(P+) | + T_{-2}^{P_1}(\Lambda_0) + T_{P_1}^{R}(\Lambda_0) \\ &\leqq T_{-2}^{R}(f_0) + T_{-2}^{R}(\Lambda_0). \end{aligned}$$

This establishes P_2 as a point of E' which is a contradiction proving $2=\sup E'$ and hence that

$$\sum_{j=1}^{k} | g(s_j) - g(s_{j-1}) | = H(0, 1) \leqq T_{-2}^{2}(f_0) + T_{-2}^{2}(\Lambda_0),$$

which implies

$$T_0^1(g) \leqq T_{-2}^{2}(f_0) + T_{-2}^{2}(\Lambda_0)$$

since S was arbitrary.

† We define x_0 as a left-hand limit point of E if every interval $[x, x_0]$, where $x<x_0$, contains a point of E as an inner point.

Now let $t_1, t_2, t_3, \cdots$ be a denumerable set of points on $[0, 1]$ which include all points of discontinuity of Λ_0 and define $\Lambda_n(t) = \Lambda_0(t)$ for $t = t_1, t_2, \cdots, t_n$ and $\Lambda_n(t) = 0$ for all other t on $[-2, 2]$. Clearly $\Lambda_n(t) \to \Lambda_0(t)$ for all t on $[-2, 2]$ and inasmuch as

$$T_{\sigma_0(t')}^{\sigma_0(t'')}(g) \geqq T_{t'}^{t''}(f_0) \qquad (-2 \leqq t' \leqq t'' \leqq 2),$$

it follows by induction that

$$T_{\sigma_0(t')}^{\sigma_0(t'')}(g) \geqq T_{t'}^{t''}(f_0) + T_{t'}^{t''}(\Lambda_n)$$

for $t' \leqq t''$ with t' and t'' both in the set obtained by deleting the points $t_1, t_2, t_3, \cdots$ from $[-2, 2]$. This of course implies

$$T_0^1(g) \geqq T_{-2}^2(f_0) + T_{-2}^2(\Lambda_n)$$

whence, by semi-continuity

$$T_0^1(g) \geqq T_{-2}^2(f_0) + T_{-2}^2(\Lambda_0)$$

and the proof of the lemma is complete.

LEMMA 4.4. *Let $f = X + iY$ be in BV and denote $T_0^1(f)$ by μ. Let σ be a non-decreasing function in RBV such that $\sigma(0) = 0$, $\sigma(1) = 1$ and $\mu\sigma(t) = T_0^t(f)$ for t on $[0, 1]$. There exists a function g in BV having the following properties:*

(i) *$f = g{:}\sigma$ with $\|f\| = \|g\|$;*

(ii) *$|g(s_2) - g(s_1)| \leqq \mu|s_2 - s_1|$, $(0 \leqq s_1, s_2 \leqq 1)$, the sign of equality holding if $\sigma(t-) \leqq s_1, s_2 \leqq \sigma(t)$ or $\sigma(t) \leqq s_1, s_2 \leqq \sigma(t+)$, where $0 \leqq t \leqq 1$;*

(iii) *if ϕ is any function in CC which is applicable to g, then*

$$\|\phi{:}g\| = \|\phi{:}f\| + T_0^1(\Lambda),$$

where

$$\Lambda(t) = 2^{\operatorname{sgn} t(t-1)}\Big\{T_{\lambda=0}^1[\phi\{\lambda f(t-) + (1-\lambda)f(t)\}] - |\phi{:}f(t) - \phi{:}f(t-1)|$$
$$+ T_{\lambda=0}^1[\phi\{\lambda f(t) + (1-\lambda)f(t+)\}] - |\phi{:}f(t+) - \phi{:}f(t)|\Big\} \quad (0 \leqq t \leqq 1).$$

Define $g = \xi + i\eta$ as follows. Let s_0 be any point on $[0, 1]$ and let $t_0 = \inf \underset{t}{E}[\sigma(t) \geqq s_0]$. Now $\sigma(t_0-) \leqq s_0 \leqq \sigma(t_0+)$ and we define

$$g(s_0) = f(t_0) \quad \text{if} \quad s_0 = \sigma(t_0);$$

$$g(s_0) = \frac{f(t_0-)\{\sigma(t_0) - s_0\} + f(t_0)\{s_0 - \sigma(t_0-)\}}{\sigma(t_0) - \sigma(t_0-)} \quad \text{if} \quad \sigma(t_0-) \leqq s_0 < \sigma(t_0);$$

$$g(s_0) = \frac{f(t_0)\{\sigma(t_0+) - s\} + f(t_0+)\{s_0 - \sigma(t_0)\}}{\sigma(t_0+) - \sigma(t_0)} \quad \text{if} \quad \sigma(t_0) < s_0 \leqq \sigma(t_0+).$$

As a consequence of

$$(1) \qquad |f(t'') - f(t')| \leqq \mu |\sigma(t'') - \sigma(t')| \qquad (0 \leqq t', t'' \leqq 1),$$

it is easily verified that $g\{\sigma(t)\} = f(t)$ and $g\{\sigma(t\pm)\} = f(t\pm)$ for t on $[0, 1]$. Combining these last two relations with the definition of g and the relation

$$|f(t \pm) - f(t)| = \mu |\sigma(t \pm) - \sigma(t)| \qquad (0 \leqq t \leqq 1),$$

we obtain

$$(2) \qquad |g(s_2) - g(s_1)| = \mu |s_2 - s_1|$$

if $\sigma(t-) \leqq s_1, s_2 \leqq \sigma(t)$ or $\sigma(t) \leqq s_1, s_2 \leqq \sigma(t+)$, where $0 \leqq t \leqq 1$. Hence

$$\begin{aligned} T_{\sigma(t-)}^{\sigma(t+)}(g) &= \mu |\sigma(t) - \sigma(t-)| + \mu |\sigma(t+) - \sigma(t)| \\ &= |f(t) - f(t-)| + |f(t+) - f(t)| \qquad (0 \leqq t \leqq 1) \end{aligned}$$

while, on the other hand, from Lemma 3.8 it follows that

$$\begin{aligned} \Lambda(t) &= 2^{\operatorname{sgn} t(t-1)}\big[T_{\sigma(t-)}^{\sigma(t)}(\phi:g) - |\phi:f(t) - \phi:f(t-)| \\ &\qquad + T_{\sigma(t)}^{\sigma(t+)}(\phi:g) - |\phi:f(t+) - \phi:f(t)|\big] \\ &= 2^{\operatorname{sgn} t(t-1)}\big[T_{\sigma(t-)}^{\sigma(t+)}(\phi:g) - |\phi:f(t) - \phi:f(t-)| - |\phi:f(t+) - \phi:f(t)|\big]. \end{aligned}$$

Viewing the last two relations in the light of Lemma 4.3 establishes $\|g\| = \|f\|$ and $\|\phi:g\| = \|\phi:f\| + T_0^1(\Lambda)$. It also follows that monotonicity of X or Y implies monotonicity of ξ or η respectively so that from Definition 4.2 we conclude ϕ is applicable to g.

To complete the proof of (ii) let $s_1' \leqq s_2'$ be any two numbers on $[0, 1]$ and let $t_1' = \inf \underset{t}{E}[\sigma(t) \geqq s_1']$ and $t_2' = \sup \underset{t}{E}[\sigma(t) \leqq s_2']$. Now $t_1' \leqq t_2'$. If $t_1' = t_2'$ then $\sigma(t_1'-) \leqq s_1' \leqq s_2' \leqq \sigma(t_1'+)$, so that the relation (2) implies the inequality $|g(s_2') - g(s_1')| \leqq \mu |s_2' - s_1'|$. However, if $t_1' < t_2'$ then

$$\sigma(t_1'-) \leqq s_1' \leqq \sigma(t_1'+) \leqq \sigma(t_2'-) \leqq s_2' \leqq \sigma(t_2'+),$$

so that (1) and (2) yield

$$\begin{aligned} |g(s_2') - g(s_1')| &\leqq |g\{\sigma(t_1'+)\} - g(s_1')| + |g\{\sigma(t_2'-)\} - g\{\sigma(t_1'+)\}| \\ &\qquad + |g(s_2') - g\{\sigma(t_2'-)\}| \\ &\leqq \mu\{\sigma(t_1'+) - s_1'\} + |f(t_2'-) - f(t_1'+)| + \mu\{s_2' - \sigma(t_2'-)\} \\ &\leqq \mu\{\sigma(t_1'+) - s_1'\} + \mu\{\sigma(t_2'-) - \sigma(t_1'+)\} \\ &\qquad + \mu\{s_2' - \sigma(t_2'-)\} = \mu(s_2' - s_1'). \end{aligned}$$

The truth of (ii) is now apparent and the proof of the lemma is complete.

We introduce here the notion of pseudo-absolute continuity.

DEFINITION 4.6. A function f defined on $[a, b]$ is said to be *pseudo-absolutely continuous* there if corresponding to every $\epsilon(>0)$ there exists a $\delta(>0)$ and a finite point set E such that if $\{[a_n, b_n]\}$ is any denumerable set of non-overlapping intervals on $[a, b]$ with $E\cdot\sum_{n=1}^{\infty}[a_n, b_n]$ empty and $\sum_{n=1}^{\infty}|b_n-a_n|<\delta$, then $\sum_{n=1}^{\infty}|f(b_n)-f(a_n)|<\epsilon$.

We observe that a pseudo-a.c. function is of b.v. and is expressible as the sum of an a.c. function and a singular function of the saltus type (see Definition 6.2 below).

LEMMA 4.5. *If f is a pseudo-absolutely continuous point in BV and if ϕ is any function in CC which is applicable to f, then $\phi:g$ is a pseudo-absolutely continuous point in BV.*

There is no loss in generality in assuming ϕ to be real valued.

Let $X+iY=f$. From Definitions 4.1 and 4.2 there exist functions A and B on one-space to n_1- and n_2-spaces respectively which are a.c. on every finite interval and a function U on n_3-space $(n_3=n_1+n_2)$ to one-space, all of whose first partial derivatives are continuous, such that

$$\phi(x+iy) = U\{A(x)\circ B(y)\} \qquad (-\infty<x, y<\infty)$$

with A satisfying a Lipschitz condition on every finite interval if X is not monotone and B satisfying a Lipschitz condition on every finite interval if Y is not monotone. Let

$$C(t) = A\{X(t)\}\circ B\{Y(t)\} \qquad (0\leqq t\leqq 1).$$

Readily seen is the pseudo-absolute continuity† of C. Let S be a sphere in n_3-space which includes the range of C. Now U satisfies a Lipschitz condition on S; i.e., there exists a constant $M(>0)$ such that if γ and γ' are any two points of S, then

$$|U(\gamma')-U(\gamma)| \leqq M\{\text{Euclidean distance between } \gamma \text{ and } \gamma'\},$$

so that, since $\phi:f(t)=U\{C(t)\}$ for t on $[0, 1]$, the pseudo-absolute continuity of $\phi:f$ becomes apparent.

Since a continuous pseudo-absolutely continuous function is a.c. we have

COROLLARY 4.1. *If f is an a.c. point in BV and if ϕ is any function in CC which is applicable to f, then $\phi:g$ is an a.c. point in BV.*

LEMMA 4.6. *If $\{g_n\}$ is a sequence of points in BV satisfying the relation*

$$|g_n(t_2)-g_n(t_1)| \leqq M|t_2-t_1| \qquad (0\leqq t_1, t_2\leqq 1; n=0, 1, 2, \cdots).$$

† See footnote on page 59.

and if ϕ is any function in CC which is applicable to g_n for $n=0, 1, 2, \cdots$, then the relation $\|g_n-g_0\|\to 0$ implies $\|\phi:g_n-\phi:g_0\|\to 0$.

There is no loss in generality in assuming ϕ real valued. Let $\{g_{n,1}\}$ be any subsequence of $\{g_n\}$ wherein $g_{0,1}=g_0$. To prove $\|\phi:g_n-\phi:g_0\|\to 0$ it is merely necessary to establish the existence of a sequence $\{g_{n,2}\}$ which is a subsequence of $\{g_{n,1}\}$, such that $\|\phi:g_{n,2}-\phi:g_0\|\to 0$. First we note that of the sequence $\{g_{n,1}\}$ there exists a subsequence $\{g_{n,2}\}$ wherein $g_{0,2}=g_0$ and $X_n+iY_n=g_{n,2}$ for $n=0, 1, 2, \cdots$, which enjoys one of the following four properties:

(i) X_1 and Y_1 are both not monotone;

(ii) X_n is monotone ($n=0, 1, 2, \cdots$) and Y_1 is not;

(iii) X_n and Y_n are monotone ($n=0, 1, 2, \cdots$);

(iv) X_1 is not monotone and Y_n is monotone ($n=0, 1, 2, \cdots$).

Now since ϕ is applicable to $\{g_{1,2}\}$, there exist functions A and B on one-space to n_1- and n_2-spaces respectively and a real function U defined on n_3-space ($n_3=n_1+n_2$), all of whose first partial derivatives are continuous, such that

$$\phi(x+iy) = U\{A(x)\circ B(y)\} \qquad (-\infty < x, y < \infty)$$

and such that C_n defined by $C_n(t)=A\{X_n(t)\}\circ B\{Y_n(t)\}$ $(0\leqq t\leqq 1)$, is an a.c. function on $[0, 1]$ to n_3-space not only for $n=1$, but for $n=0, 1, 2, \cdots$. Thus upon defining

$$(C_{n,1}(t), C_{n,2}(t), \cdots, C_{n,n_3}(t)) = C_n(t) \text{ and } H_n(t) = U\{C_n(t)\}$$
$$(0\leqq t\leqq 1, n=0, 1, 2, \cdots),$$

we conclude H_n is an a.c. point in RBV with $H_n=\phi:g_{n,2}$ for $n=0, 1, 2, \cdots$. Also follows the existence of continuous functions $D_1, D_2, \cdots, D_{n_3}$ on n_3-space to one-space such that (for $n=0, 1, 2, \cdots$)

$$H_n'(t) = \sum_{p=1}^{n_3} D_p\{C_n(t)\}C_{n,p}'(t) \qquad (\text{almost all } t \text{ on } [0, 1]).$$

Since g_0 is continuous, we conclude $(X_n+iY_n)-uv\to(X_0+iY_0)$ and hence $C_n(t)\to C_0(t)$ uniformly for t on $[0, 1]$, so that

$$\lim_{n\to\infty} D_p\{C_n(t)\} = D_p\{C_0(t)\} \qquad (p=1, 2, \cdots, n_3)$$

uniformly for t on $[0, 1]$. Combining this with the relation

$$\lim_{n\to\infty}\int_0^1 |C_{n,p}'(t) - C_{0,p}'(t)|\, dt = 0 \quad (p=1, 2, \cdots, n_3),$$

which is a corollary of Lemma 4.1 and Lemma 4.2, yields

$$T_0^1(H_n - H_0) = \int_0^1 |H_n'(t) - H_0'(t)| \, dt \to 0.$$

Hence, since $H_n(0) \to H_0(0)$, we conclude $\|\phi:g_{n,2} - \phi:g_0\| \to 0$. This completes the proof.

With this background we now turn to the proofs of the following two theorems.

Theorem 4.1. *If ϕ is applicable to f_n for $n=0, 1, 2, \cdots$ and in addition if ϕ is strictly applicable to f_0, then the relation $f_n - v \to f_0$ implies*

$$\phi:f_n - v \to \phi:f_0.$$

Theorem 4.2. *If ϕ is applicable to f_n for $n=0, 1, 2, \cdots$, then the relation $f_n - uv \to f_0$ implies*

$$\phi:f_n - uv \to \phi:f_0.$$

Let $\mu_n = T_0^1(f_n)$, $(n=0, 1, 2, \cdots)$. From Lemma 4.4 we conclude the existence (for $n=0, 1, 2, \cdots$) of functions g_n and σ_n in CR having the following properties:

(i) σ_n is a non-decreasing function with $\sigma_n(0)=0$, $\sigma_n(1)=1$ and $\mu_n\sigma_n(t) = T_0^t(f_n)$ for $0 \leqq t \leqq 1$;

(ii) $f_n = g_n:\sigma_n$ with $\|f_n\| = \|g_n\|$;

(iii) $|g_n(s_2) - g_n(s_1)| \leqq \mu_n |s_2 - s_1|$, $(0 \leqq s_1, s_2 \leqq 1)$, the sign of equality holding if $\sigma_n(t-) \leqq s_1, s_2 \leqq \sigma_n(t)$ or $\sigma_n(t) \leqq s_1, s_2 \leqq \sigma_n(t+)$, where $0 \leqq t \leqq 1$;

(iv) ϕ is applicable to g_n and

$$\|\phi:g_n\| = \|\phi:f_n\| + T_0^1(\Lambda_n),$$

where

$$\Lambda_n(t) = 2^{\operatorname{sgn} t(t-1)}\Big\{T_{\lambda=0}^1[\phi\{\lambda f_n(t-) + (1-\lambda)f_n(t)\}] - |\phi:f_n(t) - \phi:f_n(t-)| + T_{\lambda=0}^1[\phi\{\lambda f_n(t) + (1-\lambda)f_n(t+)\}] - |\phi:f_n(t+) - \phi:f_n(t)|\Big\} \quad (0 \leqq t \leqq 1).$$

We divide the remainder of the proof of Theorems 4.1 and 4.2 into three parts.

Part I. *$\phi:g_n$ is a.c. for each $n=0, 1, 2, \cdots$ and $\|\phi:g_n - \phi:g_0\| \to 0$.*

From (i) and Lemma 3.1 follows the relation, $\mu_n\sigma_n(t) \to \mu_0\sigma_0(t)$ for t on $[0, 1]$ so that

$$|g_n(\sigma_n(t)) - g_n(\sigma_0(t))| \leqq |\mu_n\sigma_n(t) - \mu_n\sigma_0(t)| \to 0$$

since $\mu_n \to \mu_0$. From (ii) follows the relation $g_n\{\sigma_n(t)\} \to g_0\{\sigma_0(t)\}$ for t on $[0, 1]$, so that upon combining the above relations we conclude

$$g_n\{\sigma_0(t)\} \to g_0\{\sigma_0(t)\} \qquad (0 \leqq t \leqq 1).$$

Now let s_0 be any point on $[0, 1]$ and note that there exists a point t_0 on $[0, 1]$ for which $\sigma_0(t_0-) \leqq s_0 \leqq \sigma_0(t_0+)$, so that either $s_0 = \sigma_0(t_0)$, or $\sigma_0(t_0-) \leqq s_0 < \sigma_0(t_0)$, or $\sigma_0(t_0) < s_0 \leqq \sigma_0(t_0+)$. If $s_0 = \sigma_0(t_0)$ then follows immediately the conclusion $g_n(s_0) \to g_0(s_0)$. Supposing now that $\sigma_0(t_0-) \leqq s_0 < \sigma_0(t_0)$, let $0 < t_1 < t_2 < t_3 < \cdots$ with $t_p \to t_0$ and denote by P any limit point of the sequence $\{g_n(s_0)\}$. Also let $s_0' = \sigma_0(t_0-)$ and $s_0'' = \sigma_0(t_0)$, $P_0' = g_0(s_0')$, $P_0 = g_0(s_0)$, and $P_0'' = g_0(s_0'')$. Thus

$$\begin{aligned} &| g_n(s_0) - g_n\{\sigma_0(t_p)\} | \leqq \mu_n\{s_0 - \sigma_0(t_p)\}, \\ &\quad | g_n(s_0'') - g_n(s_0) | \leqq \mu_n\{s_0'' - s_0\} \qquad (n = 0, 1, 2, \cdots; p = 1, 2, 3, \cdots). \end{aligned}$$

Letting $n \to \infty$ and then $p \to \infty$ and then using (iii) establishes

$$\begin{aligned} | P - P_0' | &\leqq \mu_0\{s_0 - s_0'\} = | P_0 - P_0' | \\ | P_0'' - P | &\leqq \mu_0\{s_0'' - s_0\} = | P_0'' - P_0 |. \end{aligned}$$

Adding and using (iii) again yields

$$| P_0'' - P_0' | \leqq | P - P_0' | + | P_0'' - P | \leqq \mu_0(s_0'' - s_0') = | P_0'' - P_0' |$$

which implies equality in the last three relations which in turn implies $P_0 = P$. Thus $g_n(s_0) \to g_0(s_0)$ if $\sigma_0(t_0-) \leqq s_0 < \sigma_0(t_0)$. A similar proof of this relation holds if $\sigma_0(t_0) < s_0 \leqq \sigma_0(t_0+)$, so that finally it is established that $g_n(s) \to g_0(s)$ for s on $[0, 1]$.

From (iii) we conclude g_n is a.c. with $|g_n'(s)| \leqq \mu_n$, for $0 \leqq s \leqq 1$; $n = 0, 1, 2, \cdots$. Hence (for $n = 0, 1, 2, \cdots$)

$$\int_0^1 \Big| \mu_n - | g_n'(s) | \Big| ds = \int_0^1 \{\mu_n - | g_n'(s) | \} ds = 0,$$

so that $\mu_n = |g_n'(s)|$ for almost all s on $[0, 1]$. Applying Theorem 3.3 yields $\|g_n - g_0\| \to 0$. Letting $M(>0)$ be such that $\mu_n < M$ for $n = 0, 1, 2, \cdots$ we see from (iii) that

$$| g_n(s_2) - g_n(s_1) | \leqq M | s_2 - s_1 | \qquad (0 \leqq s_1, s_2 \leqq 1; n = 0, 1, 2, \cdots),$$

so that Lemma 4.6 and Corollary 4.1 complete the proof of Part I.

Part II. (Proof of Theorem 4.1.) Since ϕ is strictly applicable to f_0 it is apparent from (iv) that $\Lambda_0(t) = 0$, $(0 \leqq t \leqq 1)$, and hence

$$\|\phi : g_n\| \geqq \|\phi : f_n\| \qquad (n = 0, 1, 2, \cdots),$$

the equality holding if $n=0$. Thus $\phi:f_n$ is in BV for $n=0, 1, 2, \cdots$ and further, since $\phi:f_n(t)\to\phi:f_0(t)$ for t on $[0, 1]$, we have

$$0 = \limsup_{n\to\infty} ||\phi:g_n - \phi:g_0|| \geqq \limsup_{n\to\infty} ||\phi:g_n|| - ||\phi:g_0||$$
$$\geqq \limsup_{n\to\infty} ||\phi:f_n|| - ||\phi:f_0|| \geqq \liminf_{n\to\infty} ||\phi:f_n|| - ||\phi:f_0|| \geqq 0$$

by the semi-continuity property of total variation.

The proof of Part II is now complete.

PART III. (Proof of Theorem 4.2.) Since $||\phi:g_n|| \geqq ||\phi:f_n||$ we conclude as before that $\phi:f_n$ is a point of BV for each $n=0, 1, 2, \cdots$. Since, however, $f_n - uv \to f_0$ it becomes clear that in this case

$$f_n(t \pm) \to f_0(t \pm) \quad \text{uniformly for } t \text{ on } [0, 1],$$

which used in connection with Lemma 4.6 readily establishes

$$\Lambda_n(t) \to \Lambda_0(t) \qquad (0 \leqq t \leqq 1),$$

so that $\liminf_{n\to\infty} T_0^1(\Lambda_n) - T_0^1(\Lambda_0) \geqq 0$. Hence

$$0 = \limsup_{n\to\infty} ||\phi:g_n - \phi:g_0|| \geqq \limsup_{n\to\infty} ||\phi:g_n|| - ||\phi:g_0||$$
$$\geqq \limsup_{n\to\infty} ||\phi:f_n|| - ||\phi:f_0|| + \liminf_{n\to\infty} T_0^1(\Lambda_n) - T_0^1(\Lambda_0)$$
$$\geqq \limsup_{n\to\infty} ||\phi:f_n|| - ||\phi:f_0|| \geqq \liminf_{n\to\infty} ||\phi:f_n|| - ||\phi:f_0|| \geqq 0.$$

Since $\phi:f_n(t)\to\phi:f_0(t)$ uniformly for t on $[0, 1]$, the proof of Part III is now complete.

As a corollary we have

COROLLARY 4.2. *If the function ϕ is in CC with $\phi(x+iy)=u(x, y)+iv(x, y)$ for $-\infty < x, y < \infty$, where u, v have continuous first partial derivatives, then the relation $f_n - v \to f_0$ with f_0 continuous implies*

$$\phi:f_n - v \to \phi:f_0.$$

If the transformation is that of raising to a positive integer power, Theorems 4.1 and 4.2 lead to

COROLLARY 4.3. *Let k be a positive integer. If $f_n - v \to f_0$ and if corresponding to each t on $[0, 1]$ there is a ray through the origin (of the complex plane) on which lie the points $f_0(t)$, $f_0(t+)$, and $f_0(t-)$, then*

$$f_n^k - v \to f_0^k.$$

COROLLARY 4.4. *If $f_n - uv \to f_0$, then $f_n^k - uv \to f_0^k$.*

Before concluding this section the following remark seems in order. If ϕ is in CC and satisfies a Lipschitz condition on every bounded set in the complex plane, then $f\varepsilon BV$ implies $\phi:f\varepsilon BV$. Hence it is natural to inquire into the truth of the following statement.

If $\phi\varepsilon CC$ and satisfies a Lipschitz condition on every finite set and if f_0 is continuous, then the relation $f_n - v \to f_0$ implies $\phi:f_n - v \to \phi:f_0$.

That the statement is not true is illustrated by the following example. Let $f_n(t) = t + i/n$ for $0 \leqq t \leqq 1$; $n = 1, 2, 3, \cdots$. Let $f_0(t) = t$. It is clear that $f_n - v \to f_0$. Define

$$u(x, y) = \frac{\sin n^4\pi x}{n^4\pi} + \left\{\frac{\sin (n-1)^4\pi x}{(n-1)^4\pi} - \frac{\sin n^4\pi x}{n^4\pi}\right\} \sin^2 \left\{\frac{\pi n(n-1)}{2}\left(y - \frac{1}{n}\right)\right\},$$

if $1/n \leqq y < 1/(n-1)$, where $n \geqq 2$;

$$u(x, y) = \frac{\sin \pi x}{\pi} \quad \text{if} \quad y \geqq 1; \qquad u(x, y) = 0 \quad \text{if} \quad y \leqq 0;$$

and let $\phi(x+iy) = u(x, y)$ for $-\infty < x, y < \infty$. Since the first partial derivatives of u exist everywhere and are dominated by 3, it follows that ϕ satisfies a Lipschitz condition on the complex plane. But

$$\|\phi:f_n\| = \int_0^1 |\cos n^4\pi x| \, dx \geqq \int_0^1 \cos^2 n^4\pi x \, dx = \tfrac{1}{2} \qquad (n = 1, 2, 3, \cdots)$$

with $\|\phi:f_0\| = 0$ so that it is not true that $\phi:f_n - v \to \phi:f_0$. As a rough appraisal of the generality of Theorems 4.1 and 4.2 it is interesting to note that a function in CC may be applicable to f_n for $n = 0, 1, 2,$ without satisfying a Lipschitz condition on every bounded set in the complex plane.

5. **Convergence in length.** As our first application of preceding results we have the following theorem, which is a result obtained in a different way in AL.

THEOREM 5.1. *The relation $Y_n - l \to Y_0$ implies $Y_n - v \to Y_0$.*

Let $\phi(x+iy) = y$, $(-\infty < x, y < \infty)$. Now ϕ is strictly applicable to $I + iY_n$ for $n = 0, 1, 2, \cdots$ and since $(I + iY_n) - v \to (I + iY_0)$ we conclude that $\phi:(I + iY_n) - v \to \phi:(I + iY_0)$. Hence the theorem is established.

COROLLARY 5.1. *The relation $Y_n - ul \to Y_0$ implies $Y_n - uv \to Y_0$.*

We introduce here the notion of a singular function.

DEFINITION 5.1. If f is of b.v. on $[a, b]$ with $f'(t)=0$ for almost all t on $[a, b]$ then f is said to be *singular* on $[a, b]$.

A well known property of singular functions is this: Let f be a singular function in BV and assume g an a.c. function in BV. Then

$$T_0^1(f+g) = T_0^1(f) + T_0^1(g),$$

so that if either $f(0)=0$ or $g(0)=0$, it is clear that $\|f+g\| = \|f\| + \|g\|$. We are now prepared to prove the following

THEOREM 5.2. *If Y_0 is a singular function in RBV, then the relations $Y_n - l \to Y_0$ and $Y_n - v \to Y_0$ are equivalent.*

Supposing $Y_n - v \to Y_0$ we deduce the relation

$$\|I + iY_0\| \leqq \liminf_{n\to\infty} \|I + iY_n\| \leqq \limsup_{n\to\infty} \|I + iY_n\| \leqq \limsup_{n\to\infty} \|Y_n\| + \|I\|$$
$$= \|I\| + \|Y_0\| = \|I\| + \|iY_0\| = \|I + iY_0\|,$$

which proves $Y_n - l \to Y_0$.

Application of Theorem 5.1 completes the proof.

THEOREM 5.3. *If u is applicable (R) to Y_n for $n=0, 1, 2, \cdots$ and in addition if u is strictly applicable (R) to Y_0, then the relation $Y_n - l \to Y_0$ implies*

$$(u|Y_n) - l \to (u|Y_0).$$

Define $\phi(x+iy) = x + iu(x, y)$ for $-\infty < x, y < \infty$. Now ϕ is applicable to $(I+iY_n)$ for $n=0, 1, 2, \cdots$. It is also strictly applicable to $(I+iY_0)$. Hence $\phi:(I+iY_n) - v \to \phi:(I+iY_0)$ or $\{I+i(u|Y_n)\} - v \to \{I+i(u|Y_0)\}$.

THEOREM 5.4. *If X_0 is a.c., then the relations $X_n - l \to X_0$ and $Y_n - l \to Y_0$ imply the relations*†

$$(X_n + Y_n) - l \to (X_0 + Y_0),$$
$$X_n Y_n - l \to X_0 Y_0.$$

Let Ψ be a.c. on $\underset{x}{E}[-\infty < x < \infty]$ with $\Psi(x) = X_0(x)$ for x on $[0, 1]$. Define $u_1(x, y) = y - \Psi(x)$, $u_2(x, y) = y + \Psi(x)$; $-\infty < x, y < \infty$. Thus by Theorem 5.3 we have $(u_1|X_n) - l \to \theta$, which implies (Theorem 5.2) that $\|X_n - X_0\| \to 0$ (see §2 for definition of θ). Hence

$$\|I + i(Y_n + X_n - X_0)\| \leqq \|I + iY_n\| + \|X_n - X_0\| \to \|I + iY_0\|,$$

whence, by using the semi-continuity property of total variation, we deduce

$$(Y_n + X_n - X_0) - l \to Y_0.$$

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† This is a generalization of Theorem 6 in AL.

If the assumption that X_0 is a.c. is deleted the theorem ceases to be true. See AL, page 23.

This gives, in view of Theorem 5.3,

$$[u_2|(Y_n + X_n - X_0)] - l \to (u_2|Y_0) \quad \text{or} \quad (Y_n + X_n) - l \to (Y_0 + X_0).$$

Now by Lemma 3.2 we have

$$\|Y_n \cdot (X_n - X_0)\| \leqq \|Y_n\| \cdot \|X_n - X_0\| \to 0,$$

so that upon defining $u_3(x, y) = y\Psi(x)$, $-\infty < x, y < \infty$, it is seen by Theorem 5.3 that

$$(u_3|Y_n) - l \to (u_3|Y_0) \quad \text{or} \quad X_0Y_n - l \to X_0Y_0$$

and since (Theorem 5.2)

$$(X_nY_n - X_0Y_n) = [Y_n \cdot (X_n - X_0)] - l \to \theta,$$

we conclude upon adding, θ being a.c., that $X_nY_n - l \to X_0Y_0$.

LEMMA 5.1. *Let X be an a.c. point in the space RBV. The relation $(cI + Y_n) - v \to (cI + Y_0)$ for all real c implies $(X + Y_n) - v \to (X + Y_0)$.*

Let $[a, b]$ be a subinterval of $[0, 1]$. From Lemma 3.1 it follows that

$$T_{t=a}^{b}(c_1 + c_2t + Y_n(t)) \to T_{t=a}^{b}(c_1 + c_2t + Y_0(t))$$

for all real c_1 and c_2, whence we conclude

$$(\beta + Y_n) - v \to (\beta + Y_0),$$

where β is any polygonal function in RBV.

Let $\{\beta_p\}$ be a sequence of polygonal functions in RBV such that as $p \to \infty$ $\beta_p - l \to X$. From Theorems 5.4 and 5.1 follow the relations $\|\beta_p - X\| \to 0$ and $(\beta_p + Y_0) - v \to (X + Y_0)$ as $p \to \infty$. Hence

$$\begin{aligned}\|X + Y_0\| &\leqq \liminf_{n\to\infty} \|X + Y_n\| \leqq \limsup_{n\to\infty} \|X + Y_n\| \\ &\leqq \limsup_{n\to\infty} \|\beta_p + Y_n\| + \lim_{n\to\infty} \|X - \beta_p\| \\ &= \|\beta_p + Y_0\| + \|X - \beta_p\| \to \|X + Y_0\| \quad \text{as} \quad p \to \infty\end{aligned}$$

and the lemma is proved.

THEOREM 5.5. *The relation† $(cI + Y_n) - v \to (cI + Y_0)$ for all real numbers c and the relation $Y_n - l \to Y_0$ are equivalent.*

Let $\alpha(t) = \int_0^t Y_0'(s)ds$ and $\beta(t) = Y_0(t) - \alpha(t)$ for t on $[0, 1]$. From the preceding lemma follows

† We are indebted to Professor E. J. McShane for raising the question as to whether the relation $(cI + Y_n) - v \to (cI + Y_0)$ for all real numbers c implies $Y_n - l \to Y_0$.

$$(Y_n - \alpha) - v \to \beta.$$

Since β is singular we deduce from Theorems 5.2 and 5.4 that

$$(Y_n - \alpha) - l \to \beta \quad \text{and} \quad Y_n - l \to (\alpha + \beta) = Y_0.$$

From Theorem 5.4 the converse follows immediately.

If Y_0 is a.c. then it appears, as a consequence of Theorems 5.1 and 5.4, that the relation $Y_n - l \to Y_0$ implies

$$\int_0^1 | Y_n'(t) - Y_0'(t) | \, dt \to 0.$$

However, if Y_0 is not a.c., this conclusion need not be true. It is true, nevertheless, that Y_n' converges to Y_0' in a manner intermediate between convergence in the mean and convergence in measure. To characterize this type of convergence we introduce the following definitions which may have some intrinsic interest.

DEFINITION 5.2. If f_n is measurable on a set E for $n = 1, 2, 3, \cdots$ and if corresponding to every $\epsilon > 0$ there exists a measurable set $E_1 \subset E$ of measure $> |E| - \epsilon$ such that

$$\lim_{m,n\to\infty} \int_{E_1} | f_m(t) - f_n(t) | \, dt = 0,$$

then $\{f_n\}$ is said to be convergent *almost in the mean* on E.

DEFINITION 5.3. If f_n is measurable on a set E for $n = 0, 1, 2, \cdots$ and if corresponding to every $\epsilon > 0$ there exists a measurable set $E_1 \subset E$ of measure $> |E| - \epsilon$ such that

$$\int_{E_1} | f_n(t) - f_0(t) | \, dt \to 0,$$

then f_n is said to converge *almost in the mean to* f_0 on E.

If $\{f_n\}$ is convergent almost in the mean on E then it is easily seen that there exists a function f_0 defined on E such that f_n converges almost in the mean to f_0 on E.

DEFINITION 5.4. By $f_n - \mu \to f_0$ is meant this: f_n is in CR for $n = 0, 1, 2, \cdots$ and f_n converges almost in the mean to f_0 on $[0, 1]$.

THEOREM 5.6. *The relation* $Y_n - l \to Y_0$ *implies and is implied by the two relations* $Y_n - v \to Y_0$ *and* $Y_n' - \mu \to Y_0'$.

We have already seen (Theorem 5.1) that $Y_n - l \to Y_0$ implies $Y_n - v \to Y_0$. We now propose to show that $Y_n - l \to Y_0$ implies $Y_n' - \mu \to Y_0'$.

Define

$$\alpha(t) = \int_0^t Y_0(s)ds, \qquad \beta(t) = Y_0(t) - \alpha(t) \text{ for } t \text{ on } [0, 1],$$

and let ϵ be any positive number. The singularity of β implies, as is well known, the existence for each $m=1, 2, 3, \cdots$ of non-overlapping intervals $[a_{m,1}, b_{m,1}], [a_{m,2}, b_{m,2}], \cdots, [a_{m,N_m}, b_{m,N_m}]$ contained in $[0, 1]$ such that

$$\sum_{j=1}^{N_m} T_{a_{m,j}}^{b_{m,j}}(\beta) < \frac{1}{m}, \qquad |A_m| > 1 - \frac{\epsilon}{2^m},$$

where $A_m = \sum_{j=1}^{N_m}[a_{m,j}, b_{m,j}]$. The absolute continuity of the function α implies that $(Y_n - \alpha) - l \rightarrow \beta$ which in turn implies $(Y_n - \alpha) - v \rightarrow \beta$. Letting $A = A_1A_2A_3 \cdots$ we conclude from Lemmas 3.1 and 3.7 that

$$\limsup_{n\to\infty} \int_A |Y_n'(t) - Y_0'(t)| dt = \limsup_{n\to\infty} \int_A |Y_n'(t) - \alpha'(t)| dt$$

$$\leqq \limsup_{n\to\infty} \int_{A_m} |Y_n'(t) - \alpha'(t)| dt = \limsup_{n\to\infty} \sum_{j=1}^{N_m} \int_{a_{m,j}}^{b_{m,j}} |Y_n'(t) - \alpha'(t)| dt$$

$$\leqq \limsup_{n\to\infty} \sum_{j=1}^{N_m} T_{a_{m,j}}^{b_{m,j}}(Y_n - \alpha) = \sum_{j=1}^{N_m} T_{a_{m,j}}^{b_{m,j}}(\beta) < \frac{1}{m} \qquad (m = 1, 2, 3, \cdots)$$

which implies $\int_A |Y_n'(t) - Y_0'(t)| dt \rightarrow 0$. Clearly $|A| > 1 - \epsilon$ so that from Definition 5.4 follows the relation $Y_n' - \mu \rightarrow Y_0'$.

Let us assume now that $Y_n - v \rightarrow Y_0$ and $Y_n' - \mu \rightarrow Y_0'$. Define

$$\alpha_n(t) = \int_0^t Y_n(s)ds, \quad \beta_n(t) = Y_n(t) - \alpha_n(t) \quad \text{for} \quad 0 \leqq t \leqq 1; n = 0, 1, 2, \cdots,$$

and let $\epsilon > 0$. There exists a set $E \subset [0, 1]$ such that $|E| > 1 - \epsilon$ and

$$\int_E |\alpha_n'(t) - \alpha_0'(t)| dt = \int_E |Y_n'(t) - Y_0'(t)| dt \rightarrow 0.$$

Denoting by E' the complement of E with respect to $[0, 1]$ it is seen that the last relation combines with

$$\int_E |\alpha_n'(t)| dt + \int_{E'} |\alpha_n'(t)| dt + T_0^1(\beta_n) - \int_E |\alpha_0'(t)| dt$$
$$- \int_{E'} |\alpha_0'(t)| dt - T_0^1(\beta_0) = \{T_0^1(Y_n) - T_0^1(Y_0)\} \rightarrow 0$$

to yield

$$\left\{ \int_{E'} |\alpha_n'(t)| dt + T_0^1(\beta_n) - \int_{E'} |\alpha_0'(t)| dt - T_0^1(\beta_0) \right\} \rightarrow 0.$$

Thus from the relation

$$\begin{aligned} &T_{t=0}^{1}\{t + iY_n(t)\} - T_{t=0}^{1}\{t + iY_0(t)\} \\ &= T_{t=0}^{1}\{t + i\alpha_n(t)\} + T_0^1(\beta_n) - T_{t=0}^{1}\{t + i\alpha_0(t)\} - T_0^1(\beta_0) \\ &= \int_E |1 + i\alpha_n'(t)|\,dt + \int_{E'} |1 + \alpha_n'(t)|\,dt + T_0^1(\beta_n) \\ &\quad - \int_E |1 + i\alpha_0'(t)|\,dt - \int_{E'} |1 + \alpha_0'(t)|\,dt - T_0^1(\beta_0) \\ &\leqq \int_E |\alpha_n'(t) - \alpha_0'(t)|\,dt + \int_{E'} |\alpha_n'(t)|\,dt + T_0^1(\beta_n) \\ &\quad - \int_{E'} |\alpha_0'(t)|\,dt - T_0^1(\beta_0) + 2|E'|, \end{aligned}$$

which holds for $n=1, 2, 3, \cdots$ we conclude

$$\begin{aligned} 0 &\leqq \liminf_{n\to\infty} T_{t=0}^{1}\{t + iY_n(t)\} - T_{t=0}^{1}\{t + iY_0(t)\} \\ &\leqq \limsup_{n\to\infty} T_{t=0}^{1}\{t + iY_n(t)\} - T_{t=0}^{1}\{t + iY_0(t)\} \leqq 2|E'| < 2\epsilon. \end{aligned}$$

The arbitrariness of ϵ completes the proof.

COROLLARY 5.2. *The relation $Y_n - l \to Y_0$ implies and is implied by the two relations $Y_n - v \to Y_0$ and Y_n' converges in measure to Y_0' on $[0, 1]$.*

Convergence in measure implies almost convergence in the mean of a subsequence.

An immediate consequence is

THEOREM 5.7. *Let $X_n - l \to X_0$ and $Y_n - l \to Y_0$. Then a necessary and sufficient condition for $(X_n + Y_n) - l \to (X_0 + Y_0)$ is that $(X_n + Y_n) - v \to (X_0 + Y_0)$; furthermore, a necessary and sufficient condition for $X_n Y_n - l \to X_0 Y_0$ is that $X_n Y_n - v \to X_0 Y_0$.*

THEOREM 5.8. *Let Y_n be in RBV with $P_n(t)$ and $N_n(t)$ denoting the positive and negative variations of Y_n on $[0, 1]$, $(n=0, 1, 2, \cdots; 0 \leqq t \leqq 1)$. Then the relation $Y_n - l \to Y_0$ implies the relations $P_n - l \to P_0$ and $N_n - l \to N_0$.*

To prove this theorem verify first the relations† $P_n - v \to P_0$, $N_n - v \to N_0$, and

$$\tfrac{1}{2}(P_n - N_n) - l \to \tfrac{1}{2}(P_0 - N_0), \qquad \tfrac{1}{2}(P_n + N_n) - l \to \tfrac{1}{2}(P_0 + N_0).$$

The desired conclusions are now immediate consequences of Theorem 5.7.

† AC, Theorem 1; AL, Corollary to Theorem 1.

6. **Uniform convergence in length.** Theorem 4.2 combined with the methods used in proving Theorem 5.3 yields

THEOREM 6.1. *If u is applicable (R) to Y_n for $n=0, 1, 2, \cdots$, then the relation $Y_n - ul \to Y_0$ implies*

$$(u|Y_n) - ul \to (u|Y_0).$$

We now recall the concept of an elementary step-function and of a singular function of the saltus type.

DEFINITION 6.1. A function f defined on $[a, b]$ is said to be an *elementary step-function* there if there exists a real number c on $[a, b]$ and complex numbers $\gamma_1, \gamma_2, \gamma_3$, such that

$$f(t) = \gamma_1, \text{ if } 0 \leqq t < c; \; f(c) = \gamma_2; \; f(t) = \gamma_3 \text{ if } c < t \leqq b.$$

DEFINITION 6.2. A function f of b.v. on $[a, b]$ is said to be a *singular function of the saltus type* on $[a, b]$ if there exist elementary step-functions $f_1, f_2, f_3, \cdots$ defined on $[a, b]$ such that

$$f(t) = \sum_{n=1}^{\infty} f_n(t), \quad a \leqq t \leqq b; \qquad T_a^b(f) = \sum_{n=1}^{\infty} T_a^b(f_n).$$

It is readily seen that a singular function of the saltus type is singular, though we shall not make explicit use of this property. From the definition of pseudo-absolute continuity it follows that if f is a pseudo-absolutely continuous function in BV, then there exists an a.c. function α in BV and a singular function β of the saltus type in BV such that $f = \alpha + \beta$.

LEMMA 6.1. *If β is an elementary step-function in BV, then the relation $f_n - uv \to f_0$ implies the relation*

$$(f_n + \beta) - uv \to (f_0 + \beta).$$

There exist a real number c on $[0, 1]$ and complex numbers $\gamma_1, \gamma_2, \gamma_3$, such that $\beta(t) = \gamma_1$ if $0 \leqq t < c$, $\beta(c) = \gamma_2$, and $\beta(t) = \gamma_3$ if $c < t \leqq 1$, so that as a consequence of Lemma 3.1 it may be deduced that

$$\begin{aligned}
T_0^{c-}(f_n + \beta) &= T_{t=0}^{c-}\{f_n(t) + \gamma_1\} = T_0^{c-}(f_n) \to T_0^{c-}(f_0) \\
&= T_{t=0}^{c-}\{f_0(t) + \gamma_1\} = T_0^{c}(f_0 + \beta), \\
|f_n(c) + \gamma_2 - &f_n(c-) - \gamma_1| + |f_n(c+) + \gamma_3 - f_n(c) - \gamma_2| \\
&\to |f_0(c) + \gamma_2 - f_0(c-) - \gamma_1| + |f_0(c+) + \gamma_3 - f_0(c) - \gamma_2|, \\
T_{c+}^{1}(f_n + \beta) &= T_{t=c+}^{1}\{f_n(t) + \gamma_3\} = T_{c+}^{1}(f_n) \to T_{c+}^{1}(f_0) \\
&= T_{t=c+}^{1}\{f_0(t) + \gamma_3\} = T_{c+}^{1}(f_0 + \beta).
\end{aligned}$$

Combining these three relations establishes the lemma.

LEMMA 6.2. *If β (in BV) is a singular function of the saltus type, then the relation $f_n - uv \to f_0$ implies the relation $(f_n + \beta) - uv \to (f_0 + \beta)$.*

There exist elementary step-functions $\beta_1, \beta_2, \beta_3, \cdots$ such that $\beta(t) = \sum_{j=1}^{\infty} \beta_j(t)$ for t on $[0, 1]$ with $T_0^1(\beta) = \sum_{j=1}^{\infty} T_0^1(\beta_j)$. Hence $T_0^1(B_p - \beta) \to 0$, where $B_p(t) = \sum_{j=1}^{p} \beta_j(t)$. From the preceding lemma we conclude (by induction) that

$$\lim_{n \to \infty} \|f_n + B_p\| = \|f_0 + B_p\| \qquad (p = 1, 2, 3, \cdots).$$

Thus follows

$$\begin{aligned} \|f_0 + \beta\| &\leqq \liminf_{n \to \infty} \|f_n + \beta\| \leqq \limsup_{n \to \infty} \|f_n + \beta\| \\ &\leqq \limsup_{n \to \infty} \|f_n + B_p\| + \|\beta - B_p\| = \|f_0 + B_p\| + \|\beta - B_p\| \\ &\leqq \|f_0 + \beta\| + 2\|\beta - B_p\| \to \|f_0 + \beta\| \quad \text{as} \quad p \to \infty, \end{aligned}$$

and the proof is completed.

LEMMA 6.3. *If X_0 is an a.c. function in RBV, then the relations $X_n - l \to X_0$ and $Y_n - l \to Y_0$ imply the relation $(X_n + iY_n) - v \to (X_0 + iY_0)$.*

Define $\alpha(t) = Y_0(0) + \int_0^t Y'(s)ds$, $\beta(t) = Y(t) - \alpha(t)$ for t on $[0, 1]$ noting that β is singular with $\beta(0) = 0$. From Theorems 5.1 and 5.4 we have $\|Y_n - \alpha\| \to \|\beta\|$ and since $\|X_n - X_0\| \to 0$ we deduce $\|X_n + i\alpha\| \to \|X_0 + i\alpha\|$, so that

$$\begin{aligned} \|X_n + iY_n\| &= \|X_n + i(Y_n - \alpha + \alpha)\| \\ &\leqq \|X_n + i\alpha\| + \|i(Y_n - \alpha)\| \to \|X_0 + i\alpha\| + \|i\beta\| \\ &= \|X_0 + i(\alpha + \beta)\| = \|X_0 + iY_0\|, \end{aligned}$$

and the proof is complete (by semi-continuity).

THEOREM 6.2. *If X_0 is a pseudo-absolutely continuous function in RBV, then the relations $X_n - ul \to X_0$ and $Y_n - ul \to Y_0$ imply the relations*

$$(X_n + Y_n) - ul \to (X_0 + Y_0) \quad \textit{and} \quad X_n Y_n - ul \to X_0 Y_0.$$

Since X_0 is pseudo-absolutely continuous we conclude the existence of an a.c. function α in RBV and a singular function β of the saltus type likewise in RBV for which $X_0 = \alpha + \beta$. As a corollary of Lemma 6.2, $(X_n - \beta) - ul \to \alpha$, and since $Y_n - ul \to Y_0$ it is seen from Lemma 6.3 that

$$(X_n - \beta + iY_n) - uv \to (\alpha + iY_0),$$

so that using Lemma 6.2 again, yields

$$(X_n + iY_n) - uv \to (\alpha + \beta + iY_0) = (X_0 + iY_0).$$

Letting $\phi_1(x+iy)=x+y$ and $\phi_2(x+iy)=xy$ for $-\infty<x,\ y<\infty$ we conclude from Theorem 4.2 that

$$\phi_1:(X_n+iY_n)-uv\rightarrow\phi_1:(X_0+iY_0),$$

$$\phi_2:(X_n+iY_n)-uv\rightarrow\phi_2:(X_0+iY_0).$$

That is,

$$(X_n+Y_n)-uv\rightarrow(X_0+Y_0)\quad\text{and}\quad X_nY_n-uv\rightarrow X_0Y_0.$$

Application of Theorem 5.7 completes the proof.

7. **Strong convergence.** It is at once apparent that strong convergence implies every other type considered in this paper, and also that it is invariant under addition and multiplication. It is natural to ask if Theorem 6.1 likewise holds for strong convergence. The answer is yes, but before proving this we state as an obvious corollary of Lemma 4.5 the following

LEMMA 7.1. *If Y is a pseudo-absolutely continuous function in RBV and u is applicable (R) to Y, then $(u|Y)$ is likewise pseudo-absolutely continuous.*

We are now prepared to prove

THEOREM 7.1. *If u is applicable (R) to Y_n for $n=0, 1, 2, \cdots$, then the relation $Y_n-s\rightarrow Y_0$ implies*

$$(u|Y_n)-s\rightarrow(u|Y_0).$$

Since Y_0 is in RBV there exist a continuous function α in RBV and a singular function β of the saltus type likewise in RBV for which $Y_0=\alpha+\beta$. Defining

$$S(t)=\frac{t+T_0^t(\alpha)}{1+T_0^1(\alpha)}\qquad(0\leqq t\leqq 1),$$

it is seen that S is a continuous increasing function in RBV. Clearly there exists an increasing function Ψ satisfying a Lipschitz condition on $\underset{x}{E}[-\infty<x<\infty]$ for which $\Psi\{S(t)\}=t,\ 0\leqq t\leqq 1$. Let $A(s)=\alpha\{\Psi(s)\}$, $B(s)=\beta\{\Psi(s)\}$, $\eta_n(s)=Y_n\{\Psi(s)\}$, $(0\leqq s\leqq 1;\ n=0, 1, 2, \cdots)$. Finally let $u_1(x, y)=u(\Psi(x), y)$, $-\infty<x,\ y<\infty$.

First notice that

$$\begin{aligned}|A(s_2)-A(s_1)|&\leqq T_{\Psi(s_1)}^{\Psi(s_2)}(\alpha)<\Psi(s_2)-\Psi(s_1)+T_{\Psi(s_1)}^{\Psi(s_2)}(\alpha)\\&=(1+T_0^1(\alpha))(S\{\Psi(s_2)\}-S\{\Psi(s_1)\})=(1+T_0^1(\alpha))(s_2-s_1)\end{aligned}$$

if $0\leqq s_1\leqq s_2\leqq 1$, which implies absolute continuity of A. Since β is a singular function of the saltus type, it may be seen from Lemma 3.8 that B is likewise a singular function of the saltus type so that noting $\eta_0 = A + B$ establishes the pseudo-absolute continuity of η_0. Now from the definitions involved and the fact that Ψ is increasing and absolutely continuous it becomes apparent that if u is $\mathfrak{R}$, then u_1 is $\mathfrak{R}$; if u is $\mathfrak{R}_2$, then u_1 is $\mathfrak{R}_2$; and since monotonicity of Y_n implies monotonicity of η_n for $n=0, 1, 2, \cdots$, we conclude that u_1 is applicable (R) to η_n for $n=0, 1, 2, \cdots$. Another application of Lemma 3.8 yields the relation

$$\left\| \eta_n - \eta_0 \right\| = \left\| Y_n - Y_0 \right\| \to 0,$$

so that, since strong convergence implies uniform convergence in length, we may deduce successively (with the help of Lemma 7.1, Theorems 6.1 and 6.2) the following relations,

$$\eta_n - ul \to \eta_0, \quad (u_1|\eta_n) - ul \to (u_1|\eta_0), \quad \{(u_1|\eta_n) - (u_1|\eta_0)\} - ul \to \theta,$$
$$\left\| (u_1|\eta_n) - (u_1|\eta_0) \right\| \to 0.$$

Thus (Lemma 3.8)

$$\begin{aligned} T^1_{t=0}\{u(t, Y_n(t)) - u(t, Y_0(t))\} &= T^1_{s=0}\{u[\Psi(s), Y_n(\Psi(s))] - u[\Psi(s), Y_0(\Psi(s))]\} \\ &= T^1_{s=0}\{u(\Psi(s), \eta_n(s)) - u(\Psi(s), \eta_0(s))\} = T^1_{s=0}\{u_1(s, \eta_n(s)) - u_1(s, \eta_0(s))\} \\ &\leqq \left\| (u_1|\eta_n) - (u_1|\eta_0) \right\| \to 0, \end{aligned}$$

which implies immediately

$$\left\| (u|Y_n) - (u|Y_0) \right\| \to 0$$

as was to be proved.

THEOREM 7.2. *If f_0 is an a.c. function in BV, then a necessary and sufficient condition that $f_n - s \to f_0$ is that*

$$(cI + f_n) - v \to (cI + f_0),$$

for all real numbers c.

The necessity being obvious we turn to the sufficiency. Let $X_n + iY_n = f_n$ for $n=0, 1, 2, \cdots$, and define

$$\phi_1(x+iy) = x, \ \phi_2(x+iy) = x+y, \text{ for } -\infty < x, y < \infty.$$

Since ϕ_1 and ϕ_2 are applicable to $(cI+f_n)$ for $n=0, 1, 2, \cdots$ and strictly applicable to $(cI+f_0)$ whatever real number c may be, we conclude

$$(cI + X_n) - v \to (cI + X_0), \quad (cI + X_n + Y_n) - v \to (cI + X_0 + Y_0)$$

for all real numbers c. Whence, with the help of Theorems 5.5 and 5.4 follow successively the relations

$$X_n - l \to X_0, \quad (X_n + Y_n) - l \to (X_0 + Y_0), \quad Y_n - l \to Y_0,$$

$$\|X_n - X_0\| \to 0, \quad \|Y_n - Y_0\| \to 0, \quad \|f_n - f_0\| \to 0$$

and the proof is complete.

8. Applications; generalizations of a theorem of Plessner and its converse. It is well known that if $p>1$ then the two relations

$$\int_0^t f_n(s)ds \to \int_0^t f_0(s)ds \qquad (0 \leqq t \leqq 1)$$

and

$$\int_0^1 | f_n(t) |^p dt \to \int_0^1 | f_0(t) |^p dt$$

imply and are implied by the relation

$$\int_0^1 | f_n(t) - f_0(t) |^p dt \to 0.$$

It is likewise well known that the theorem is not true for $p=1$. The following theorem would therefore seem of some interest.

THEOREM 8.1. *If f_n is a summable function in CR for $n=0, 1, 2, \cdots$, then the two relations*

$$\int_0^t f_n(s)ds \to \int_0^t f_0(s)ds \qquad (0 \leqq t \leqq 1),$$

$$\int_0^1 | c + f_n(t) | dt \to \int_0^1 | c + f_0(t) | dt \textit{ for all real numbers } c,$$

imply and are implied by the relation

$$\int_0^1 | f_n(t) - f_0(t) | dt \to 0.$$

Obviously the last relation implies the first two. Assuming the first two relations to be true and defining $F_n(t) = \int_0^t f_n(s)ds$ (for $0 \leqq t \leqq 1$; $n=0, 1, 2, \cdots$), we note from the first relation that $(ct+F_n(t)) \to (ct+F_0(t))$ for t on $[0, 1]$, and from the second relation that

$$T_{t=0}^1\{ct + F_n(t)\} = \int_0^1 | c + f_n(t) | dt \to \int_0^1 | c + f_0(t) | dt = T_{t=0}^1\{ct + F_0(t)\}$$

for all real numbers c. Hence

$$(cI + F_n) - v \to (cI + F_0) \text{ for all real numbers } c,$$

so that Theorem 7.2 implies

$$\int_0^1 | f_n(t) - f_0(t) | \, dt = T_0^1(F_n - F_0) \to 0$$

which completes the proof.

A result of Ursell combined with a theorem of Plessner establishes the following theorem:†

Let f be a finite, real-valued, measurable function with period 1. *If*

$$\lim_{h\to 0} T_{t=0}^1\{f(t+h) - f(t)\} = 0,$$

then f is a.c. on $[0, 1]$.

We now propose to generalize this theorem. First, however, it is convenient to prove the following

THEOREM 8.2. *Let f be a finite, real-valued, function with period* 1 *which is measurable on a set of positive measure. If there exists a non-vanishing function g in RBV such that*

$$\lim_{h\to 0} T_{t=0}^1\{f(t) + hg(t)\} = 0,$$

then f is continuous on $[0, 1]$.

Clearly there exists a closed set D on $[0, 1]$ of positive measure relative to which f is continuous. Let β be the characteristic function of this set and denote $\int_0^y \beta(x)dx$ by $B(y)$ for $-\infty < y < \infty$. Now B satisfies a Lipschitz condition so that use of Theorems 7.1 and 3.1, Lemma 3.7, and Corollary 3.1 establishes the relation

$$\begin{aligned} 0 &= \lim_{h\to 0} T_{t=0}^1\{B(t + hg(t)) - B(t)\} \\ &\geqq \limsup_{h\to 0} \int_0^1 | B'(t + hg(t))(1 + hg'(t)) - B'(t) | \, dt \\ &= \limsup_{h\to 0} \int_0^1 | \beta(t + hg(t))(1 + hg'(t)) - \beta(t) | \, dt. \end{aligned}$$

Hence there exists a $\delta_0 > 0$ such that $|h| < \delta_0$ implies

$$\int_0^1 | \beta(t + hg(t))(1 + hg'(t)) - \beta(t) | \, dt < \int_0^1 |\beta(t)| dt,$$

† An elementary proof was given by N. Dunford, Bulletin of the American Mathematical Society, vol. 41 (1935), pp. 356–358.

so that corresponding to each h^* for which $|h^*| < \delta_0$ there exists a point t^* in D, such that $t^* + h^* g(t^*)$ is likewise in D; for assuming the contrary leads immediately to a contradiction of the above relation.

Let ϵ be any positive number. Since D is closed there exists a $\delta_1 > 0$ such that, $|h| < \delta_1$ implies

$$|f(t + hg(t)) - f(t)| < \frac{\epsilon}{2}$$

for all t on $[0, 1]$ for which t and $t + hg(t)$ are both in D. By hypothesis there exists a $\delta_2 > 0$ such that $|h| < \delta_2$ implies

$$T_{t=0}^{1}\{f(t + hg(t)) - f(t)\} < \frac{\epsilon}{2}.$$

Let δ be the least of the numbers δ_0, δ_1, δ_2, and let h_0 be any number $< \delta$ in absolute value. As we have seen, there exists a point t_0 in D such that $t_0 + h_0 g(t_0)$ is likewise in D. Hence

$$\begin{aligned} |f(t_1 + h_0 g(t_1)) - f(t_1)| &\leqq |f(t_1 + h_0 g(t_1)) - f(t_1) - f(t_0 + h_0 g(t_0)) + f(t_0)| \\ &\quad + |f(t_0 + h_0 g(t_0)) - f(t_0)| \\ &< T_{t=0}^{1}\{f(t + h_0 g(t)) - f(t)\} + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

for t_1 on $[0, 1]$. Hence f is continuous on $[0, 1]$ and the proof is complete.

It should be noted that the only place in the proof where a result of this paper is used is in proving

$$\lim_{h \to 0} \int_0^1 |\beta(t + hg(t))(1 + hg'(t)) - \beta(t)| \, dt = 0.$$

However if $g(t) = 1$ for t on $[0, 1]$, this relation is an immediate consequence of a well known theorem of Lebesgue, which in connection with the method used in proving Theorem 8.3 leads to a proof of the Plessner theorem which is independent of the preceding results in this paper.

We now turn to

THEOREM 8.3. *Let f be a finite, real-valued function with period* 1 *which is measurable on a set of positive measure. If there exist a function g in RBV and a positive number r such that*

$$\lim_{h \to 0} T_{t=0}^{1}\{f(t + hg(t)) - f(t)\} = 0$$

with $|g(t)| > r$ for t on $[0, 1]$, *then f is a.c. on* $[0, 1]$.

Let $\delta>0$ be such that $T_{t=0}^{1}U(t, h)<1$ for $|h|\leqq\delta$, where $U(t, h) = [f(t+hg(t))-f(t)]g(t)$ for $0\leqq t\leqq 1$, $h\leqq\delta$; let $\alpha(t)=1/g(t)$, $(0\leqq t\leqq 1)$; let $r_0=r^{-1}$ and note that $\|\alpha\|<\infty$, $|\alpha(t)|<r_0$ for t on $[0, 1]$; let $V(h)=T_{t=0}^{1}U(t, h)$ for $|h|\leqq\delta$. Since V is bounded on $[-\delta, \delta]$ and f is continuous by the previous theorem, the semi-continuity property of total variation shows that V is lower semi-continuous and hence summable on $[-\delta, \delta]$. Let

$$M(h) = \sup_{0\leqq t\leqq 1, |s|\leqq h} U(t, s) \qquad (0 \leqq h \leqq \delta)$$

remarking that M is monotone on $[0, \delta]$ and $M(h)\to 0$ as $h\to 0+$. Define

$$F_h(t) = \frac{1}{h}\int_t^{t+h} f(s)ds \qquad (0 \leqq t \leqq 1, 0 \leqq h \leqq r_1),$$

where $r_1=\delta r$, and let S be any partition of $[0, 1]$ with $S\equiv(0=t_0<t_1<t_2 < \cdots <t_N=1)$.

From the relation

$$\begin{aligned} F_h(t) - f(t) &= \frac{1}{h}\int_0^h \{f(t+s) - f(t)\}ds \\ &= \frac{1}{h}\int_0^{h\alpha(t)} \{f(t+sg(t)) - f(t)\}g(t)ds \\ &= \frac{1}{h}\int_0^{h\alpha(t)} U(t, s)ds \qquad (0 \leqq t \leqq 1, 0 \leqq h \leqq r_1) \end{aligned}$$

and the relation

$$\begin{aligned} &\sum_{j=1}^{N}\left|\frac{1}{h}\int_0^{h\alpha(t_j)} U(t_j, s)ds - \frac{1}{h}\int_0^{h\alpha(t_{j-1})} U(t_{j-1}, s)ds\right| \\ &\quad = \sum_{j=1}^{N}\left|\frac{1}{h}\int_0^{h\alpha(t_j)} U(t_j, s) - U(t_{j-1}, s)ds + \frac{1}{h}\int_{h\alpha(t_{j-1})}^{h\alpha(t_j)} U(t_{j-1}, s)ds\right| \\ &\quad \leqq \sum_{j=1}^{N}\left\{\frac{1}{h}\int_{-hr_0}^{hr_0} | U(t_j, s) - U(t_{j-1}, s) | ds + \frac{1}{h}\left|\int_{h\alpha(t_{j-1})}^{h\alpha(t_j)} M(hr_0)ds\right|\right\} \\ &\quad \leqq \frac{1}{h}\int_{-hr_0}^{hr_0} \sum_{j=1}^{N} | U(t_j, s) - U(t_{j-1}, s) | ds + \sum_{j=1}^{N} M(hr_0) | \alpha(t_j) - \alpha(t_{j-1}) | \\ &\quad \leqq \frac{1}{h}\int_{-hr_0}^{hr_0} V(s)ds + M(hr_0)\|\alpha\| \qquad (0 \leqq h \leqq r_1), \end{aligned}$$

we deduce (since S was arbitrary) the relation

$$T_0^1(F_h - f) \leqq \frac{1}{h}\int_{-hr_0}^{hr_0} V(s)ds + M(hr_0)\|\alpha\| \qquad (0 \leqq h \leqq r_1).$$

Now, Lemma 3.2 yields the relation

$$V(s) \leqq [\,|f(sg(0)) - f(0)| + T_{t=0}^{1}\{f(t + sg(t)) - f(t)\}\,]\|g\| \qquad (|s| \leqq r_1),$$

so that $V(s) \to 0$ as $s \to 0$. Hence

$$\lim_{h\to 0+} T_0^1\{F_h - f\} = 0,$$

where F_h is a.c. for $0 \leqq h \leqq r_1$ which implies, as is well known, the absolute continuity of f on [0, 1]. This completes the proof.

From the results in §7 it is clear that a variety of theorems concerning the behavior of $T_{t=0}^1\{f(t+hg(t)) - f(t)\}$ as $h \to 0$ (or as $h \to 0+$, $h \to 0-$) can be readily proved. Among these is one which can be proved directly without great difficulty, and which forms the necessity part of the next and concluding theorem. This theorem is a simultaneous extension of Plessner's theorem and its converse.

THEOREM 8.4. *Let f be a finite, real-valued function with period* 1 *which is measurable on set of positive measure. Let g be a non-vanishing function in RBV which satisfies a Lipschitz condition. Then a necessary and sufficient condition that f be a.c. on* [0, 1] *is that*

$$\lim_{h\to 0} T_{t=0}^{1}\{f(t + hg(t)) - f(t)\} = 0.$$

Simply note that $t + hg(t)$ increases with t for h sufficiently small and apply Theorem 7.1.

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