GENERALIZED WEIGHT PROPERTIES OF THE RESULTANT OF $n+1$ POLYNOMIALS IN n INDETERMINATES†

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1. **Introduction.** *The multiplicity of intersection of two plane algebraic curves, $f(x, y)=0$ and $g(x, y)=0$, at a common point $O(a, b)$, r-fold for f and s-fold for g, is not less than rs, and is greater than rs if and only if the two curves have in common a principal tangent at O.* The standard proof of this well known theorem of the theory of higher plane curves makes use of Puiseux expansions. If, namely, $R(x)=R(f, g)$ denotes the resultant of f and g, considered as polynomials in y, and if $y_1, y_2, \cdots, y_n$ and $\bar{y}_1, \bar{y}_2, \cdots, \bar{y}_m$ are the roots of $f=0$ and $g=0$ respectively, then, the axes being in generic position, the intersection multiplicity at O is defined as the multiplicity of the root $x=a$ of the resultant $R(x)$, and this multiplicity is found by substituting into the product $\prod_{i=1}^{n}\prod_{j=1}^{m}(y_i-\bar{y}_j)$ the Puiseux expansions of the roots y_i and $\bar{y}_j$. A less known proof, in which the multiplicity to which the factor $x-a$ occurs in $R(x)$ is derived in a purely algebraic manner, was given by C. Segre.‡ Following a procedure due to A. Voss,§ Segre uses the Sylvester determinant and arrives at the required result by a skillful manipulation of the rows and columns.

In the first part of this paper (§§2, 3), we give a new proof of the property of the resultant $R(f, g)$ (see Theorem 1), which is implicitly contained in the quoted paper by C. Segre and of which the above intersection theorem is an immediate corollary. This proof makes use only of the intrinsic properties of the resultant and so contains the germ of an extension to the case of $n+1$ polynomials in n variables. In the second part (§§4–9) we extend Theorem 1 to the resultant of $n+1$ polynomials (Theorem 6). From Theorem 6 follows as a corollary the analogous intersection theorem for hypersurfaces in S_{n+1} (§9).

I. Two polynomials in one variable

2. **A generalized weight property of the resultant.** Let

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‡ C. Segre, *Le molteplicità nelle intersezioni delle curve piane algebriche con alcune applicazioni ai principi della teoria di tali curve*, Giornale de Matematiche di Battaglini, vol. 36 (1898).

§ A. Voss, *Über einen Fundamentalsatz aus der Theorie der algebraischen Functionen*, Mathematische Annalen, vol. 27 (1886).

$$f = a_0 y^n + a_1 y^{n-1} + \cdots + a_n,$$
$$g = b_0 y^m + b_1 y^{m-1} + \cdots + b_m,$$

be two polynomials, with literal coefficients, and let $R(f, g)$ be their resultant:

$$R(f, g) = \sum a_0^{i_0} a_1^{i_1} \cdots a_n^{i_n} b_0^{j_0} b_1^{j_1} \cdots b_m^{j_m},$$

where, by well known properties of R, we have

$$i_0 + i_1 + \cdots + i_n = m, \quad j_0 + j_1 + \cdots + j_m = n,$$
$$i_1 + 2i_2 + \cdots + ni_n + j_1 + 2j_2 + \cdots + mj_m = mn.$$

THEOREM 1. *Let r and s be two non-negative integers, $r \leqq n$, $s \leqq m$. If we give to each coefficient a_i (b_j) the weight $r-i$ ($s-j$) or zero, according as $r-i \geqq 0$ ($s-j \geqq 0$) or $r-i \leqq 0$ ($s-j \leqq 0$), then the weight of any term in the resultant $R(f, g)$ is $\geqq rs$. The sum of terms of weight rs is given by the following expression:*

$$(-1)^{(m-s)r} R(f_r, g_s) R(f^*_{n-r}, g^*_{m-s}),$$

where

$$f_r = a_0 y^r + \cdots + a_r, \qquad f^*_{n-r} = a_r y^{n-r} + \cdots + a_n;$$
$$g_s = b_0 y^s + \cdots + b_s, \qquad g^*_{m-s} = b_s y^{m-s} + \cdots + b_m.$$

We consider the polynomials

$$\bar{f} = a_0 t^r y^n + \cdots + a_{r-1} t y^{n-r+1} + a_r y^{n-r} + \cdots + a_n,$$
$$\bar{g} = b_0 t^s y^m + \cdots + b_{s-1} t y^{m-s+1} + b_s y^{m-s} + \cdots + b_m,$$

where t is a new indeterminate. Let t^k be the highest power of t which divides the resultant $R(\bar{f}, \bar{g})$, $\bar{f}$ and $\bar{g}$ being considered as polynomials in y:

$$(1) \qquad R(\bar{f}, \bar{g}) = t^k R_1(a_i, b_j, t), \qquad R_1(a_i, b_j, 0) \neq 0.$$

By a well known property of the resultant, we have $R(\bar{f}, \bar{g}) = A\bar{f} + B\bar{g}$, where A and B are polynomials in y, a_i, b_j, t, with integral coefficients; or using a familiar notation: $R(\bar{f}, \bar{g}) \equiv 0(\bar{f}, \bar{g})$. We put $\bar{f} = \bar{f}^* + a_n$, $\bar{g} = \bar{g}^* + b_m$. If we make the substitution $a_n = -\bar{f}^*$, $b_m = -\bar{g}^*$ in the identity $R(\bar{f}, \bar{g}) = A\bar{f} + B\bar{g}$, then $\bar{f}$ and $\bar{g}$ vanish, and therefore also $t^k R_1(a_i, b_j, t)$ must vanish. Since t is unaltered by the substitution, we have

$$R_1(a_0, \cdots, a_{n-1}, -\bar{f}^*; b_0, \cdots, b_{m-1}, -\bar{g}^*; t) = 0.$$

If we now order $R_1(a_i, b_j, t)$ according to the powers of $a_n + \bar{f}^*$ and $b_m + \bar{g}^*$, the constant term vanishes, and hence $R_1(a_i, b_j, t) \equiv 0(\bar{f}, \bar{g})$.† Putting $t = 0$,

† This proof that $t^k R_1 \equiv 0(\bar{f}, \bar{g})$ implies $R_1 \equiv 0(\bar{f}, \bar{g})$ is taken from van der Waerden, *Moderne Algebra*, II, p. 15 (quoted in the sequel as W.). Further on we shall use frequently the notion and properties of *inertia forms* as given in W., pp. 15–21.

we find $R_{10}=R_1(a_i,\ b_j,\ 0)\equiv 0(f^*_{n-r},\ g^*_{m-s})$, and hence R_{10} vanishes whenever f^*_{n-r} and g^*_{m-s} have a common factor of degree $\geqq 1$ in y. Consequently, by a well known property of the resultant, *R_{10} is divisible by $R(f^*_{n-r}, g^*_{m-s})$*, provided, however, that $n\neq r$ and $m\neq s$ (inequalities assuring the irreducibility of $R(f^*_{n-r},\ g^*_{m-s})$).

If we now consider the resultant $R(\bar{f},\ \bar{g})$ of the following polynomials:

$$\bar{f} = a_0y^n + \cdots + a_ry^{n-r} + ta_{r+1}y^{n-r-1} + \cdots + t^{n-r}a_n,$$
$$\bar{g} = b_0y^m + \cdots + b_sy^{m-s} + tb_{s+1}y^{m-s-1} + \cdots + t^{m-s}b_m,$$

or, what is the same, the resultant of the polynomials

$$a_nt^{n-r}y^n + \cdots + a_{r+1}ty^{r+1} + a_ry^r + \cdots + a_0,$$
$$b_mt^{m-s}y^m + \cdots + b_{s+1}ty^{s+1} + b_sy^s + \cdots + b_0,$$

and if we put $R(\bar{f},\ \bar{g}) = t^lR_2(a_i,\ b_j,\ t)$ and $R_{20}=R_2(a_i,\ b_j,\ 0)$, where t^l is the highest power of t which divides $R(\bar{f},\ \bar{g})$, we conclude as before that R_{20} is divisible by the resultant of the polynomials

$$a_ry^r + a_{r-1}y^{r-1} + \cdots + a_0,$$
$$b_sy^s + b_{s-1}y^{s-1} + \cdots + b_0;$$

i.e., R_{20} is divisible by $R(f_r,\ g_s)$, provided $r\neq 0$ and $s\neq 0$. But since

$$\bar{f} = t^{n-r}\bar{f}\left(\frac{y}{t}\right), \qquad \bar{g} = t^{m-s}\bar{g}\left(\frac{y}{t}\right),$$

we have

$$t^{rs}R(\bar{f},\ \bar{g}) = t^{(n-r)(m-s)}R(\bar{f},\ \bar{g}),$$

i.e., $R(\bar{f},\ \bar{g})$ and $R(\bar{f},\ \bar{g})$ differ only by a factor which is a power of t. Hence $R_{20}=R_{10}$, and therefore *R_{10} is divisible by both $R(f^*_{n-r},\ g^*_{m-s})$ and $R(f_r,\ g_s)$.*

Assuming that $r\neq 0,\ n,\ s\neq 0,\ m$, we have that $R(f^*_{n-r},\ g^*_{m-s})$ and $R(f_r, g_s)$ are irreducible and distinct polynomials in the coefficients $a_i,\ b_j$. [a_0, for instance, actually occurs in $R(f_r,\ g_s)$, but does not occur in $R(f^*_{n-r},\ g_{m-s})$.] Hence *R_{10} is divisible by the product $R(f_r,\ g_s)\cdot R(f^*_{n-r},\ g^*_{m-s})$*. Since $R(\bar{f},\ \bar{g})$, and hence also R_{10}, is of degree m in the coefficients of f and of degree n in the coefficients of g, we conclude that

$$R_{10} = cR(f_r,\ g_s)R(f^*_{n-r},\ g^*_{m-s}),$$

where c is a numerical factor (an integer).

Assume $r=0,\ s\neq m$. Then $f^*_{n-r}=f$ and $R(f^*_{n-r},\ g^*_{m-s})$ is irreducible and of degree n in the coefficients of g, and consequently the quotient $R_{10}/R(f^*_{n-r},\ g^*_{m-s})$ is independent of the coefficients of g. On the other hand, $R(\bar{f},\ \bar{g})$ contains the term $a_0^m b_m^n$, so that the exponent k of t in (1) equals 0, and therefore R_{10} can vanish only if f and g^*_{m-s} have a common zero or if $a_0=0$. It follows that also in this case $R_{10}=ca_0^s R(f, g^*_{m-s})=c\cdot R(f_0, g_s)\, R(f, g^*_{m-s})$, where c is an integer.

The case $s=0$, $r\neq n$ is treated in a similar manner.

R_{10} is visibly given by the product $a_0^m b_m^n=R(f_0, g)R(f, g_0^*)$ if $r=0$, $s=m$, and a similar remark holds in the case $r=n$, $s=0$. Hence we have proved that in all cases

$$R_{10} = c\cdot R(f_r, g_s)R(f^*_{n-r}, g^*_{m-s}), \tag{2}$$

where c is an integer.

The resultant $R(\bar{f}, \bar{g})$ can be obtained from $R(f, g)$ by replacing $a_0, a_1, \cdots, a_{r-1}$ and $b_0, b_1, \cdots, b_{s-1}$ by $a_0t^r, a_1t^{r-1}, \cdots, a_{r-1}t$ and $b_0t^s, b_1t^{s-1}, \cdots, b_{s-1}t$ respectively. Every term of $R(f, g)$ acquires then a factor t^w, where w is the weight of this term as specified in the statement of Theorem 1. By (1), $R_{10}(=R_1(a_i, b_j, 0))$ is the sum of all terms of $R(f, g)$ of lowest weight k, and since, always according to our definition of the weight, $R(f_r, g_s)$ is isobaric of weight rs, while $R(f^*_{-r}, g^*_{m-s})$ is of weight zero, it follows that $k=rs$. This and the identity (2) complete the proof of our theorem.

To determine the numerical constant c, we take a special case, say $f=a_0y^n+a_n$, $g=b_sy^{m-s}$. Then $f_r=a_0y^r$, $g_s=b_s$, $f^*_{n-r}=a_n$, $g^*_{m-s}=b_sy^{m-s}$, and

$$R(f, g) = (-1)^{(m-s)n}a_0^s a_n^{m-s} b_s^n,$$
$$R(f_r, g_s) = a_0^s b_s^r, \quad R(f^*_{n-r}, g^*_{m-s}) = (-1)^{(n-r)(m-s)}a_n^{m-s}b_s^{n-r}.$$

Hence, in this case we have

$$R(f, g) = (-1)^{(m-s)r}R(f_r, g_s)R(f^*_{n-r}, g^*_{m-s}),$$

and consequently $c=(-1)^{(m-s)r}$.

Remark. The resultant of the polynomials f and g coincides, to within the sign, with the resultant of the polynomials $a_ny^n+\cdots+a_0$, $b_my^m+\cdots+b_0$. Applying our theorem to the last two polynomials, we see that it is permissible to interchange, in the statement of Theorem 1, a_i with a_{n-i} and b_j with b_{n-j}. This is equivalent to attaching the weights $r, r-1, \cdots, 1, s, s-1, \cdots, 1$ to $a_n, a_{n-1}, \cdots, a_{n-r+1}, b_m, b_{m-1}, \cdots, b_{m-s+1}$ respectively, and the weight 0 to the remaining coefficients.

3. **The intersection multiplicity of two curves at a common point.** The application of Theorem 1 toward the determination of the intersection multiplicity of two curves at a common point is immediate. If the coefficients a_i and b_j of the polynomials f and g are polynomials in x, and if the origin O is a common point of the two curves $f=f(x, y)=0$, and $g=g(x, y)=0$,

r-fold for f and s-fold for g, then $a_n, a_{n-1}, \cdots, a_{n-r+1}$ are divisible by x^r, $x^{r-1}, \cdots, x$, respectively and $b_m, b_{m-1}, \cdots, b_{m-s+1}$ are divisible by $x^s, x^{s-1}, \cdots, x$, respectively. Hence every term of the resultant $R(f, g) = R(x)$ is divisible by x^w, where w is the weight of the term as specified in the remark at the end of the preceding section. Since $w \geqq rs$, x^{rs} divides $R(x)$. Let

$$R(x) = \alpha x^{rs} + \text{terms of higher degree},$$

where α is a constant.

Let $f = \sum c_{ij} x^i y^j$, $g = \sum d_{ij} x^i y^j$. Then

$$c_{ij} = \left[\frac{a_{n-j}(x)}{x^{r-j}}\right]_{x=0},$$

for all i and j such that $i+j=r$; similarly

$$d_{ij} = \left[\frac{b_{m-j}(x)}{x^{s-j}}\right]_{x=0},$$

if $i+j=s$. Moreover, $c_{0,j} = a_{n-j}(0)$, $j = r, r+1, \cdots, n$, and $d_{0,j} = b_{m-j}(0)$, $j = s, s+1, \cdots, m$. Applying Theorem 1, we find

$$\alpha = \pm R(f_r, g_s) R(f^*_{n-r}, g^*_{m-s}),$$

where

$$f_r = c_{r0} x^r + c_{r-1,1} x^{r-1} y + \cdots + c_{0r} y^r,$$

$$g_s = d_{s0} x^s + d_{s-1,1} x^{s-1} y + \cdots + d_{0s} y^s,$$

and

$$f^*_{n-r} = c_{0r} + c_{0,r+1} y + \cdots + c_{0,n} y^{n-r},$$

$$g^*_{m-s} = d_{0s} + d_{0,s+1} y + \cdots + d_{0,m} y^{m-s}.$$

Here $R(f_r, g_s) = 0$ if and only if the curves f and g have at the origin a common principal tangent. If $R(f_r, g_s) \neq 0$, then $R(f^*_{n-r}, g^*_{m-s}) = 0$ if and only if the two curves have a common point on the y-axis outside the origin. Hence if the y-axis is generic and if there are no common principal tangents at O, then $\alpha \neq 0$ and the intersection multiplicity at O equals rs.

II. The general case of $n+1$ polynomials in n indeterminates

4. Preliminary remarks on forms of inertia. Let K be an underlying domain of integrity, and let $f_1, f_2, \cdots, f_m$ be polynomials in $x_1, x_2, \cdots, x_n$, with coefficients in a polynomial ring $K[t] = K[t_1, t_2, \cdots, t_s]$, where $t_1, \cdots, t_s$ are indeterminates. A polynomial T in $K[t]$ is an *inertia form* of the polynomials $f_1, \cdots, f_m$, if it has the property:

$$x_i^{\tau} T \equiv 0(f_1, \cdots, f_m), \qquad (3)$$

for $i=1, 2, \cdots, n$ and for some integer τ, i.e., if $x_i^\tau T$ belongs to the polynomial ideal generated by $f_1, \cdots, f_m$ in $K[t_1, \cdots, t_s; x_1, \cdots, x_n]$. It follows from the definition that the inertia forms of $f_1, \cdots, f_m$ form an ideal $\mathfrak{T}$ in $K[t]$.

THEOREM 2. *If for $\alpha=1, 2, \cdots, n$ each polynomial f_j is of the form: $f_j=t_{\alpha_j}x_\alpha^{\sigma_{j\alpha}}+f_{j\alpha}^*$, $\sigma_{j\alpha}\geqq 0$, where $t_{\alpha_1}, \cdots, t_{\alpha_m}$ are distinct indeterminates in the set $t_1, \cdots, t_s$ and where $f_{j\alpha}^*$ is a polynomial independent of $t_{\alpha_1}, \cdots, t_{\alpha_m}$, then $\mathfrak{T}$ is a prime ideal, and* (3) *holds for $i=1, 2, \cdots, n$ if it holds for one value of i.*

In order to prove† the theorem let, for instance, $f_j=t_jx_1^{\sigma_j}+f_j^*$. If (3) holds for a given i and for a given polynomial $T(t_1, t_2, \cdots, t_m, \cdots, t_s)$, then it follows by the substitution $t_j=-f_j^*/x_1^{\sigma_j}$:

$$T\left(-\frac{f_1^*}{x_1^{\sigma_1}}, -\frac{f_2^*}{x_1^{\sigma_2}}, \cdots, -\frac{f_m^*}{x_1^{\sigma_m}}, \cdots, t_s\right) = 0. \tag{4}$$

Conversely, if a polynomial $T(t_1, \cdots, t_s)$ vanishes identically after the substitution $t_j=-f_j^*/x_1^{\sigma_j}$, then T satisfies (3) for $i=1$. Under the assumption made in the above theorem, it follows immediately that (4) is a necessary and sufficient condition in order that T be a form of inertia. Hence $\mathfrak{T}$ is a prime ideal and T is a form of inertia if (3) holds for $i=1$.

COROLLARY. *If $\sigma_1=\sigma_2=\cdots=\sigma_m=0$, then any form of inertia T satisfies* (3) *with $\tau=0$.*

If the polynomials $f_1, f_2, \cdots, f_m$ are homogeneous in $x_1, \cdots, x_n$, then it is well known that the vanishing of all the inertia forms for special values t_j^0 of the parameters t_j is a necessary and sufficient condition that the equations $f_1(x_i; t_j^0)=0, f_2(x_i; t_j^0)=0, \cdots, f_m(x_i; t_j^0)=0$ have a non-trivial solution (not all $x_i=0$) (see W., p. 16).

For non-homogeneous polynomials the following theorem holds:

THEOREM 3.1. *Let f_i contain terms of lowest degree s_i in $x_1, \cdots, x_n$:*

$$f_i = f_{i,s_i}(x_1, \cdots, x_n) + f_{i,s_i+1}(x_1, \cdots, x_n) + \cdots,$$

where $f_{i,k}$ is homogeneous of degree k in $x_1, \cdots, x_n$, and let us consider the homogeneous polynomials:

$$\bar{f}_i = x_0^{l_i-s_i} f_{i,s_i}(x_1, \cdots, x_n) + x_0^{l_i-s_i-1} f_{i,s_i+1}(x_1, \cdots, x_n) + \cdots, \tag{5}$$

where x_0 is an indeterminate and l_i is the degree of f_i. The vanishing of all the inertia forms of $f_1, f_2, \cdots, f_m$ for special values of the parameters t_j is a sufficient condition in order that (a) *either the equations $\bar{f}_1=0, \cdots, \bar{f}_m=0$ have a non-*

† Compare W., p. 15.

trivial solution different from $x_0=1, x_1=\cdots=x_n=0$; *or that* (b) *the equations* $f_{1,s_1}(x_1,\cdots,x_n)=0,\cdots,f_{m,s_m}(x_1,\cdots,x_n)=0$ *have a non-trivial solution* (*in a suitable extension field of* K).

The converse holds only under certain restrictions:

THEOREM 3.2. *If* (a) *holds and if the coefficients of* $x_1^{l_i},\cdots,x_n^{l_i}$ *in* f_i $(i=1,2,\cdots,m)$ *are indeterminates which do not occur in other terms of* f_i, *then the inertia forms of* $f_1,\cdots,f_m$ *all vanish.*

THEOREM 3.3. *If* (b) *holds, and if the coefficients of* $x_1^{s_i},\cdots,x_n^{s_i}$ *in* f_i $(i=1,2,\cdots,m)$ *are indeterminates which do not occur in other terms of* f_i, *then the inertia forms of* $f_1,\cdots,f_m$ *all vanish.*

$\bar{f}_1,\cdots,\bar{f}_m$, considered as polynomials in x_0, possess a resultant system

$$\phi_1(x_1,\cdots,x_n),\cdots,\phi_h(x_1,\cdots,x_n),$$

where the ϕ_i's are homogeneous polynomials. Since $\phi_i\equiv 0(\bar{f}_1,\cdots,\bar{f}_m)$, we have for every inertia form T of the polynomials ϕ_i: $x_j^\sigma T\equiv 0(\bar{f}_1,\cdots,\bar{f}_m)$, $j=1,2,\cdots,n$. Putting $x_0=1$, we see that T is also an inertia form of the polynomials $f_1,\cdots,f_m$.

Let all the inertia forms of $f_1,\cdots,f_m$ vanish for special values of the parameters t_j. Then for these special values of the t_j's also the inertia forms of $\phi_1,\cdots,\phi_h$ all vanish, the *homogeneous equations* $\phi_1=0,\cdots,\phi_h=0$ have a non-trivial solution, and consequently, by known properties of the resultant system $\phi_1,\cdots,\phi_h$, the alternatives (a) and (b) of Theorem 3.1 follow.

If T is an inertia form of $f_1,\cdots,f_m$, then passing to the homogeneous polynomials $\bar{f}_1,\cdots,\bar{f}_m$, it is found that $(x_0x_i)^\sigma T\equiv 0(\bar{f}_1,\cdots,\bar{f}_m)$, for $i=1,2,\cdots,n$ and for some σ. Under the hypothesis of Theorem 3.2 concerning the coefficients of $x_1^{l_i},\cdots,x_n^{l_i}$, we can repeat the reasoning of the proof of Theorem 2, and it follows that $x_i^\rho T\equiv 0(\bar{f}_1,\cdots,\bar{f}_m)$, for $i=1,2,\cdots,n$ and for some ρ. Hence if (a) holds, then $T=0$.

For the proof of Theorem 3.3, let $x_1^0,\cdots,x_n^0$ be a non-trivial solution of the equations $f_{1,s_1}=0,\cdots,f_{m,s_m}=0$, and let, for instance, $x_1^0\neq 0$. We make the following change of indeterminates:

$$x_1=y_1,\ x_2=y_2y_1,\cdots,x_n=y_ny_1.$$

Then

$$f_i=y_1^{s_i}f_{1,s_i}(1,y_2,\cdots,y_n)+y_1^{s_i+1}f_{1,s_i+1}(1,y_2,\cdots,y_n)+\cdots$$
$$=y_1^{s_i}\psi_i(y_1,\cdots,y_n),$$

and if T is an inertia form of $f_1(x),\cdots,f_m(x)$, then $y_1^\sigma T\equiv 0(\psi_1,\cdots,\psi_m)$. Under the hypothesis of Theorem 3.3, the constant terms in $\psi_1,\psi_2,\cdots,\psi_m$ are indeterminates, and hence, by the corollary to Theorem 2, $T\equiv 0(\psi_1, \cdots, \psi_m)$. Since for $t_i=t_i^0$, the equations $\psi_1=0, \cdots, \psi_m=0$ have the solution $y_1^0=0$, $y_2^0=x_2^0/x_1^0, \cdots, y_n=x_n^0/x_1^0$, it follows that $T(t_1^0, \cdots, t_s^0)=0$.

5. **The inertia forms of some special set of $n+1$ polynomials in n indeterminates.** The theorems of the preceding section are applicable in the special case when $m=n+1$ and when each f_i is a polynomial with literal coefficients in which all the terms of degree $<s_i\leqq l_i$ are missing, l_i being the degree of f_i:

$$(6)\qquad f_i=\sum_{(j)} a^{(i)}_{j_1j_2\cdots j_n} x_1^{j_1}x_2^{j_2}\cdots x_n^{j_n}, \qquad s_i\leqq j_1+\cdots+j_n\leqq l_i.$$

If $s_1=s_2=\cdots=s_{n+1}=0$, then the ideal of the inertia forms is a principal ideal (R), where R is the resultant of $f_1, f_2, \cdots, f_{n+1}$. R is an irreducible polynomial homogeneous of degree $l_2\cdots l_{n+1}$ in the coefficients of f_1, homogeneous of degree $l_1l_3\cdots l_{n+1}$ in the coefficients of f_2, etc. Finally, by the corollary to Theorem 2, $R\equiv 0(f_1, f_2, \cdots, f_{n+1})$, and the vanishing of R for special values of the coefficients $a^{(i)}_{(j)}$ is a necessary and sufficient condition in order that the polynomials $f_1, f_2, \cdots, f_{n+1}$, rendered homogeneous, have a common non-trivial zero (see W., p. 20).

We prove the following theorems in the case when $s_1, s_2, \cdots, s_{n+1}$ are not necessarily all zero:

THEOREM 4. *Let e_i $(=a^{(i)}_{l_i0\cdots0})$ be the coefficient of $x_1^{l_i}$ in f_i. If $s_{n+1}<l_{n+1}$, then any inertia form of $f_1, f_2, \cdots, f_{n+1}$ which does not vanish identically, must be of degree >0 in each of the coefficients $e_1, e_2, \cdots, e_n$.*

COROLLARY. *If one at least of the polynomials $f_1, \cdots, f_{n+1}$ is non-homogeneous, the ideal $\mathfrak{T}$ of their inertia forms is a principal ideal.*†

The proof is similar to the one given in W., pp. 16–17, in the case $s_1=\cdots=s_{n+1}=0$, only with a slightly different specialization of the coefficients $a^{(i)}_{(j)}$. Assume that there exists an inertia form T, not identically zero, which is independent of e_1. Putting $f_i=e_ix_1^{l_i}+f_i^*$, and applying (4) (where σ_i should be replaced by l_i), we see that T cannot be independent of all the coefficients $e_2, \cdots, e_{n+1}$ (since T is not identically zero) and we conclude that the quotients

$$f_2^*/x_1^{l_2}, \cdots, f_{n+1}^*/x_1^{l_{n+1}}$$

are algebraically dependent in $K[a^{(i)}_{(j)}]$, K being the ring of natural integers. By a lemma proved in W., p. 17, these quotients remain algebraically depend-

† If all the polynomials f_i are homogeneous, then $\mathfrak{T}$ contains the resultant of any n of these polynomials and is therefore not a principal ideal.

ent after an arbitrary specialization $a_{(j)}^{(i)}=\alpha_{(j)}^{(i)}$ $(\alpha_{(j)}^{(i)}<K)$. Let us take for $f_1, f_2, \cdots, f_{n+1}$ the special set of polynomials $x_1^{l_1}, x_1^{l_2-1}x_2, \cdots, x_1^{l_n-1}x_n, x_1^{l_{n+1}-1}$, observing that the specialization $f_{n+1}=x_1^{l_{n+1}-1}$ is permissible, since, by hypothesis, f_{n+1} is not homogeneous. The above quotients become

$$x_2/x_1, \cdots, x_n/x_1, 1/x_1,$$

and since these are evidently algebraically independent, our assumption that T is independent of e_1 leads to a contradiction.

The corollary now follows in exactly the same manner as in W., p. 19.

Let (D) be the principal ideal of the inertia forms of the polynomials $f_1, f_2, \cdots, f_{n+1}$. D, if it is not identically zero, is an irreducible polynomial in the coefficients $a_{(j)}^{(i)}$. We next prove that indeed *D is not identically zero*, i.e., that there exist inertia forms of $f_1, \cdots, f_{n+1}$ which are not identically zero.

If $\phi_1, \cdots, \phi_{n+1}$ denote general polynomials in $x_1, \cdots, x_n$ with literal coefficients, of degree $l_1, l_2, \cdots, l_{n+1}$ respectively, we can write $\phi_i=\psi_i+f_i$, where $f_1, \cdots, f_{n+1}$ are our given polynomials and where ψ_i is of degree s_i-1. Let $\phi_i=\sum a_{j_1\cdots j_n}^{(i)}x_1^{j_1}\cdots x_n^{j_n}$, $0\leqq j_1+\cdots+j_n\leqq l_i$. Let t be a parameter, and let ϕ_i^t be the polynomial obtained from ϕ_i by replacing each coefficient $a_{j_1\cdots j_n}^{(i)}$ by $t^{s_i-j_1-\cdots-j_n}a_{j_1\cdots j_n}^{(i)}$, if $s_i>j_1+\cdots+j_n$, i.e., if $a_{j_1\cdots j_n}^{(i)}$ is the coefficient of a term of the polynomial ψ_i, while the coefficients of f_i remain unaltered. Let $R_t=R(\phi_1^t, \cdots, \phi_{n+1}^t)$ be the resultant of the ϕ_i^t's considered as polynomials in $x_1, \cdots, x_n$, and let t^α, $\alpha\geqq 0$, be the highest power of t which divides R_t:

$$(7) \qquad R_t = t^\alpha R^{(1)}(t, a_{(j)}^{(i)}) = t^\alpha R_t^{(1)}.$$

Since each polynomial ϕ_i^t contains the terms $x_1^{l_i}, \cdots, x_n^{l_i}$, whose coefficients are indeterminates, it follows by Theorem 2, that the ideal of the inertia forms of $\phi_1^t, \cdots, \phi_{n+1}^t$ is prime. Now, no power of t is an inertia form of $\phi_1^t, \cdots, \phi_{n+1}^t$, because otherwise, for $t=1$, it would follow that 1 is an inertia form of $\phi_1, \cdots, \phi_{n+1}$, and this is impossible. Hence, since $t^\alpha R_t^{(1)}$ is an inertia form of $\phi_1^t, \cdots, \phi_{n+1}^t$, it follows that also $R_t^{(1)}$ is an inertia form. For $t=0$, we have $\phi_i^0=f_i$, and $R_0^{(1)}$ is therefore an inertia form of $f_1, \cdots, f_{n+1}$ which does not vanish identically.

6. The resultant $R(\phi_1, \cdots, \phi_{n+1})$ as an isobaric function of the coefficients $a_{(j)}^{(i)}$. Let $\phi_1, \cdots, \phi_{n+1}$ denote, as in the preceding section, general polynomials in the n variables $x_1, \cdots, x_n$, of degree $l_1, \cdots, l_{n+1}$ respectively, and let $R(\phi_1, \cdots, \phi_{n+1})=R(a_{(j)}^{(i)})$ be their resultant. It is clear that $R(\cdots, t^{j_1+\cdots+j_n}a_{j_1\cdots j_n}^{(i)}, \cdots)$ is the resultant of $\phi_1(x_1t, \cdots, x_{n+1}t), \cdots, \phi_{n+1}(x_1t, \cdots, x_{n+1}t)$ and is therefore divisible by $R(a_j^{(i)})$, since the ideal of the inertia forms of these $n+1$ polynomials is, by the preceding theorems, a principal ideal and since the irreducible polynomial $R(a^{(i)}_{(j)})$ obviously belongs to this ideal. It follows that $R(\cdots, t^{j_1+\cdots+j_n}a^{(i)}_{j_1\cdots j_n}, \cdots)$ differs from $R(a_j^{(i)})$ only by a factor which is a power of t, say by t^σ. Hence $R(a_j^{(i)})$ *is an isobaric function of the coefficients of* $a_j^{(i)}$, *of weight* σ, *provided that we attach to* $a^{(i)}_{j_1\cdots j_n}$ *the weight* $j_1+\cdots+j_n$.

To find σ, we specialize the polynomials ϕ_i as follows

$$\phi_1 = a_1x_1^{l_1}, \cdots, \phi_n = a_nx_n^{l_n}, \phi_{n+1} = a_{n+1}.$$

The resultant R does not vanish identically, since the equations $\phi_1=0, \cdots, \phi_{n+1}=0$ have no common solution if $a_1, \cdots, a_{n+1}$ are indeterminates. Taking into account the degree of R in the coefficients of each ϕ_i, we deduce that $R=c\cdot a_1^{l_2\cdots l_{n+1}}\cdots a_{n+1}^{l_1l_2\cdots l_n}$, where c is a numerical factor. Since $a_1, \cdots, a_n$ are of weight $l_1, l_2, \cdots, l_n$ respectively and a_{n+1} is of weight zero, it follows that $\sigma=nl_1l_2\cdots l_{n+1}$.

As an immediate corollary of this last result and of the fact that $R(\phi_1, \cdots, \phi_{n+1})$ is homogeneous of degree $l_1\cdots l_{i-1}\,l_{i+1}\cdots l_{n+1}$ in the coefficients of ϕ_i, it follows that *if we attach to* $a^{(i)}_{j_1\cdots j_n}$ *the weight* $l_i-j_1-\cdots-j_{n+1}$, *then* $R(\phi_1, \cdots, \phi_{n+1})$ *is isobaric of weight* $l_1l_2\cdots l_{n+1}$.

7. **Properties of R based on a more general definition of the weights of the coefficients $a^{(i)}_{(j)}$.** We separate in ϕ_i the terms of degree $\leqq s_i$ from those of degree $>s_i$, and we put $\phi_i=\bar{\psi}_i+\bar{f}_i$, where $\bar{\psi}_i$ is of degree s_i and $\bar{f}_i$ contains all the terms of degree $>s_i$. While in §5 we have replaced $a^{(i)}_{j_1\cdots j_n}$ by $t^{s_i-j_1-\cdots-j_n}a^{(i)}_{j_1\cdots j_n}$, if $s_i>j_1-\cdots-j_n$, we now instead replace $a^{(i)}_{j_1\cdots j_n}$ by $t^{j_1+\cdots+j_n-s_1}a^{(i)}_{j_1\cdots j_n}$, if $j_1+\cdots+j_n>s_i$, i.e., if $a^{(i)}_{j_1\cdots j_n}$ is the coefficient of a term in $\bar{f}_i$, and leave the coefficients of $\bar{\psi}_1, \cdots, \bar{\psi}_{n+1}$ unaltered.

Let $\bar{\phi}_1^t, \cdots, \bar{\phi}_{n+1}^t$ be the polynomials obtained in this manner, and let

$$R(\bar{\phi}_1^t, \cdots, \bar{\phi}_{n+1}^t) = t^\beta R_t^{(2)} = t^\beta R^{(2)}(t; a^{(i)}_{(j)}) \tag{8}$$

be the resultant of the polynomials $\bar{\phi}_i^t$. Here t^β is the highest power of t which divides $R(\bar{\phi}_1^t, \cdots, \bar{\phi}_{n+1}^t)$, so that $R_0^{(2)}=R^{(2)}(0; a_j^{(i)})$ does not vanish identically. As in the case of the polynomials ϕ_i^t of §5, we conclude also here that $R_0^{(2)}$ is a form of inertia of the polynomials $\bar{\psi}_1, \cdots, \bar{\psi}_{n+1}$, and since these are general polynomials of degree $s_1, \cdots, s_{n+1}$ respectively, we deduce that $R_0^{(2)}$ *is divisible by* $R(\bar{\psi}_1, \cdots, \bar{\psi}_{n+1})$.

Now the polynomials ϕ_i^t and $\bar{\phi}_i^t$ are related in the following way: $\bar{\phi}_i^t=\phi_i^t(tx_1, \cdots, tx_{n+1})/t^{s_i}$. From this it follows, in view of the isobaric property of R given in the preceding section, that their resultants differ only by a factor which is a power of t. Hence, by (7) and (8), we have $R^{(1)}(t, a^{(i)}_{(j)}) = R^{(2)}(t, a^{(i)}_{(j)})$, and in particular for $t=0$, we have $R_0^{(1)}=R_0^{(2)}$. Let $R_0=R_0^{(1)} =R_0^{(2)}$. R_0 is divisible by $R(\bar{\psi}_1, \cdots, \bar{\psi}_{n+1})$ *and* by D, where D is the base of the principal ideal of the inertia forms of $f_1, \cdots, f_{n+1}$.† Hence R_0 *is divisible by the product* $D \times R(\bar{\psi}_1, \cdots, \bar{\psi}_{n+1})$, since both factors are irreducible and distinct polynomials. ($R(\bar{\psi}_1, \cdots, \bar{\psi}_{n+1})$ is of degree >0 in each constant term $a^{(1)}_{00\cdots 0}$, while, except in the trivial case $s_1 = \cdots = s_{n+1} = 0$, where $f_1, \cdots, f_{n+1}$ coincide with $\phi_1, \cdots, \phi_{n+1}$, at least one of the polynomials f, say f_i, and hence also D, is independent of $a^{(1)}_{00\cdots 0}$.)

The precise relationship between R_0 and $D \cdot R(\bar{\psi}_1, \cdots, \bar{\psi}_{n+1})$ is given by the following theorems:

THEOREM 5.1. *If two at least of the polynomials f_i are non-homogeneous, then*

$$R_0 = c \cdot D \cdot R(\bar{\psi}_1, \cdots, \bar{\psi}_{n+1}), \tag{9.1}$$

where c is a numerical factor (an integer).

THEOREM 5.2. *If $f_2, \cdots, f_{n+1}$ are homogeneous, then*

$$R_0 = c \cdot D^{l_1 - s_1} \cdot R(\bar{\psi}_1, \cdots, \bar{\psi}_{n+1}), \tag{9.2}$$

where c is a numerical factor (an integer). In this case D is simply the resultant of $f_2, \cdots, f_{n+1}$.

Before proving these theorems, let us first derive an immediate consequence. From the meaning of $R_0 = R_0^{(1)}$ [cf. (7)] it follows that if to each coefficient $a^{(i)}_{j_1 \cdots j_n}$ in ϕ_i we attach the weight $s_i - j_1 - \cdots - j_n$, if $j_1 + \cdots + j_n \leqq s_i$, and the weight zero if $j_1 + \cdots + j_n > s_i$, then R_0 is the sum of terms of owest weight α in the resultant $R(\phi_1, \cdots, \phi_{n+1})$. According to this definition of the weight, each term in D is of weight zero, while $R(\bar{\psi}_1, \cdots, \bar{\psi}_{n+1})$, by §6, is of weight $s_1 \cdots s_{n+1}$. Hence we may state the following theorem:

THEOREM 6. *Let $\phi_1, \cdots, \phi_{n+1}$ be general polynomials in $x_1 \cdots x_n$, of degree $l_1, \cdots, l_{n+1}$ respectively, and let $s_1, \cdots, s_{n+1}$ be integers such that $0 \leqq s_i \leqq l_i$. If we attach to each coefficient $a^{(i)}_{j_1 \cdots j_n}$ in ϕ_i the weight $s_i - j_1 - \cdots - j_n$ or the weight zero, according as $j_1 + \cdots + j_n \leqq s_i$ or $j_1 + \cdots + j_n > s_i$, then each term of the resultant $R(\phi_1, \cdots, \phi_{n+1})$ is of weight $\geqq s_1 s_2 \cdots s_{n+1}$. The sum of terms of lowest weight $s_1 s_2 \cdots s_{n+1}$ is given by the product $cD^\sigma R(\bar{\psi}_1, \cdots, \bar{\psi}_{n+1})$, where c is a numerical factor. The symbols have the following meaning: $\bar{\psi}_i$ is the sum of terms of ϕ_i which are of degree $\leqq s_i$ and f_i is the sum of terms of ϕ_i of degree $\geqq s_i$; $R(\bar{\psi}_1, \cdots, \bar{\psi}_{n+1})$ is the resultant of $\bar{\psi}_1, \cdots, \bar{\psi}_{n+1}$; if not all $s_i = l_i$, then D is the base of the principal ideal of the inertia forms of $f_1, \cdots, f_{n+1}$; if all $s_i = l_i$, then $D = 1$; finally, $\sigma = 1$, except when all the integers s_i but one, say s_1, coincide with the corresponding integers l_i, in which case $\sigma = l_1 - s_1$.*

† In the trivial case when $f_1, \cdots, f_{n+1}$ are all homogeneous polynomials, D is not defined, but then R_0 evidently coincides with $R(\phi_1, \cdots, \phi_{n+1})$.

Remark. Again from the meaning of $R_0(=R_0^{(2)})$ it follows that if we attach to $a^{(i)}_{j_1\cdots j_n}$ the weight $j_1+\cdots+j_n-s_i$ or zero, according as $j_1+\cdots+j_n\geqq s_i$ or $j_1+\cdots+j_n<s_i$, then R_0 is also the sum of terms of lowest weight, β, in the resultant $R(\phi_1,\cdots,\phi_{n+1})$ [cf. (8)]. According to this definition of the weight, each term of $R(\bar\psi_1,\cdots,\bar\psi_{n+1})$ is of weight zero, and D^σ has to be isobaric of weight β.

To find β, we observe that Theorem 5.1 implies that D is homogeneous of degree $l_2\cdots l_{n+1}-s_2\cdots s_{n+1}$ in the coefficients of f_1, homogeneous of degree $l_1l_3\cdots l_{n+1}-s_1s_3\cdots s_{n+1}$ in the coefficients of f_2, etc. On the other hand, if $j_1+\cdots+j_n$ is taken as the weight of $a^{(i)}_{j_1\cdots j_n}$, then R_0 and $R(\bar\psi_1,\cdots,\bar\psi_{n+1})$ are isobaric forms of weight $nl_1\cdots l_{n+1}$ and $ns_1\cdots s_{n+1}$ respectively, whence D is of weight $n(l_1\cdots l_{n+1}-s_1\cdots s_{n+1})$. It follows that if we replace in the polynomial D each coefficient $a^{(i)}_{j_1\cdots j_n}$ by $a^{(i)}_{j_1\cdots j_n}t^{s_i-j_1-\cdots-j_n}$, D acquires the factor t^β, where

$$\beta = -n(l_1\cdots l_{n+1}-s_1\cdots s_{n+1})+s_1(l_2\cdots l_{n+1}-s_2\cdots s_{n+1})$$
$$+s_2(l_1l_3\cdots l_{n+1}-s_1s_3\cdots s_{n+1})+\cdots+s_{n+1}(l_1\cdots l_n-s_1\cdots s_n),$$

or

$$\beta = s_1s_2\cdots s_{n+1}+\left(\frac{l_1-s_1}{l_1}+\cdots+\frac{l_{n+1}-s_{n+1}}{l_{n+1}}-1\right)l_1l_2\cdots l_{n+1}.$$

If $s_2=l_2,\cdots,s_{n+1}=l_{n+1}$, then it is seen that $\beta=0$, and this agrees with Theorem 5.2, because in this case the coefficients of $f_2,\cdots,f_{n+1}$ are of weight zero.

Proof of Theorems 5.1 and 5.2. We begin with Theorem 5.2, whose proof is simpler. We have in this case $\bar\psi_i=\phi_i$, $i=2,\cdots,n+1$, and hence $R(\bar\psi_1,\cdots,\bar\psi_{n+1})$ is of degree $l_2\cdots l_{n+1}$ in the coefficients of $\bar\psi_1$. Hence, if we put

$$R_0 = D^\sigma R(\bar\psi_1,\phi_2,\cdots,\phi_{n+1})\cdot P,$$

then P is independent of the coefficients of ϕ_1.

Now in the present case $\beta=0$, and R_0 is what becomes of the resultant $R(\bar\phi_1^t,\cdots,\bar\phi_{n+1}^t)$ if we put $t=0$, where now $\phi_i^t=\phi_i$, $i=2,\cdots,n+1$, and $\bar\phi_1^t=\bar\psi_1+\bar f_1(tx_1,\cdots,tx_n)/t^{s_1}$. It follows that $R_0=0$ implies that either the equations $\bar\psi_1=0$, $\phi_2=0,\cdots,\phi_{n+1}=0$, rendered homogeneous, have a non-trivial solution, or that the homogeneous equations $f_2=0,\cdots,f_{n+1}=0$ have a non-trivial solution. Hence $R(\bar\psi_1,\phi_2,\cdots,\phi_{n+1})$ and $R(f_2,\cdots,f_{n+1})$ are the only irreducible factors which can occur in R_0. Since the *irreducible* polynomial $R(f_2, \cdots, f_{n+1})$ obviously coincides with D, Theorem 5.2 follows by comparing the degrees of the first and second member of (9.2).

For the proof of Theorem 5.1, it is sufficient to show that *D is of degree $l_2l_3 \cdots l_{n+1} - s_2s_3 \cdots s_{n+1}$ in the coefficients of f_1, of degree $l_1l_3 \cdots l_{n+1} - s_1s_3 \cdots s_{n+1}$ in the coefficients of f_2*, etc. Since $DR(\bar{\psi}_1, \cdots, \bar{\psi}_{n+1})$ divides R_0, D cannot be of higher degree in the coefficients of $f_1, f_2, \cdots, f_{n+1}$, and therefore it remains to show that D is of degree *not less* than $l_2l_3 \cdots l_{n+1} - s_2s_3 \cdots s_{n+1}$ in the coefficients of f_1, etc. We prove this in the following section.

8. **The degree of D.** We wish to show in this section that *if at least two of the polynomials $f_1, \cdots, f_{n+1}$ are non-homogeneous, then D is of degree $\geqq l_2 \cdots l_{n+1} - s_2 \cdots s_{n+1}$ in the coefficients of f_1, of degree $\geqq l_1l_3 \cdots l_{n+1} - s_1s_3 \cdots s_{n+1}$ in the coefficients of f_2*, etc. Obviously, the condition that at least two of the polynomials f_i be non-homogeneous, is necessary. In fact, if only one of the polynomials f_i, say f_{n+1}, is non-homogeneous, then D coincides with the resultant $R(f_1, \cdots, f_n)$ of the forms $f_1, \cdots, f_n$, and its degree in the coefficients of f_1 is not $l_2 \cdots l_{n+1} - s_2 \cdots s_{n+1}$ $[= l_2 \cdots l_n(l_{n+1} - s_{n+1})]$, but $l_2 \cdots l_n$. If all the polynomials f_i are homogeneous, then the ideal of their inertia forms is not a principal ideal and D is not defined.

If for special values of the coefficients $a_{(j)}^{(i)}$, one of the polynomials f_i, say f_{n+1}, factors into a product gh of two polynomials, then D becomes an inertia form of both sets of polynomials $f_1, \cdots, f_n, g$ and $f_1, \cdots, f_n, h$. Hence, assuming that the ideals of inertia forms of these two sets of polynomials are principal ideals, say (D_1) and (D_2) respectively, then for those special values of the coefficients $a_{(j)}^{(i)}$, D is divisible by both D_1 and D_2. This remark shall be used in the sequel.

Let f_n and f_{n+1} be the non-homogeneous polynomials. We first consider the case in which $f_1, \cdots, f_{n-1}$ are polynomials of degree 1, and in this case we examine separately three possibilities.

(a) *At least two of the polynomials $f_1, \cdots, f_{n-1}$ are non-homogeneous* (and hence two at least of the integers $s_1, \cdots, s_{n-1}$ vanish). We specialize the coefficients of f_n and f_{n+1} in such a manner that f_n becomes the product of l_n general polynomials $f_{n,i}$ of the first degree, of which s_n are linear forms, and that f_{n+1} becomes similarly the product of l_{n+1} linear factors, $f_{n+1,i}$. The l_nl_{n+1} $(n+1)$-row coefficient determinants relative to the sets of polynomials $f_1, \cdots, f_{n-1}, f_{n,i}, f_{n+1,j}$ are all distinct and irreducible inertia forms, since at least two of the polynomials of each set are non-homogeneous. Hence D is divisible by the product of these determinants and is therefore of degree $\geqq l_nl_{n+1}$ in the coefficients of f_i, $i = 1, 2, \cdots, n-1$, and of degree $\geqq l_{n+1}$ (l_n) in the coefficients of f_n (f_{n+1}).

We observe that this proves that in the present case D coincides with the resultant $R(f_1, \cdots, f_{n+1})$, or, what is the same, that this resultant is irreducible.

(b) *All but one of the polynomials* $f_1, \cdots, f_{n-1}$ *are homogeneous.* Let, for instance, f_1 be non-homogeneous. With the same specialization of f_n and f_{n+1} as in the preceding case, let $f_{n,1}, \cdots, f_{n,s_n}$; $f_{n+1,1}, \cdots, f_{n+1,s_{n+1}}$ be the homogeneous linear factors of f_n and of f_{n+1} respectively. The $(n+1)$-row coefficient determinants of $f_1, \cdots, f_{n-1}, f_{n,i}, f_{n+1,j}$ remain irreducible, except when simultaneously $1 \leqq i \leqq s_n$ *and* $1 \leqq j \leqq s_{n+1}$, in which case the determinant factors into the constant term of f_1 and into the n-row determinant of the coefficients of $x_1, \cdots, x_n$ in $f_2, \cdots, f_{n-1}, f_{n,i}, f_{n+1,j}$. Hence D is divisible by the product of $l_n l_{n+1} - s_n s_{n+1}$ $(n+1)$-row determinants and $s_n s_{n+1}$ n-row determinants, these last ones being independent of the coefficients of f_1. Hence D is of degree $\geqq l_n l_{n+1} - s_n s_{n+1}$ in the coefficients of f_1, of degree $\geqq l_n l_{n+1}$ in the coefficients of f_i, $i = 2, \cdots, n-1$, and of degree $\geqq l_{n+1}$ (l_n) in the coefficients of f_n (f_{n+1}).

(c) *All the polynomials* $f_1, \cdots, f_{n-1}$ *are homogeneous.* Let

$$f_i = \sum_{j=1}^{n} a_{ij} x_j, \qquad i = 1, 2, \cdots, n-1, \tag{10}$$

$$f_i = f_{i,s_i} + f_{i,s_i+1} + \cdots + f_{i,l_i}, \qquad i = n, n+1, \tag{10'}$$

where f_{i,s_i+k} is homogeneous of degree $s_i + k$. Solving (10) for $x_2, \cdots, x_n$ we get

$$A_1 x_i \equiv A_i x_1 (f_1, f_2, \cdots, f_{n-1}), \tag{11}$$

where $A_1, \cdots, A_n$ are $(n-1)$-row minors of the matrix (a_{ij}) and hence homogeneous of degree 1 in the coefficients of each of the polynomials $f_1, \cdots, f_{n-1}$.

Substituting (11) into (10′) we get

$$A_1^{l_n} f_n \equiv x_1^{s_n} \phi_n(x_1)(f_1, \cdots, f_{n-1}); \; A_1^{l_{n+1}} f_{n+1} \equiv x_1^{s_{n+1}} \phi_{n+1}(x_1)(f_1, \cdots, f_{n-1}), \tag{12}$$

where

$$\begin{aligned} \phi_n(x_1) &= A_1^{l_n - s_n} f_{n,s_n}(A_1, \cdots, A_n) + x_1 A_1^{l_n - s_n - 1} f_{n,s_n+1}(A_1, \cdots, A_n) + \cdots \\ &\quad + x_1^{l_n - s_n} f_{n,l_n}(A_1, \cdots, A_n); \\ \phi_{n+1}(x_1) &= A_1^{l_{n+1} - s_{n+1}} f_{n+1,s_{n+1}}(A_1, \cdots, A_n) \\ &\quad + x_1 A_1^{l_{n+1} - s_{n+1} - 1} f_{n+1,s_{n+1}+1}(A_1, \cdots, A_n) + \cdots \\ &\quad + x_1^{l_{n+1} - s_{n+1}} f_{n+1,l_{n+1}}(A_1, \cdots, A_n). \end{aligned}$$

Let $R=R(\phi_n, \phi_{n+1})$ be the resultant of $\phi_n(x_1)$ and $\phi_{n+1}(x_1)$. We have $R\equiv 0(\phi_n, \phi_{n+1})$ and hence, by (12), $x_1^\sigma R\equiv 0(f_1, \cdots, f_{n-1}, f_n, f_{n+1})$, for some σ. It follows, by Theorem 2, that R is a form of inertia of the polynomials f_i. From the form of the coefficients of $\phi_n(x_1)$ and $\phi_{n+1}(x_1)$ and from the fact that R is an isobaric form of weight $(l_n-s_n)(l_{n+1}-s_{n+1})$ in these coefficients, it follows that $A_1^{(l_n-s_n)(l_{n+1}-s_{n+1})}$ is a factor of R. Let $R=A_1^{(l_n-s_n)(l_{n+1}-s_{n+1})}\cdot P$. Now A_1 is independent of the coefficients of f_n and f_{n+1} and hence, by Theorem 4, is not a form of inertia of $f_1, \cdots, f_{n+1}$. Consequently *P is a form of inertia of $f_1, \cdots, f_{n+1}$*. The coefficients of $\phi_i (i=n, n+1)$ are homogeneous of degree 1 in the coefficients of f_i and homogeneous of degree l_i in $A_1, \cdots, A_n$, hence homogeneous of degree l_i in the coefficients of each of the polynomials $f_1, \cdots, f_{n-1}$. Hence R is homogeneous of degree l_n-s_n and $l_{n+1}-s_{n+1}$ in the coefficients of f_{n+1} and f_n respectively, and homogeneous of degree $l_n(l_{n+1}-s_{n+1})+l_{n+1}(l_n-s_n)$ in the coefficients of f_i, $i=1, 2, \cdots, n-1$. It follows that *P is homogeneous of degree $l_{n+1}-s_{n+1}$ and l_n-s_n in the coefficients of f_n and f_{n+1} respectively, and homogeneous of degree $l_nl_{n+1}-s_ns_{n+1}$ in the coefficients of each of the polynomials $f_1, \cdots, f_{n-1}$.*

It remains to prove that $P=D$, or, what is the same, that P is an irreducible polynomial in the coefficients of $f_1, \cdots, f_{n+1}$. We observe that P is the resultant of the following polynomials

$$\begin{aligned}\psi_n(x_1; A_1, \cdots, A_n) &= f_{n,s_n}(A_1, \cdots, A_n) + x_1 f_{n,s_n+1}(A_1, \cdots, A_n)\\ &\quad + \cdots + x_1^{l_n-s_n} f_{n,l_n}(A_1, \cdots, A_n),\\ \psi_{n+1}(x_1; A_1, \cdots, A_n) &= f_{n+1,s_{n+1}}(A_1, \cdots, A_n) + x_1 f_{n+1,s_{n+1}+1}(A_1, \cdots, A_n)\\ &\quad + \cdots + x_1^{l_{n+1}-s_{n+1}} f_{n+1,l_{n+1}}(A_1, \cdots, A_n).\end{aligned}$$

For the special polynomials $f_1=x_2$, $f_2=x_3, \cdots, f_{n-1}=x_n$, we have $A_1=1$, $A_2=\cdots=A_n=0$, and ψ_n, ψ_{n+1} become general polynomials with literal coefficients in x_1, of degree l_n-s_n and $l_{n+1}-s_{n+1}$ respectively, and their resultant is irreducible. Hence P cannot be divisible by two factors or by the square of a factor in which the coefficients of f_n or of f_{n+1} actually occur. On the other hand, for the special polynomials

$$\begin{aligned}\psi_n &= x_1^{l_n-s_n} f_{n,l_n}(A_1, \cdots, A_n),\\ \psi_{n+1} &= f_{n+1,s_{n+1}}(A_1, \cdots, A_n) + x_1^{l_{n+1}-s_{n+1}} f_{n+1,l_{n+1}}(A_1, \cdots, A_n)\end{aligned}$$

we get $P=\pm f_{n,l_n}^{l_{n+1}-s_{n+1}} f_{n+1,l_{n+1}}^{l_n-s_n}$, and hence P cannot have a factor independent of the coefficients of both f_n and f_{n+1}. Hence P is irreducible, $P=D$.

Passing to the general case where $f_1, \cdots, f_{n-1}$ are of arbitrary degrees $l_1, \cdots, l_{n-1}$, while f_n, f_{n+1} are non-homogeneous polynomials, we specialize each polynomial f_i, $i=1, 2, \cdots, n-1$, into the product of l_i linear factors, of which s_i are linear forms: $f_i=f_{i1}f_{i2}\cdots f_{il_i}$. By the special case considered above, the irreducible form of inertia $D_{j_1\cdots j_{n-1}}$ of the $n+1$ polynomials $f_{1j_1}, f_{2j_2}, \cdots, f_{n-1,j_{n-1}}, f_n, f_{n+1}$ $(1\leqq j_i\leqq l_i)$ actually contains the coefficients of each factor. Hence we get $l_1l_2\cdots l_{n-1}$ distinct irreducible forms of inertia and their product must divide D. Now $D_{j_1\cdots j_{n-1}}$ is of degree l_nl_{n+1} in the coefficients of f_{1j_1}, if the polynomials $f_{2j_2}, \cdots, f_{n-1,j_{n-1}}$ are not all homogeneous, and is of degree $l_nl_{n+1}-s_ns_{n+1}$ in the coefficients of f_{1j_1}, if all the polynomials $f_{2j_2}, \cdots, f_{n-1,j_{n-1}}$ are homogeneous. It follows that D is of degree $\geqq l_2\cdots l_{n+1}-s_2\cdots s_{n+1}$ in the coefficients of f_1. Similarly D is of degree $\geqq l_1\cdots l_{i-1}l_{i+1}\cdots l_{n+1}-s_1\cdots s_{i-1}s_{i+1}\cdots s_{n+1}$ in the coefficients of f_i, $i=1, 2, \cdots, n-1$. $D_{j_1\cdots j_{n-1}}$ is of degree l_{n+1} in the coefficients of f_n, if $f_{1j_1}, \cdots, f_{n-1,j_{n-1}}$ are not all homogeneous, and is of degree $l_{n+1}-s_{n+1}$ in the contrary case. Hence D is of degree $l_1\cdots l_{n-1}l_{n+1}-s_1\cdots s_{n-1}s_{n+1}$ in the coefficients of f_n. Similarly, D is of degree $l_1\cdots l_n-s_1\cdots s_n$ in the coefficients of f_{n+1}.

9. **An application to the intersection theory of algebraic hypersurfaces.** Let

$$\phi_1(x_1, \cdots, x_n, x_{n+1}) = 0,$$
$$\cdots\cdots\cdots\cdots$$
$$\phi_{n+1}(x_1, \cdots, x_n, x_{n+1}) = 0,$$

be the equations of $n+1$ hypersurfaces $F_1, \cdots, F_{n+1}$ in the $(n+1)$-dimensional projective space. Let l_i be the order of F_i. Let the origin $O(0, \cdots, 0)$ be a common point of these hypersurfaces, and let it be an s_i-fold point of F_i. We regard ϕ_i as a polynomial in $x_1, \cdots, x_n$, and we write $\phi_i=\sum a^{(i)}_{j_1\cdots j_n}x_1^{j_1}\cdots x_n^{j_n}$, where the coefficients $a^{(i)}_{(j)}$ are polynomials in x_{n+1}. Since O is an s_i-fold point of F_i, $a^{(i)}_{j_1\cdots j_n}$ is divisible by $x_1^{s_i-j_1-\cdots-j_n}$, if $j_1+\cdots+j_n\leqq s_i$. Hence, by Theorem 6, every term of the resultant $R(\phi_1, \cdots, \phi_{n+1})=R(x_{n+1})$ is divisible by $x_{n+1}^{s_1\cdots s_{n+1}}$. Let

$$R(x_{n+1}) = \alpha x^{s_1\cdots s_{n+1}} + \text{terms of higher degree},$$

where α is a constant. Let g_i $(i=1, 2, \cdots, n+1)$ denote the sum of terms of lowest degree (s_i) in ϕ_i. Then we have, by Theorem 6, $\alpha=cD_0{}^\sigma R(g_1, \cdots, g_{n+1})$, where c is an integer, and $D_0=[D]_{x_{n+1}=0}$. The homogeneous equation $g_i=0$ represents the tangent hypercone of the hypersurface F_i at the point O. Hence $R(g_1, \cdots, g_{n+1})$ vanishes, if and only if the $n+1$ hypersurfaces F_i have a common principal tangent line at O. Assume that $R(g_1, \cdots, g_{n+1})\neq 0$. If f_i

denotes, as in Theorem 6, the sum of terms in ϕ_i which are of degree $\geqq s_i$ in $x_1, \cdots, x_n$, then $f_i = [g_i]_{x_{n+1}=0}$ + terms of degree $> s_i$ in $x_1, \cdots, x_n$. It follows, by Theorem 3.1, that if $R(g_1, \cdots, g_{n+1}) \neq 0$, then $D_0 = 0$ implies that the hypersurfaces F_i have a common point on the hyperplane $x_{n+1} = 0$, outside the origin; and conversely, by Theorem 3.2. Assuming that the hypersurfaces F_i meet in a finite number of points, we see that if the coordinate axes are in generic position and if the hypersurfaces F_i have no principal tangent in common at the point O, then $\alpha \neq 0$. According to the usual definition of the intersection multiplicity of the hypersurfaces F_i at a common point, it follows that *the intersection multiplicity at O is* $\geqq s_1 \cdots s_{n+1}$ *and equals* $s_1 \cdots s_{n+1}$ *if and only if the hypersurfaces* F_i *have no common principal tangent at O.*

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