GENERALIZATIONS OF THE GAUSS LAW OF THE SPHERICAL MEAN*

BY
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1. **Introduction.** The nature of the generalizations of the Gauss law of the spherical mean considered in this paper is illustrated by the following theorem (§3):

If in three-space the function u satisfies the differential equation

$$\nabla^2 u - \lambda u = 0, \qquad \lambda = \text{const.} \tag{1}$$

on and within a three-dimensional sphere S of radius r, then

$$A(u) = u_0 \sinh (r\lambda^{1/2})/(r\lambda^{1/2}), \tag{2}$$

where $A(u)$ is the average or arithmetic mean of u over S, that is

$$A(u) = \int udS \Big/ \int dS, \tag{3}$$

while u_0 is the value of u at the center of S.

A similar result holds for any number of dimensions n. Thus for $n=2$, if u is a two-dimensional solution of (1),

$$A(u) = u_0 I_0(r\lambda^{1/2}), \tag{4}$$

where $A(u)$ is the mean of u over S, a circle of radius r. The general case of n dimensions is given by

$$A(u) = u_0\left[1 + \frac{r^2\lambda}{2\cdot n} + \frac{r^4\lambda^2}{2\cdot 4\cdot n(n+2)} + \cdots\right]. \tag{5}$$

We shall denote the bracket in (5) by $\phi_n(r, \lambda)$:

$$\phi_n(r, \lambda) = \sum_{k=0}^{\infty} (r^2\lambda)^k/2\cdot 4 \cdots 2k\cdot n(n+2)\cdots(n+2k-2). \tag{6}$$

This function is expressible in terms of Bessel functions of order $n/2-1$ as follows:

$$\phi_n(r, \lambda) = \Gamma(n/2)(r\lambda^{1/2}/2)^{1-n/2} I_{n/1-2}(r\lambda^{1/2}). \tag{7}$$

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The familiar Gauss law of the mean for harmonic functions in n dimensions in obtained by putting $\lambda=0$ in (1) and (5).

Similar laws are derived below for solutions of

$$(\nabla^2-\lambda)^p u = 0 \tag{8}$$

(§3); for these, it is shown that

$$\begin{aligned} A(u) = u_0\phi_n(r, \lambda) + (\nabla^2-\lambda)u]_0 \frac{\partial\phi_n(r,\lambda)}{\partial\lambda} + \cdots \\ + \frac{(\nabla^2-\lambda)^{p-1}u]_0}{(p-1)!}\frac{\partial^{p-1}\phi_n(r,\lambda)}{\partial\lambda^{p-1}}, \end{aligned} \tag{9}$$

where the subscripts 0 following the brackets indicate evaluation at the center of the sphere. Alternative forms for the mean, say in terms of Bessel functions, are obtained from the relations

$$\begin{aligned} \frac{\partial^k\phi_n(r,\lambda)}{\partial\lambda^k} &= \Gamma(n/2)\lambda^{-k}(r\lambda^{1/2}/2)^{k-n/2+1}I_{n/2+k-1}(r\lambda^{1/2}) \\ &= [r^{2k}/2^k n(n+2)\cdots(n+2k-2)]\phi_{n+2k}(r,\lambda). \end{aligned} \tag{10}$$

Again the case $\lambda=0$ is of special interest; now (8) becomes the repeated Laplace equation:

$$\nabla^{2p}u = 0, \tag{11}$$

whose solutions are sometimes known as "p-harmonic" functions, while (9) reduces to

$$\begin{aligned} A(u) = u_0 + (\nabla^2 u)_0 r^2/2n + \cdots \\ + (\nabla^{2p-2}u)_0 r^{2p-2}/2\cdot 4\cdots(2p-2)n(n+2)\cdots(n+2p-4), \end{aligned} \tag{12}$$

so that $A(u)$ is a polynomial in r^2.

The most general extension of these laws of the mean considered in this paper is for solutions of the differential equation

$$\sum_{i=0}^{p} c_i\nabla^{2i}u = 0 \qquad (c_p \neq 0), \tag{13}$$

where c_i are constants. In this case, the following law of the mean holds (§3):

$$\begin{vmatrix} A(u) & \phi_n(r,\lambda_1) & \cdots & \phi_n(r,\lambda_p) \\ u_0 & 1 & \cdots & 1 \\ (\nabla^2 u)_0 & \lambda_1 & \cdots & \lambda_p \\ \cdots & \cdots & \cdots & \cdots \\ (\nabla^{2p-2}u)_0 & \lambda_1^{p-1} & \cdots & \lambda_p^{p-1} \end{vmatrix} = 0, \tag{14}$$

provided that the operator on the left of (13) factors symbolically thus:

$$\sum_{i=0}^{p} c_i \nabla^{2i} = c_p \sum_{i=1}^{p} (\nabla^2 - \lambda_i), \tag{15}$$

where no two λ_i are alike. If, on the other hand, in the factorization (15) there are repeated roots, then corresponding to each root of multiplicity m, m columns of (14) past the first one are to be replaced by the λ-derivatives of the elements in (14) of order $0, 1, \cdots, m-1$. Thus, if

$$\sum_{i=0}^{p} c_i \nabla^{2i} = c_p(\nabla^2 - \lambda_1)^{m_1}(\nabla^2 - \lambda_2)^{m_2} \cdots (\nabla^2 - \lambda_k)^{m_k}, \tag{15'}$$

where $m_1 + m_2 + \cdots + m_k = p$ and $\lambda_i \neq \lambda_j$ for $i \neq j$, then the elements of the first row in (14) from the second one on are replaced by

$$\begin{aligned} &\phi_n(r, \lambda_1), \ \frac{\partial \phi_n(r, \lambda)}{\partial \lambda}\bigg|_{\lambda=\lambda_1}, \cdots, \frac{\partial^{m_1-1}\phi_n(r, \lambda)}{\partial \lambda^{m_1-1}}\bigg|_{\lambda=\lambda_1}, \\ &\qquad\qquad \phi_n(r, \lambda_2), \cdots, \frac{\partial^{m_k-1}\phi_n(r, \lambda)}{\partial \lambda^{m_k-1}}\bigg|_{\lambda=\lambda_k}, \end{aligned} \tag{16}$$

and similarly for the elements of the other rows.

An interesting application of (5) occurs in establishing the expansion

$$A(u) = \sum_{k=0}^{\infty} (\nabla^{2k}u)_0 r^{2k}/2\cdot 4 \cdots 2k \cdot n(n+2) \cdots (n+2k-2) \tag{17}$$

of the mean $A(u)$ of an arbitrary analytic function u in powers of the radius r (§4). This series will be recognized as a series whose first p terms agree with the right-hand side of (12); it can also be given the symbolic form

$$\phi_n(r, \nabla^2)u_0. \tag{18}$$

Utilizing (17), the following illuminating interpretation is derived for $\nabla^{2k}u$ at a point O:

$$(\nabla^{2k}u)_0 = \frac{2\cdot 4 \cdots 2k \cdot n(n+2) \cdots (n+2k-2)}{(2k)!} A\left(\frac{\partial^{2k}u}{\partial r^{2k}}\right)_0, \tag{19}$$

where the last factor denotes the mean of the directional derivatives of order $2k$ of the function u in all directions through O, averaged over the solid angle through O (§4). A further interpretation for $\nabla^2, \nabla^4, \cdots$, also derived from (17), is given by

$$\begin{aligned} (\nabla^2 u)_0 &= 2n \lim_{r\to 0} [A(u) - u_0]/r^2, \\ (\nabla^4 u)_0 &= 2\cdot 4\cdot n(n+2) \lim_{r_2\to 0} \lim_{r_1 \to 0} \frac{A_2(u) - u_0 - [A_1(u) - u_0]r_2^2/r_1^2}{r_2^4}, \end{aligned} \tag{20}$$

where $A_1(u)$, $A_2(u)$ are the means over spheres of radii r_1, r_2, and the repeated limit is obtained by letting r_1 approach zero first. The interpretations (19) and (20) are valid not merely for analytic functions u, but each one also for functions u possessing a sufficient number of continuous derivatives.

As a further application of (17) are considered in §5 the functional relations existing in certain cases between the means $A_1(u)$, $A_2(u)$ of u over "subspheres" lying in two mutually totally perpendicular flats of m, $n-m$ dimensions (§5, equations (64), (70), (71), (74), (75), (77), (79)). These include a theorem of Asgeirsson and some theorems of Bateman. The functional relations of §5 are utilized in §6 for *inverting* the averaging operation A, under certain assumptions regarding the function u.

One feature that is common to the laws of the mean (2), (4), (9), (14), as well as the indicated modification of (14), is that in every case $A(u)$ is linearly dependent upon p functions $f_1(r), \cdots, f_p(r)$, which depend on the radius r but are independent of the center O, while the coefficients of dependence, C_i, are independent of the radius r but do depend upon the position of the center O. Thus,

$$A(u) = C_1 f_1(r) + \cdots + C_p f_p(r), \tag{21}$$

or more precisely,

$$A(u)\Big|_{O,r} = C_1(O) f_1(r) + \cdots + C_p(O) f_p(r). \tag{21'}$$

This property of solutions of these differential equations actually *completely characterizes* them, as is shown by the following converse (§7):

CONVERSE THEOREM. *Let there be given a function u over a region R and of class $C^{(2p)}$ there. Let*

$$f_1(r), \cdots, f_p(r) \tag{22}$$

be p linearly independent functions of a variable r, of class $C^{(2p)}$ in r for $0 \leqq r < \rho$ and whose odd derivatives $f_i^{(2k+1)}(r)$, $k=0, \cdots, p-1$, vanish at $r=0$. If* (21′) *holds for a sphere of radius r and center at O for any position of the center O in R and sufficiently small radius r, then u must satisfy an equation of the form* (13) *for proper constants c_i, while the functions $f_i(r)$ must reduce to a set of p solutions of the ordinary differential equation*

$$\sum_{i=0}^{p} c_i \left(\frac{d^2}{dr^2} + \frac{n-1}{r}\frac{d}{dr}\right)^i f = 0 \tag{23}$$

* This apparent restriction on $f_i(r)$ is actually satisfied by $A(u)$ as well as by the functions $f_i(r)$ of the preceding laws of the mean.

which are analytic at $r=0$. *The latter solutions are exhibited above in* (16) *for the general case in which the operator on the left of* (13) *factors as in* (15′).

The differential equation (23) results from (13) if it is supposed that u depends only upon r, the distance from a fixed point. We shall refer to such solutions as "symmetric" solutions.

A particularly interesting special case of this converse is given by the following result:

If, for a function u of class $C^{(2p)}$ in R

$$A(u) = \text{polynomial of degree } p \text{ in } r^2, \tag{24}$$

for any position of the center O in R and sufficiently small radius r, then u satisfies (11) (or is p-harmonic).

The above laws of the mean and their converse involve symmetric solutions of (13) (or solutions of (23)) which are analytic at $r=0$. Symmetric solutions of (13) which are not analytic at $r=0$ occur when the function u under consideration satisfies the proper differential equation not in the complete interior of a sphere S but only over a spherical shell $R_{a,b}$ between two concentric spheres S_a, S_b of radii a, b; $0>a\geqq r\geqq b$ (§3). As an example, in a three-space, if u is harmonic in such a spherical shell $R_{a,b}$, then for the various spheres concentric with S_a, S_b the mean is given by

$$A(u) = C_1 + C_2/r, \tag{25}$$

where C_1, C_2 are constants. Even this simple result does not appear to be as familiar as its simplicity warrants.

The method of proof used in deducing the various laws of the mean is, itself, of some interest, particularly in view of its elegance and simplicity. It consists in regarding the operation which replaces u over each of a concentric family of spheres by its spherical mean $A(u)$ over that surface, as a linear functional operation A, and utilizing the permutability of the operator A with the operator ∇^2. This operator A is discussed in a preceding paper* to which we shall refer briefly as I, where its above mentioned property is proved (I, Theorem 1). We shall suppose that the reader has familiarized himself with I, at least with its introductory §1, with the definition of A and of the other operators contained in it, and with the statement of the theorems. The details of the proof of I, however, are not essential for a thorough understanding of the present paper.

Some of the above results, a search of the literature has revealed, are not

* *On operations permutable with the Laplacian*, American Journal of Mathematics, vol. 45 (1932), p. 667.

new. Thus, equations (2) and (4) have been given by H. Weber,* while a result equivalent to (17) for $n=3$ has been obtained by W. D. Niven.† These results were, nevertheless, included, because it is believed that the present proofs have decided advantages both in regard to unity and simplicity.

Many of the results of this paper can be generalized by passing from the operator A to other operators considered in I, which are likewise permutable with ∇^2. These generalizations are considered briefly in §9. A formula established in §8, due to Hobson, forms a natural transition to these generalizations. At the end of §9 are considered extensions to certain discontinuous functions.

Many interesting applications of the results of this paper can be made. Thus the integral representations of $J_0(r)$:

$$J_0(r) = \int_0^\pi e^{ir\cos\theta} d\theta/\pi = \int_{-1}^1 e^{irt}(1 - t^2)^{-1/2} dt/\pi,$$

result if one starts with the simple solution of (1) in the plane for $\lambda = -1: u = e^{ix_1}$, and averages it over circles with center at the origin.

Again, the Laplace integral for the Legendre polynomial:

$$P_n(z) = \int_0^\pi [z + \cos\phi(z^2 - 1)^{1/2}]^n d\phi/\pi,$$

is obtained by starting with the elementary harmonic function $(x_1+ix_2)^n$ in three-space and averaging it over circles having the x_1-axis as their axis. Similarly, the expansion

$$(1 - 2r\cos\theta + r^2)^{-1/2} = \sum r^n P_n(\cos\theta)$$

may be proved by averaging over the above circles the geometric series

$$\frac{1}{1 - (x_1 + ix_2)} = \sum (x_1 + ix_2)^n.$$

However, a systematic application of the results of this paper is reserved for a forthcoming paper entitled *On integral representation of Bessel and related functions.*

Another forthcoming paper somewhat related to the present one is entitled *Green's formulas for analytic functions.* In this paper is proved the analyticity of solutions of (13) as well as their expansibility in spherical harmonics.

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* H. Weber, Mathematische Annalen, vol. 1 (1869), p. 7; Crelle, vol. 49 (1868), p. 222.

† W. D. Niven, Transactions of the Royal Society of London, vol. 170 (1880), p. 379.

2. **On solutions of $\sum c_i L^i(V)=0$ for general linear operators L. Application to symmetric solutions of** (13). In this section we shall obtain explicit forms for symmetric solutions of (13), that is for solutions of (23). As noted above, these solutions play an essential role in the laws of the mean considered in this paper.

Before taking up (23) and its solutions consider the equation

$$L(u) - \lambda u = 0, \tag{26}$$

where L is a linear functional operator, and λ is an arbitrary constant. It will be shown how solutions of (26) can be made to yield solutions of

$$\sum_{i=0}^{p} (c_i L^i)V = 0, \qquad c_p \neq 0, \tag{27}$$

where c_i are constants. By specializing to the case $L=d^2/dr^2+(n-1)d/dr$, (23) will be obtained. Applications to other operators L occur in §9.

The solution of (26) depends upon the parameter λ as well as upon proper independent variables; consider a solution u which is analytic in λ at $\lambda=\lambda_0$. Expanding u in powers of $\lambda-\lambda_0$:

$$u = \sum_{k=0}^{\infty} \frac{\partial^k u}{\partial \lambda^k}\bigg|_{\lambda=\lambda_0} \frac{(\lambda - \lambda_0)^k}{k!}, \tag{28}$$

substituting in (26) written in the form

$$[(L - \lambda_0) - (\lambda - \lambda_0)]u = 0, \tag{26'}$$

applying L term-wise to the right-hand of (28), and comparing coefficients of like powers of $\lambda-\lambda_0$ on both sides, we obtain the recurrence relations (omitting the subscript in λ_0):

$$(L - \lambda)\frac{1}{k!}\frac{\partial^k u}{\partial \lambda^k} = \begin{cases} 0 \text{ for } k = 0, \\ \dfrac{1}{(k-1)!}\dfrac{\partial^{k-1} u}{\partial \lambda^{k-1}} \text{ for } k > 0, \end{cases} \tag{29}$$

provided the term-wise application of L is justifiable. From (29) it follows that

$$u, \frac{\partial u}{\partial \lambda}, \cdots, \frac{\partial^{k-1} u}{\partial \lambda^{k-1}} \tag{30}$$

are solutions of the functional equation

$$(L - \lambda)^k V = 0. \tag{31}$$

Now, consider the functional equation (27). Factoring the operator in (27), there results

$$(32) \qquad \sum_{i=0}^{p} c_i L^i = c_p(L-\lambda_1)^{m_1} \cdots (L-\lambda_k)^{m_k},$$

where $\lambda_1, \lambda_2, \cdots$ are the roots of $\sum c_i \lambda^i = 0$, and $m_1, m_2, \cdots$ their respective multiplicities. Now obviously, solutions, say, of $(L-\lambda_1)^{m_1}V=0$ are also solutions of $(L-\lambda_1)^{m_1}(L-\lambda_2)^{m_2}V=0$, and, therefore, also of (27). Hence we conclude that from the solution u of (26), the following p solutions of (27) may be obtained:

$$(33) \qquad \left.\frac{\partial^i u}{\partial \lambda^i}\right|_{\lambda=\lambda_j}; \; i = 0, \cdots, m_j - 1; j = 1, \cdots, k.$$

It will be noted that the λ-differentiations in (30), (33) could be replaced by differentiation with respect to μ, where μ is a properly differentiable function of λ. Thus, the functions

$$(30') \qquad u, \frac{\partial u}{\partial \mu}, \cdots, \frac{\partial^{k-1} u}{\partial \mu^{k-1}}$$

are solutions of (31). Indeed, from the differentiation formulas

$$\frac{\partial u}{\partial \mu} = \frac{\partial u}{\partial \lambda}\frac{d\lambda}{d\mu}, \quad \frac{\partial^2 u}{\partial \mu^2} = \frac{\partial^2 u}{\partial \lambda^2}\left(\frac{d\lambda}{d\mu}\right)^2 + \left(\frac{\partial u}{\partial \lambda}\right)\frac{\partial^2 \lambda}{\partial \mu^2}, \cdots$$

it follows that the functions (30′) can be expressed linearly in terms of the functions (30).

Returning now to symmetric solutions of the differential equations (1), (8), (11), (13), we replace ∇^2 by

$$\text{I, (3)*} \qquad \nabla^2 = \frac{d^2}{dr^2} + \frac{n-1}{r}\frac{d}{dr} = r^{1-n}\frac{d}{dr}\left(r^{n-1}\frac{d}{dr}\right)$$

thus converting them into *ordinary* differential equations. Thus (1) becomes

$$(34) \qquad \frac{d^2u}{dr^2} + \frac{n-1}{r}\frac{du}{dr} - \lambda\mu = 0.$$

Two solutions of (34) are readily verified to be $\phi_n(r, \lambda)$ given by (6) and

$$(35) \; \psi_n(r, \lambda) = r^{2-n}\sum_{k=0}^{\infty} (r^2\lambda)^k F_k/2\cdot 4 \cdots (2k)[(2-n)(4-n) \cdots (2-n+2k)]',$$

where

* As explained in §1, I, (3) refers to equation (3) of the previously cited paper I.

$$F_k = \begin{cases} 1, \text{ if } n \text{ is odd, or if } n \text{ is even and } k < (n/2) - 1, \\ \log r \text{ if } n \text{ is even and } k = (n/2) - 1, \\ \log r - \left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2k - n + 2}\right) \\ \qquad - \left(\frac{1}{n} + \frac{1}{n+2} + \cdots + \frac{1}{2k}\right) \text{ if } n \text{ is even and } k > (n/2) - 1, \end{cases}$$

and where the prime following the brackets $[(2-n)(4-n) \cdots (2-n+2k)]$ indicates that the factor zero that would occur in the indicated product for even n and $k \geqq (n/2)-1$ should be omitted. These solutions result in a natural manner when one attempts to integrate (34) by means of a power series in λ admitting term-wise r-differentiation. The solution ϕ_n is analytic for all r and λ; likewise for ψ_n except for $r=0$. We denote the coefficient of λ^k in (35) by $V_{n,k+1}(r)$ or by $V_{k+1}(r)$ if the omission of the subscript n leads to no confusion, thus:

$$\psi_n(r, \lambda) = \sum_{k=0}^{\infty} \lambda^k V_{n,k+1}(r), \tag{36}$$

$$\psi(r, \lambda) = \sum_{k=0}^{\infty} \lambda^k V_{k+1}(r). \tag{36'}$$

Applying the conclusions established above regarding solutions of the functional equation (26) to the present case, there follows for $k>1$

$$\begin{cases} \nabla^2 V_k = V_{k-1}, \\ \nabla^2[r^{2k}/2\cdot 4 \cdots 2k\cdot n(n+2) \cdots (n+2k-2)] \\ \qquad = r^{2k-2}/2 \cdots (2k-2)n \cdots (n+2k-4), \end{cases} \tag{37}$$

while

$$\nabla^2 V_1 = 0, \qquad \nabla^2 1 = 0; \tag{37'}$$

there also follows that

$$V_1(r), \cdots, V_p(r); \qquad 1, \cdots, r^{2p-2} \tag{38}$$

are symmetric solutions of (11). These results are obtained for $\lambda_0 = 0$. Similarly for general λ_0 are obtained the recurrence relations

$$(\nabla^2 - \lambda)\frac{1}{k!}\frac{\partial^k(\phi, \psi)}{\partial\lambda^k} = \frac{1}{(k-1)!}\frac{\partial^{k-1}(\phi, \psi)}{\partial\lambda^{k-1}} \text{ for } k > 0, \tag{39}$$

and the following symmetric solutions of (8):

$$\partial^i(\phi, \psi)/\partial\lambda^i; \qquad i = 0, \cdots, p-1. \tag{40}$$

Finally, we conclude that symmetric solutions of (13) are given by (16) and by similar derivatives of $\psi_n(r, \lambda)$.

The various solutions of the proper (ordinary) differential equations just obtained may be shown to be linearly independent thus furnishing a complete set of such solutions upon which any solution would depend linearly.

For $\lambda \neq 0$ these solutions may be expressed in terms of Bessel functions. Indeed, by introducing $v = r^{n/2-1}u$ and letting $y = (-\lambda)^{1/2}r$ (where *either* determination of $(-\lambda)^{1/2}$ is used) equation (34) becomes

$$(-\lambda)r^{n/2}\left[\frac{d^2v}{dy^2} + \frac{1}{y}\frac{dv}{dy} + \left(1 - \frac{(n/2-1)^2}{y^2}\right)v\right] = 0.$$

The bracket will be recognized as the Bessel differential operator of order $m = n/2 - 1$, applied to v. Hence for $\lambda \neq 0$ solutions of (34) can be expressed linearly, say, in terms of

$$y^{-m}J_m(y), \qquad \begin{matrix} y^{-m}J_{-m}(y) \text{ for } n \text{ odd}, \\ y^{-m}Y_m(y) \text{ for } n \text{ even}, \end{matrix} \tag{41}$$

where

$$y = (-\lambda)^{1/2}r, \qquad m = n/2 - 1.^*$$

An advantage of the solution ψ_n over the Bessel function form lies in its analyticity at $\lambda = 0$. The relation between ϕ_n and the Bessel functions is given by (7); the expression of ψ_n in terms of the latter is given by

$$\left\{\begin{aligned} \psi_n(r, \lambda) &= \Gamma(1-m)(-\lambda/2)^m\left[y^{-m}J_{-m}(y)\right] \text{ for } n \text{ odd}, \\ \psi_n(r, \lambda) &= \frac{(-\lambda)^m}{2^{m+1}\Gamma(m+1)}\left\{y^{-m}Y_m(y) \right. \\ &\quad \left. + \left[\left(1 + \frac{1}{2} + \cdots + \frac{1}{m}\right) - 2 - 2\log\frac{(-\lambda)^{1/2}}{2}\right]y^{-m}J_m(y)\right\} \\ &\qquad \text{for } n \text{ even}, \end{aligned}\right. \tag{42}$$

where, as in (41) $y = (-\lambda)^{1/2}r$, $m = n/2 - 1$. Similarly, for non-vanishing λ, the symmetric solutions of (8) can be expressed in terms of Bessel functions, a set of solutions being furnished by

* E. W. Hobson, Proceedings of London Mathematical Society, vol. 25 (1893), p. 49. Hobson suggests the term "rank" for n.

The Bessel function notation is that of G. N. Watson's *Theory of Bessel Functions*, Cambridge University Press.

$$(43)\qquad y^{-m+i}\left[J_{m+i}(y),\quad \begin{matrix} J_{-m-i}(y) & \text{for } n \text{ odd} \\ Y_{m+i}(y) & \text{for } n \text{ even} \end{matrix}\right] \quad \begin{matrix} m = n/2 - 1, & y = (-\lambda)^{1/2}r, \\ i = 0, \cdots, p-1. \end{matrix}$$

To prove this, replace the λ-differentiations by differentiations with respect to $\log(\lambda)^{1/2}$ (or $\log(-\lambda)^{1/2}$) thus operating on (41) with the operator $2\lambda\partial/\partial\lambda = yd/dy$, and utilize the formulas

$$(44)\qquad d[z^{-\nu}J_\nu(z)]/dz = -z^{-\nu}J_{\nu+1}(z)$$

and similar formulas for $z^\nu J_\nu(z)$ and $z^{-\nu}Y_\nu(z)$.*

It may be concluded from the above that Bessel functions of order $m=n/2-1$ and argument $y=(-\lambda)^{1/2}r$ are linearly dependent on the functions $y^m\phi_n(r,\lambda)$, $y^m\psi_n(r,\lambda)$. Hence the functions appearing in (43) and therefore also in (40) can be expressed linearly in terms of $y^{2i}\phi_{n+2i}$, $y^{2i}\psi_{n+2i}$. Thus a set of solutions of (8) for $\lambda\neq 0$ is also furnished by

$$(45)\qquad r^{2i}\phi_{n+2i}(r,\lambda), \qquad r^{2i}\psi_{n+2i}(r,\lambda); \qquad i = 0, \cdots, p-1.$$

For $\lambda\neq 0$ this set is, again, linearly independent.

The symmetric solutions of (13) can now be similarly expressed in terms of Bessel functions by means of the factorization (15′), a set of the form (43) or (45) corresponding to each repeated root.

3. **Laws of the mean for solutions of** (13). Consider a solution of (1) in a spherical shell $R_{a,b}$ or in a sphere R_b. Applying the operator A to both sides of (1) for spheres concentric with boundaries there follows

$$A(\nabla^2 u) - \lambda A(u) = 0.$$

Now, by I, Theorem 1, the first term above may be replaced by $\nabla^2[A(u)]$. Hence, $A(u)$ also satisfies (1). Since $A(u)$ is symmetric, we conclude that

$$(46)\qquad A(u) = C\phi_n(r,\lambda) + D\psi_n(r,\lambda),$$

where C, D are constants. The values of the latter depend upon the particular solution u as well as upon the position of the center.

Thus it follows from I, Theorem 1, that if u satisfies (1) in a spherical region R_b, then such is also the case with $A(u)$. Since $A(u)$ is of class C'' at $r=0$, the constant D in (46) must now vanish. Putting $r=0$ to determine the remaining constant C, we obtain

$$A(u)\Big|_{r=0} = C.$$

But $A(u)|_{r=0}=u_0$. Hence (5) results. For $n=2$ and 3 it yields (4) and (2)

* Watson, loc. cit., p. 66.

respectively. For $\lambda=0$, that is for harmonic functions, (5) reduces to the Gauss law of the mean, while (46) yields

$$A(u) = C + DV_1(r), \tag{47}$$

which reduces to (25) for $n=3$.

More generally, by applying the operator A to both sides of (13) and permuting A with $\nabla^2, \nabla^4, \cdots$ (see I, §9) one proves similarly that if u is a solution of (13) in $R_{a,b}$ or in R_b, then $A(u)$, likewise, satisfies (13) there. Being symmetric $A(u)$ must therefore reduce to a linear combination of the functions (16) and of the corresponding ψ-functions. Thus, for the general case of (13), (15)

$$A(u) = C_1\phi(r, \lambda_1) + \cdots + D_p \left.\frac{\partial^{m_k-1}\psi(r, \lambda)}{(m_k - 1)!\partial\lambda^{m_k-1}}\right|_{\lambda=\lambda_k}. \tag{48}$$

We shall determine the coefficients for the case when u satisfies (13) in R_b, that is for $r \leqq b$, $r=0$ included.

Applying $(\nabla^2-\lambda_2)^{m_2} \cdots (\nabla^2-\lambda_k)^{m_k}$ to (48) and utilizing (40) it is found that the ϕ, ψ functions corresponding to the roots $\lambda_2, \lambda_3, \cdots$ drop out. Applying products of the above operator by $(\nabla^2-\lambda_1)^{m_1-1}$, and utilizing (39), there results

$$\begin{aligned}(\nabla^2 - \lambda_1)^{m_1-1}(\nabla^2 - \lambda_2)^{m_2} \cdots (\nabla^2 - \lambda_k)^{m_k}A(u)\\ = (\lambda_1 - \lambda_2)^{m_2} \cdots (\lambda_1 - \lambda_k)^{m_k}[C_{m_1}\phi(r, \lambda_1) + D_{m_1}\psi(r, \lambda_1)].\end{aligned}$$

Since $A(u)$ is of class $C^{(2p)}$ at $r=0$, while the left-hand operator is of class $C^{(2p-2)}$, it follows that $D_{m_1}=0$. Similarly applying products of $(\nabla^2-\lambda_2)^{m_2} \cdots (\nabla^2-\lambda_k)^{m_k}$ by $(\nabla^2-\lambda_1)^{m_1-2}, \cdots, (\nabla^2-\lambda_1)$, one proves that $D_{m_1-1}, \cdots, D_1$ all vanish. Putting $r=0$ in the m_1 equations obtained yields successively the values $C_{m_1}, \cdots, C_1$.

A similar procedure eliminates the D's and evaluates the C's corresponding to the repeated factors $\lambda_2, \lambda_3, \cdots$. There results

$$\begin{aligned}A(u) = S\,&\frac{(\nabla^2 - \lambda_2)^{m_2} \cdots (\nabla^2 - \lambda_k)^{m_k}}{(\lambda_1 - \lambda_2)^{m_2} \cdots (\lambda_1 - \lambda_k)^{m_k}}\\ &\cdot \left\{\phi(r, \lambda_1) + \cdots + \frac{(\nabla^2 - \lambda_1)^{m_1-1}}{(m_1 - 1)!} \left.\frac{\partial^{m_1-1}\phi(r, \lambda)}{\partial\lambda^{m-1}}\right|_{\lambda=\lambda_1}\right\} u\Bigg|_{r=0},\end{aligned} \tag{49}$$

where the operators $\nabla^2, \nabla^4, \cdots$ all operate only on u (but not on ϕ, $\partial\phi/\partial\lambda, \cdots$), and S indicates a summation extended to similarly constituted terms corresponding to the repeated roots $\lambda_2, \cdots, \lambda_k$; the operators A have been suppressed from $A(u), \nabla^2 A(u), \cdots$ at $r=0$ by applying $\nabla^2, \nabla^4, \cdots$ first and replacing the mean at a point by the function itself.

For special cases both the above proof and the results simplify. Thus if the roots are all alike, that is, in case (8), (49) reduces to (9). If the roots are all distinct, (49) becomes

$$
\begin{aligned}
A(u) &= \frac{(\nabla^2-\lambda_2)\cdots(\nabla^2-\lambda_p)}{(\lambda_1-\lambda_2)\cdots(\lambda_1-\lambda_p)}u\Big|_{r=0}\phi(r,\lambda_1)+\cdots \\
&\quad+\frac{(\nabla^2-\lambda_1)\cdots(\nabla^2-\lambda_{p-1})}{(\lambda_p-\lambda_1)\cdots(\lambda_p-\lambda_{p-1})}u\Big|_{r=0}\phi(r,\lambda_p).
\end{aligned}
\tag{50}
$$

To obtain the forms of these laws of the mean given by (14) and its modifications indicated in §1, we proceed directly from (48) after the ψ's have been eliminated, applying $\nabla^2, \cdots, \nabla^{2p-2}$ to both sides, putting $r=0$, utilizing

$$
\nabla^{2i}\frac{\partial^j\phi(r,\lambda)}{\partial\lambda^j}\Big|_{r=0}=\frac{\partial^j(\lambda^i)}{\partial\lambda^j},
\tag{51}
$$

and eliminating C_i. The relation (51) is proved by interchanging the order of the λ- and the ∇^2-differentiations, replacing $\nabla^{2i}\phi$ by $\lambda^i\phi$, and putting $r=0$ before carrying out the λ-differentiations.

4. The spherical mean of analytic and regular functions. Interpretations of the iterated Laplacians. We shall indicate two proofs of (17) for the mean of analytic functions. Consider the function

$$
u=\exp\,[c_1x_1+\cdots+c_nx_n]u_0
$$

where $c_1, \cdots, c_n$ and u_0 are constants. This function is a solution of (1) with $\lambda=c_1^2+\cdots+c_n^2$. Applying the law of the mean (5) to u with the center O of the spheres at the origin, we obtain for all r

$$
A(\exp\,[c_1x_1+\cdots+c_nx_n]u_0)=\phi_n(r,\lambda)u_0,\qquad \lambda=c_1^2+\cdots+c_n^2.
\tag{52}
$$

Now consider an arbitrary analytic function u; placing the origin at the center O, we may write the Taylor series formally thus:

$$
u=\exp\,[x_1(\partial/\partial x_1)+\cdots+x_n(\partial/\partial x_n)]u_0,
\tag{53}
$$

where the exponential is to be expanded as an n-fold power series, each term multiplied by u_0, and the formal product then replaced by the corresponding partial derivative of u at the origin. Since the convergence for r less than a proper $\rho>0$ is uniform, we may average term by term; the result is, therefore, the same as the average for the above example written as an n-fold power series in c_i where the latter are replaced by the fictitious quantities $\partial/\partial x_i$ and the resulting terms interpreted as above. Hence, for $r<\rho$ (18) follows, or, more explicitly, (17).

A more direct proof of (17) is obtained by expanding $A(u)$ in powers of r^2;

$$A(u) = C_0 + C_2r^2 + C_4r^4 + \cdots, \tag{54}$$

and determining the constants C_{2i} by applying ∇^{2i} to both sides, replacing $\nabla^{2i}A(u)$ by $A(\nabla^{2i}u)$, putting $r=0$, and utilizing the latter part of (37). To justify the expansion (54), write the Taylor series of u in the form

$$u = \sum_{k=0}^{\infty} \frac{1}{k!}\left(x_1\frac{\partial}{\partial x_1} + \cdots + x_n\frac{\partial}{\partial x_n}\right)^k u_0 = \sum_{k=0}^{\infty} \frac{r^k}{k!}\left.\frac{\partial^k u}{\partial r^k}\right|_0, \tag{55}$$

where $\partial/\partial r$ denotes differentiation along the ray from the origin through $x_1, \cdots, x_n$, with respect to the distance r from the origin. Upon taking means of both sides a series such as (54) will result, since, for odd k the contributions from opposite directions toward the mean will cancel each other.

Comparing the coefficients of the powers of r^2 in (17) with those obtained by averaging the last member of (55) we obtain (19). To derive the first relation (20), transpose the first term on the right of (17) ($k=0$) to the left, divide by r^2, and let r approach zero; the further relations (20) are derived in a similar fashion by utilizing the preceding ones.

The interpretations (19), (20), for $\nabla^2, \nabla^4, \cdots$ are proved for non-analytic functions of class $C^{(2k)}$ for proper k by replacing (53), (55) by finite sums with a remainder term of the form $o(r^{2k})$ and deriving from these a similar modification of (17).

From (17) are readily derived the formulas*

$$\int u\,dS = K_n \sum_{k=0}^{\infty} \frac{(\nabla^{2k}u)_0 r^{2k+n-1}}{2\cdot 4 \cdots 2k\cdot n(n+2)\cdots(n+2k-2)}, \tag{56}$$

$$\int u\,dv = K_n \sum_{k=0}^{\infty} \frac{(\nabla^{2k}u)_0 r^{2k+n}}{2\cdot 4 \cdots 2k\cdot n(n+2)\cdots(n+2k)} \tag{57}$$

for the integrals of u over the surface and the volume of S. Likewise by differentiating (17) with respect to r, then multiplying by S, one may obtain similar series for the surface integrals over S of the normal derivative of u of any order.

5. **Applications of** (17). As a further application of (17), consider the means $A_1(u)$, $A_2(u)$ of analytic functions u over "subspheres" Σ_1, Σ_2 lying in two mutually totally perpendicular flats of m, $n-m$ dimensions:

$$\Sigma_1 : x_1^2 + x_2^2 + \cdots + x_m^2 = r^2, \quad x_{m+1} = x_{m+2} = \cdots = x_n = 0, \tag{58}$$

$$\Sigma_2 : x_{m+1}^2 + \cdots + x_n^2 = r^2, \quad x_1 = x_2 = \cdots = x_m = 0. \tag{59}$$

Application of (17) (for sufficiently small r) yields

* Here, as in I, K_n denotes the "area" of a unit sphere in n-dimensions.

$$(60)\qquad A_1(u) = \sum_{k=0}^{\infty} (D_1^k u)_0 r^{2k}/2\cdot 4 \cdots 2k\cdot m \cdots (m + 2k - 2),$$

$$(61)\qquad A_2(u) = \sum_{k=0}^{\infty} (D_2^k u)_0 r^{2k}/2\cdot 4 \cdots 2k\cdot (n - m) \cdots (n - m + 2k - 2),$$

where

$$(62)\qquad D_1 = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_m^2;\quad D_2 = \partial^2/\partial x_{m+1}^2 + \cdots + \partial^2/\partial x_n^2.$$

It will be shown that in certain cases there exists a linear functional relation between the means $A_1(u)$, $A_2(u)$.

Thus, if u is harmonic,

$$0 = \nabla^2 u = D_1 u + D_2 u = 0,$$

whence

$$D_1 u = -D_2 u.$$

Applying D_1 to both sides, there results

$$D_1^2 u = -D_1 D_2 u = -D_2 D_1 u = D_2^2 u,$$

and, by induction, for any k

$$(63)\qquad D_1^k u = (-1)^k D_2^k u.$$

There exists thus a definite ratio between the coefficients of r^2 in (60) and (61): this proves the above statement.

Denote $A_1(u)$, $A_2(u)$ by $f_1(r)$, $f_2(r)$. For even n and $m = n/2$ (and harmonic functions u), the functional relation reduces to

$$(64)\qquad f_1(r) = f_2(ir).$$

For other cases, the relation between $f_1(r)$, $f_2(r)$ may be obtained as follows:

Consider the series

$$(65)\qquad g_p(r) = \sum_{k=0}^{\infty} \frac{r^{2k} C_k}{p(p+2)\cdots(p+2k-2)} = \Gamma\left(\frac{p}{2}\right)\sum_{k=0}^{\infty} \frac{r^{2k} C_k}{2^k \Gamma\left(\dfrac{p}{2} + k\right)}$$

for integer p, where the C_k do not vary with p. One may express $g_{p+q}(r)$ for $q > 0$ in terms of $g_p(r)$ as follows:

$$(66)\qquad g_{p+q}(r) = \frac{\Gamma[(p+q)/2]}{\Gamma(p/2)\Gamma(q/2)}\, r^{2-p-q} \int_0^{r^2} s^{p-2} g_p(s)(r^2 - s^2)^{q/2-1} d(s^2).$$

This relation is even simpler if the functions

$$(67)\qquad G_p(r) = r^{p-2} g_p(r)/\Gamma(p/2)$$

are introduced, whereupon (66) is replaced by

$$G_{p+q}(r) = \int_0^{r^2} G_p(s)(r^2 - s^2)^{q/2-1}d(s^2)/\Gamma(q-2). \tag{68}$$

Either (66) or (68) is readily established by termwise multiplication and integration, utilizing the Eulerian integral of the first kind; the results are valid at least in the circle $|r|<\rho$ in which g_p is analytic. The relation (68) will be recognized as an Abel integral equation; it is related to "fractional integrals"* and will be recognized as equivalent to

$$G_{p+q}(r) = \left[\frac{d}{d(r^2)}\right]^{-q/2} G_p(r). \tag{68'}$$

For even q the solution of (68) is given by

$$G_p(r) = \left[\frac{d}{d(r^2)}\right]^{q/2} G_{p+q}(r), \tag{69}$$

while for odd q the fractional integrations or differentiations cannot be eliminated, and the solution of (68) is given by

$$\begin{aligned} G_p(r) &= \left[\frac{d}{d(r^2)}\right]^{(q+1)/2} \left[\frac{d}{d(r^2)}\right]^{-1/2} G_{p+q}(r) \\ &= \left[\frac{d}{d(r^2)}\right]^{(q+1)/2} \int_0^{r^2} (r^2 - s^2)^{-1/2} G_{p+q}(s)d(s^2)/\pi^{1/2}.\dagger \end{aligned} \tag{69'}$$

To apply the above to the means $f_1(r)$, $f_2(r)$ for $m \neq n/2$, suppose that $m<n/2$, and put $p=m$, $p+q=n-m$, $g_p(r)=f_1(r)$, $g_{p+q}(r)=f_2(ir)$, (or else put $g_p(r)=f_1(ir)$, $g_{p+q}(r)=f_2(r)$). There results

$$f_2(r) = \frac{\Gamma[(n-m)/2]r^{2-n+m}}{\Gamma(m/2)\Gamma(n/2-m)} \int_0^{r^2} f_1(is)s^{m-2}(r^2-s^2)^{n/2-m-1}d(s^2), \tag{70}$$

and, in particular, for even n and $m=n/2-1$,

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* See, for instance, the author's paper on *Heaviside's operational calculus*, American Mathematical Monthly, vol. 43 (1936), pp. 332–334, 339.

† Another way of expressing the linear functional relation in question is by means of

$$g_{p'}(r) = \frac{1}{2\pi i}\int g_p(s) F\left(\frac{p'}{2}, 1, \frac{p}{2}; \frac{r^2}{s^2}\right) ds,$$

where F is the hypergeometric function, and the integration is carried out over a circle $|s| =$const. on and within which f_p is analytic, and for $|r|<|s|$. This relation holds for any integer p, p', $p \gtrless p'$.

$$r^{n/2-1}f_2(r) = \left(\frac{n}{2} - 1\right)\int_0^r f_1(is)s^{n/2-2}ds. \tag{71}$$

The special case of (71) $n=4$ (and hence $m=1$), as well as the special case of (64), $n=4$, $m=2$, has been noted by H. Bateman.*

Suppose next, that the function u, instead of being harmonic, is an analytic solution of $D_1u=D_2u$, that is of

$$\left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_m^2}\right)u = \left(\frac{\partial^2}{\partial x_{m+1}^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right)u. \tag{72}$$

Equation (63) is now replaced by

$$D_1^k(u) = D_2^k(u), \tag{73}$$

and the functional relations between $f_1(r)$, $f_2(r)$ become even simpler. Thus, for even n and $m=n/2$, (64) is replaced by

$$f_1(r) = f_2(r);\dagger \tag{74}$$

while in applying (67)–(69′), g_p, g_{p+q} are replaced by $f_1(r)$, $f_2(r)$ directly, so that for even n (69) yields

$$r^{m-2}f_1(r) = \frac{\Gamma(m/2)}{\Gamma[(n-m)/2]}\left[\frac{d}{d(r^2)}\right]^{n-2m}[r^{n-m-2}f_2(r)]. \tag{75}$$

The proof of the above results is based upon the analyticity of u and of $f_1(r)$, $f_2(r)$; this is necessarily the case with harmonic functions and their means; however, solutions of (72) need not be analytic in all the variables. That the results obtained for solutions of (72) do apply, whether they are analytic or not, follows from the fact that one may approximate to a function and its first and second derivatives by means of an analytic function and its derivatives.

Applying (74) to the one-dimensional‡ wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \tag{76}$$

one obtains

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* H. Bateman, *Some geometric theorems*, etc., American Journal of Mathematics, vol. 34 (1912), pp. 332–334.

† This result is due to L. Asgeirsson, Mathematische Annalen, vol. 112 (1936).

‡ A one-dimensional "sphere" along the 1-space (x-axis) is the locus $(x-x_0)^2=r^2$, where x_0 is the center, r the radius; it consists of the two points x_0+r, x_0-r. The "spherical mean" of a function $u(x)$ over it is to be understood as $[u(x_0+r)+u(x_0-r)]/2$.

$$\frac{u(x+ct', t) + u(x-ct', t)}{2} = \frac{u(x, t+t') + u(x, t-t')}{2}. \tag{77}$$

For the three-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2}\right) \tag{78}$$

a relation similar to (71) yields

$$A(u)\Big|_t = \int_{-r/c}^{r/c} u_0(t+t')dt'/(2r/c), \tag{79}$$

where the left-hand member is the mean of u at the time t over a sphere of radius r, while the right-hand member is the time-average of u at the center over the time interval $t-r/c$, $t+r/c$.*

6. **Inversion of the averaging process.** The relations between means over spheres in two mutually perpendicular directions considered in the preceding section are also of interest in connection with the *inverse* problem of spherical means. Of course, the operation A does not possess a unique inverse, since many different functions can give rise to the same spherical mean. However, if the function u is properly restricted, then the operation A may be inverted uniquely. Thus we may ask: what even function $f(x_1)$ of the single variable x_1, will, when averaged over spheres with center at the origin, give rise to a given function $f_2(r)$?

Suppose the f_2 is analytic in r^2 and f_1 in x^2. The following relations hold at the origin (that is at the center of the spheres):

$$\nabla^{2k} f_2\Big|_0 = \nabla^{2k}[A(f_1)]_0 = A(\nabla^{2k} f_1)_0 = (\nabla^{2k} f_1)_0 = \frac{d^{2k} f_1}{dx_1^{2k}}\Big|_0.$$

Hence between the coefficients of the expansions of f_1 in powers of x_1^2 and of f_2 in powers of r^2 exist the same ratios as between the coefficients of powers of r^2 in (60), (61) for the case (73) provided m, $n-m$ are replaced by 1 and n respectively.

Similarly, if it is supposed that n is an even function of

$$x = (x_1^2 + x_2^2 + \cdots + x_m^2)^{1/2}, \qquad m < n, \tag{80}$$

the coefficients of the expansions of $A(u)$ in powers of r^2 and of u in powers of x^2 will have the same ratios as the coefficients of the powers of r^2 in (60),

* The time $t-r/c$ corresponds to the moment at which spherical wavelets should start from the points of the sphere, so that as they diverge with the characteristic velocity c, they will arrive at the center at the time t; similarly for $t+r/c$ and converging spherical wavelets.

(61) for the case (73) provided m, $n-m$ are replaced by m, n. Carrying out the necessary modifications in (66), there results

$$A(u) = \frac{\Gamma(n/2)r^{2-n}}{\Gamma(m/2)\Gamma[(n-m)/2]}\int_0^{r^2} s^{m-2}u(s)(r^2-s^2)^{[(n-m)/2]-1}d(s^2). \tag{81}$$

The relation (81) may be established directly geometrically as follows: Break up the spheres of integration S, implied in A by means of the loci $x=\text{const.}$, and let

$$y = (x_{m+1}^2 + \cdots + x_n^2)^{1/2} \tag{82}$$

and

$$ds = (dx^2 + dy^2)^{1/2} = dx\, r/y,$$

since $x^2+y^2=r^2$. The cylindrical shell between x and $x+dx$ has a base lying in $x_{m+1}= \cdots =x_n=0$ of m-dimensional content

$$dB = K_m x^{m-1}dx,$$

while upon each point of this base is "projected" an $(n-m)$-dimensional subsphere of S, of radius y and of content

$$K_{n-m}y^{n-m-1}.$$

The area of S intercepted by dB is therefore

$$dS = dB\, K_{n-m}y^{n-m-1}\sec\theta,$$

where θ is the angle between the normal to S and the projecting lines, so that $\sec\theta=r/y$. Hence

$$dS = K_m K_{n-m}x^{m-1}y^{n-m-2}r\,dx$$

and

$$A(u) = \frac{K_m K_{n-m}}{K_n r^{n-2}}\int_0^{r} u(x)x^{m-1}(r^2-x^2)^{(n-m)/2-1}dx. \tag{81'}$$

This is readily reduced to (81).

The uniqueness of the averaging process may now be based upon the known facts concerning the solution of (81).

The results of this section will be applied in the paper *On integral representations*, etc., mentioned at the end of §1.

7. **Converse theorem.** We start the proof of the converse theorem described in §1 by noting that if in

$$f_i(r) = \text{polynomial of degree } p \text{ in } r^2 + O(r^2)$$

we replace r^2 by $x_1^2+ \cdots +x_n^2$, then $f_i(r)$, regarded as a space function in a

Euclidean n-dimensional space in which r is the distance from a point P, is of class $C^{(2p)}$ even at P. Applying (19) there follows

$$\nabla^{2k} f_i(r) = b_k d^{2k} f_i(r)/dr^{2k} \Big|_{r=0},$$

where $b_k \neq 0$. Now, consider the square matrix m

$$\nabla^{2j-2} f_i(r) \Big|_{r=0}; \qquad i, j = 1, 2, \cdots, p,$$

where the Laplacians are obtained by forming space functions out of $f_1(r)$ in the manner explained; suppose, for definiteness, that i represents the order of the columns. Assume that not all the elements of the first row vanish. Then, by a reversible linear transformation on $f_i(r)$ (consisting in permuting, if necessary, two functions, then dividing the first function by its value at $r=0$, and adding a constant multiple of it to the others) it is possible to transform them into a new set of functions for which the corresponding matrix has for the elements of the first row the numbers $1, 0, \cdots, 0$. Similarly, if in the new matrix not all the elements of the second row beyond the first element vanish, it is possible to obtain p new functions related to the former ones by a reversible linear transformation and such that their matrix has for its first two rows the first two rows of the unit matrix: one needs only to transform the functions beyond the first one in a fashion similar to the above and then add to the first function a proper constant multiple of the second one. This may be continued till either

(a) the matrix m has been transformed into the unit matrix, or else,

(b) the matrix m has been transformed into a matrix whose first $p'-1$ rows, $p' \leq p$, are rows of the unit matrix, while in row p' the principal diagonal term and all the terms following it vanish.

Denote the new functions of r into which $f_i(r)$ have thus been transformed by $F_i(r)$, and write in place of (21)

$$A(u) \Big|_{\text{center at } O} = D_1(O)F_1(r) + \cdots + D_p(O)F_p(r). \tag{83}$$

In case (a) consider (83) for an arbitrary but fixed O. Interpreting each member as a space function in the neighborhood of O, apply ∇^{2j} ($j=0, 1, \cdots, p-1$) to both sides, and put $r=0$. Replacing $\nabla^{2j} A(u)\big|_{r=0}$ by $A(\nabla^{2j}u)\big|_{r=0}$ hence by $\nabla^{2j}u\big|_{r=0}$, we obtain at O (hence everywhere inside R)

$$D_1(O) = u, \cdots, D_p(O) = \nabla^{2p-2}u.$$

Applying ∇^2 once more to (83), putting $r=0$, and utilizing the results obtained, we get

$$\nabla^{2p}A(u) = \nabla^{2p}u = \nabla^{2p}F_1(r)\Big|_{r=0} u + \cdots + \nabla^{2p}F_p(r)\Big|_{r=0} \nabla^{2p-2}u. \tag{84}$$

We have thus proved that u satisfies an equation of the form (13).

In case (b) we obtain in a similar way from the first p' applications of ∇^2 a differential equation for u of the form (13) but of lower "order" p'. Applying the results of §3 it follows that $A(u)$ is linearly dependent on $p'<p$ functions of r. This case therefore reduces to case (a) with a value of p lower than the value initially used.

As an example, suppose that for u of class C'', $A(u)$ is proportional to the same function $f(r)$ for any concentric spherical family irrespective of the position of the center; f is supposed to be of class C'' in r and $f'(0)=0$. Then (84) yields

$$\nabla^2 u = \left(\nabla^2 F(r)\Big|_{r=0}\right) u, \qquad F(r) = f(r)/f(0)$$

or

$$\nabla^2 u = [nf''(0)/f(0)]u.$$

In particular, if $f''(0)=0$, u is harmonic.

We close this section by considering the interesting question as to whether there exists a theorem similar to the one just proved but converse to those results of §3 for which the averages $A(u)$ are taken over spheres lying in a spherical shell $R_{a,b}$ *and enclosing the inner sphere*, so that $A(u)$ is linearly dependent on functions $f_i(r)$, not all of which are regular at $r=0$. Thus, in the plane, if u is harmonic in a ring $R_{a,b}$, the average over concentric circles of radius r and enclosing the inner boundary $r=a$ is given by $A \log r+B$, where A and B are constants whose value depends upon the position of the center. Now suppose conversely, that u is of class C'' in $R_{a,b}$ and that the average for circles of above description is given by $A \log r+B$; could one infer that u is harmonic? That such need *not* be the case can be seen from the following example.

Let $u=x_1 \log (x_1^2+x_2^2)$, so that $\nabla^2 u=4x_1/(x_1^2+x_2^2)$, $\nabla^4 u=0$; u is biharmonic but not harmonic. It will be shown that in spite of u not being harmonic, $A(u)$ is of the form $A+B \log r$ for any family of circles enclosing the origin.

Since $\nabla^4 u=0$, $A(u)=AV_1(r)+BV_2(r)+C+Dr^2$, where A, B, C, D, are constants depending upon the position of the center. Holding the latter fixed, applying ∇^2 to $A(u)$, and replacing $\nabla^2 A(u)$ by $A(\nabla^2 u)$, there results

$$A(\nabla^2 u) = BV_1(r) + 4D = B \log r + 4D.$$

Letting r become infinite, it follows that $B=D=0$, since $\nabla^2 u$ vanishes at infinity.

8. **A generalization of** (17). In this section is established the following generalization of the law of the mean (17):

$$(85) \qquad A(uP_k) = \sum_{i=0}^{[k/2]} \frac{2^{k-2i}}{i!} \left\{ \frac{\partial^{k-i}\phi_n(r, \lambda)}{\partial \lambda^{k-i}} \bigg|_{\lambda=\nabla^2} (\nabla^{2i} P_k) \left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n} \right) \right\} u_0.$$

Here u is analytic at the origin, which point will be supposed to be the center of the spherical family implied in the averaging operation A; P_k is a homogeneous polynomial of degree k; the indicated term on the right-hand side is obtained by replacing λ by ∇^2 in the λ-power expansion of $\partial^{k-i}\phi/\partial\lambda^{k-i}$, multiplying the resulting series termwise by $\nabla^{2i}P_k$ in which $x_1, x_2, \cdots$ have been replaced by $\partial/\partial x_1, \partial/\partial x_2, \cdots$, multiplying the result by u_0, then interpreting each term by applying the indicated differentiation to u at the origin; $[x]$, as usual, denotes "the greatest integer in x."

For the case

$$P_k = H_k,$$

where H_k is harmonic, the above reduces to

$$(86) \qquad A(uH_k) = 2^k \left\{ \frac{\partial^k \phi_n(r, \lambda)}{\partial \lambda^k} \bigg|_{\lambda=\nabla^2} H_k \left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n} \right) \right\} u_0,$$

and introducing Bessel functions by means of (10),

$$(87) \qquad \begin{aligned} A(uH_k) = \Gamma\left(\frac{n}{2}\right) r^{k+1-n/2} \{ &(\lambda^{1/2})^{1-k-n/2} 2^{(n/2-1)} \\ &\cdot I_{n/2+k-1}(r\lambda^{1/2}) H_k(\partial/\partial x_1, \cdots) \} \Big|_{\lambda=\nabla^2} u_0. \end{aligned}$$

To prove the above, consider as in §4, the special case

$$u = \exp\,[c_1x_1 + c_2x_2 + \cdots + c_nx_n]u_0,$$

putting again

$$\lambda = c_1^2 + \cdots + c_n^2.$$

Since

$$\begin{aligned} &\nabla^2(P_k \exp\,[c_1x_1 + c_2x_2 + \cdots + c_nx_n]) \\ &\qquad = \exp\,[c_1x_1 + \cdots\,][\lambda^2 P_k + 2(c_1\partial/\partial x_1 + \cdots + c_n\partial/\partial x_n)P_k + \lambda P_k], \end{aligned}$$

it follows that

$$(\nabla^2 - \lambda)(P_k \exp [c_1x_1 + \cdots])$$
$$= \exp [c_1x_1 + \cdots][\nabla^2 + 2(c_1\partial/\partial x_1 + \cdots)]P_k$$

and hence by induction

$$(\nabla^2 - \lambda)^i(P_k \exp [c_1x_1 + \cdots])$$
$$= \exp [c_1x_1 + \cdots][\nabla^2 + 2(c_1\partial/\partial x_1 + \cdots)]^iP_k.$$

In particular

$$(\nabla^2 - \lambda)^{k+1}(P_k \exp [c_1x_1 + \cdots]) = 0.$$

Hence, the mean of $P_k\exp[c_1x_1+ \cdots]$ may be evaluated by means of (9). The last term of the right-hand sum of (9) reduces to

$$\frac{\partial^k\phi}{\partial\lambda^k}\frac{1}{k!}2^k(c_1\partial/\partial x_1 + \cdots)^kP_k\Big|_0 = \frac{\partial^k\phi}{\partial\lambda^k}2^kP_k(c_1, \cdots , c_n)$$

since P_k is a homogeneous polynomial of degree k. A similar reduction in the other terms results in

$$A(P_k \exp [c_1x_1 + \cdots]) = \sum_{i=0}^{[k/2]}\frac{2^{k-2i}}{i!}\frac{\partial^{k-i}\phi_n}{\partial\lambda^{k-i}}(\nabla^{2i}P_k)(c_1, \cdots , c_n)u_0,$$

where $[x]$ denotes the greatest integer not exceeding x, and $(\nabla^{2i}P_k)(c_1, \cdots, c_n)$ is the result of replacing the x's by the c's in the $(k-2i)$th degree polynomial $\nabla^{2i}P_k$.

Having thus established (85) for an exponential function, the case of a general analytic function is now deduced from the above, as in (24), by means of the formal representation of the Taylor series by means of an exponential.

The result (86) (for $n=3$) is due essentially to Hobson.* It will be utilized in the following section in connection with obtaining analogues of the results thus far obtained for means and the functional operator A, to other functional operators considered in I.

An interesting application of (85), (86) consists in examining the order of vanishing with r of the means $A(uH_k)$, $A(uP_k)$. It is found that

$$(88)\qquad \begin{cases} A(uH_k) = O(r^{2k}), \\ A(uP_k) = O[r^{(k,k+1)}] \text{ according as } k \text{ is (even, odd)}. \end{cases}$$

More precisely,

* Hobson, Proceedings of the London Mathematical Society, vol. 24 (1892–1893), p. 80.

$$(89)\quad \lim_{r\to 0}\frac{A(uH_k)}{r^{2k}}=\frac{2^k}{n(n+2)\cdots(n+2k-2)}H_k\left(\frac{\partial}{\partial x_1},\cdots,\frac{\partial}{\partial x_n}\right)u_0,$$

$$(90)\quad \begin{cases} \lim\limits_{r\to 0}\dfrac{A(uP_k)}{r^k}=\dfrac{u_0(\nabla^2)^{k/2}P_k}{2\cdot 4\cdots k\cdot n(n+2)\cdots(n+k-2)} & \text{for even } k,\\[2ex] \lim\limits_{r\to 0}\dfrac{A(uP_k)}{r^{k+1}}=\dfrac{(\nabla^2)^{(k-1)/2}P_k\left(\dfrac{\partial}{\partial x_1},\cdots,\dfrac{\partial}{\partial x_n}\right)u_0}{2\cdot 4\cdots k\cdot n(n+2)\cdots(n+k-1)} & \text{for odd } k.\end{cases}$$

9. **Extension to other operators.** Let L be any linear functional operator which is permutable with ∇^2. If u is a solution of (1), then $L(u)$ likewise satisfies (1). This follows (as in §3 for the case $L=A$) by applying L to both sides of (1) and permuting L with ∇^2:

$$L(\nabla^2 u)-\lambda L(u)=0=\nabla^2(L(u))-\lambda L(u).$$

Similarly, it is shown that $L(u)$ satisfies (8), (11), or (13) if u does and if L is permutable with ∇^2, ∇^4, $\cdots$. In many cases this leads to results as definite as the laws of the mean proved above (properly speaking, however, the results are not covered by the title of this paper).

Consider first the operator L_k (see I, (5)):

$$(91)\quad L_k(u)=h_k(\omega)\int h_k'(\omega')u(r,\omega')d\omega'=h_k(\omega)I(r).$$

Since (see I, (7))

$$\nabla^2[h_kI(r)]=h_k\left[\frac{d^2}{dr^2}+\frac{n-1}{r}\frac{d}{dr}+\frac{k(2-n-k)}{r^2}\right]I,$$

it follows that if u satisfies (1), $I(r)$ will satisfy the ordinary differential equation obtained from (1) by replacing ∇^2 by

$$(92)\quad \frac{d^2}{dr^2}+\frac{n-1}{r}\frac{d}{dr}+\frac{k(2-n-k)}{r^2};$$

similarly for solutions of (8), (11), (13). Thus in case of solutions of (8), $I(r)$ satisfies

$$(93)\quad \left[\frac{d^2}{dr^2}+\frac{n-1}{r}\frac{d}{dr}+\frac{k(2-k-n)}{r^2}-\lambda\right]^p I=0.$$

Proceeding in a manner similar to that employed in §2 for the case $k=0$, the following alternative forms for solutions of (93) may be obtained

$$
\left.\begin{aligned}
&y^{-n/2+i+1}B_{n/2+k+i-1}(y), \qquad y = (-\lambda)^{1/2}r,\\
&r^{k+2i}(\phi, \psi)_{n+2k+2i}(r, \lambda),\\
&r^{k}\frac{\partial^{i}}{\partial\lambda^{i}}(\phi, \psi)_{n+2k}(r, \lambda),
\end{aligned}\right\} \qquad i = 0, \cdots, p-1. \tag{94}
$$

Here $B_s(x)$ denotes a Bessel function of order s and argument x. All the three forms (91) apply for $\lambda \neq 0$; the two latter also for $\lambda = 0$. For $\lambda = 0$ the above yield the p-harmonic polynomials

$$
H_k,\ H_k r^2, \cdots, H_k r^{2p-2},^{*} \tag{95}
$$

while from the last form (94) are obtained the p-harmonic functions

$$
F r^{-n-k+2+2i} h_k; \qquad i = 0, \cdots, p-1, \tag{96}
$$

where

$$
F = \begin{cases} \ln r \text{ if the power of } r \text{ is even and non-negative,}\\ 1 \text{ otherwise.}\end{cases} \tag{97}
$$

The general case (13) can now be handled by factorization (see (15), (15′)).

The operation L_k, or more explicitly, its component

$$
I(r) = \int h_k'(\omega')u(r, \omega')d\omega',
$$

differs only by a factor K_n/r^k from the operation $A(uH_k')$, $H_k' = h_k r^k$, considered in the preceding section.

Combining the results indicated in this section with (86) one obtains for analytic solutions of (13)

$$
\begin{vmatrix}
\dfrac{I(r)n(n+2)\cdots(n+2k-2)}{K_n r^k} & \phi_{n+2k}(r, \lambda_1) & \cdots & \phi_{n+2k}(r, \lambda_p)\\
H_k'\left(\dfrac{\partial}{\partial x_1}, \cdots, \dfrac{\partial}{\partial x_n}\right)u_0 & 1 & \cdots & 1\\
H_k'\left(\dfrac{\partial}{\partial x_1}, \cdots, \dfrac{\partial}{\partial x_n}\right)(\nabla^2 u)_0 & \lambda_1 & \cdots & \lambda_p\\
\cdots & \cdots & \cdots & \cdots\\
H_k'\left(\dfrac{\partial}{\partial x_1}, \cdots, \dfrac{\partial}{\partial x_n}\right)(\nabla^{2p-2} u)_0 & \lambda_1^{p-1} & \cdots & \lambda_p^{p-1}
\end{vmatrix} = 0 \tag{98}
$$

* It is known that such products of harmonic polynomials by powers of r^2 suffice to yield a complete set of p-harmonic polynomials of any degree. See Almansi, *Sull'integrazione*, etc., Annali di Matematica, (3), vol. 2 (1899), §3.

provided that the factorization of the left-hand member of (13) has no repeated roots. For $k=0$ this reduces to (14).

Turning to other operators discussed in I, consider the operators Λ_k (see I, §7), which generalize L_k to non-integer k:

$$\Lambda_k(u) = h_k(\Omega) \int h_k'(\Omega')u(r, \Omega')d\Omega'; \tag{99}$$

here h_k, h_k' are solutions of I, (12):

$$\Delta_2 h = -k(k+n-2)h \tag{100}$$

along (Riemann surfaces spread over) the unit sphere. It is found that for solutions of (8) the differential equation (93) still applies to $I(r)$, but of course, with the proper value of k, leading to Bessel function solutions as displayed in the first form (94), but of non-integer order.

Consider next the operators $\overset{*}{L}_{k,n-1}$ (see I, §6):

$$\overset{*}{L}_{k,n-1}(u) = h_k \int_{-\infty}^{+\infty} \cdots \int h_k' u(x_1', \cdots, x_{n-1}', x_n)dx_1' \cdots dx_{n-1}' = h_k I(x_n), \tag{101}$$

where h_k, h_k' are functions of $x_1, x_2, \cdots, x_{n-1}$ satisfying the differential equation

$$\left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_{n-1}^2}\right) h = kh. \tag{102}$$

When $\overset{*}{L}_{k,n-1}$ are applied under the conditions of I, Theorem IV to solutions of (13), it is found that $I(x_n)$ satisfies the equation obtained from (13) by replacing ∇^2 by

$$\partial^2/\partial x_n^2 + k. \tag{103}$$

Consider finally non-Euclidean spaces N_n and operators permutable with Δ_2, the second invariant differential operator of Beltrami. To them belong the operators L_k, L_k^*, L_k^{**} of I, §8, applied respectively over spheres, horospheres, and equidistant surfaces, the two latter in the Lobatchevsky space. Here L_k is given by (91), where r, ω are spherical coordinates, and h_k, h_k' satisfy the equation (100) over a Euclidean unit sphere. Applied to a solution of

$$\Delta_2 u - \lambda u = 0, \tag{104}$$

the operator L_k leads to a function I satisfying the equation

$$\frac{d^2I}{dr^2} + c(n-1)\cot cr\,\frac{dI}{dr} - \left(\frac{r^2k(k+n-2)}{\sin^2 cr} + \frac{\lambda}{c^2}\right)I = 0. \tag{105}$$

The operator L_k^* is applied over equidistant horospheres $\xi = \text{const.}$ with the element of length

$$ds^2 = e^{2c\xi}(dx_1^2 + \cdots + dx_{n-1}^2) + d\xi^2; \tag{106}$$

it is given by (101) with x_n replaced by ξ, where h, h' satisfy (100). Now, a solution of (104) leads to I satisfying

$$\frac{d^2I}{d\xi^2} + (n-1)ce^{-c\xi}\frac{dI}{d\xi} + e^{-2c\xi}kI = 0. \tag{107}$$

The operator L_k^{**} leads to (105) but with rc replaced by $\pi/2 - r/c$.

Let us map the non-Euclidean space N_n on a Euclidean sphere in E_{n+1} of radius $R = 1/c$ and define a function v in E_{n+1} thus:

$$v = R^q u. \tag{108}$$

If u satisfies (104), then

$$\begin{aligned} \nabla^2 v &= \left(\frac{\partial^2}{\partial R^2} + \frac{n}{R}\frac{\partial}{\partial R} + \frac{\Delta_2}{R^2}\right)v \\ &= [q(q+n-1) + \lambda]v/R^2. \end{aligned}$$

Hence if q is chosen as a root of

$$q(q+n-1) + \lambda = 0, \tag{109}$$

then v is harmonic in E_{n+1}. This relation will be utilized in the paper *On integral representations* etc. cited in §1.

In concluding we recall briefly the extensions of some of the above results to certain discontinuous functions. Consider, for instance, for $n=3$ the function $1/r_1$, where r_1 is the distance from a fixed point P_1. The operations A or L_k applied over spheres with center at P yield harmonic results to each side of the sphere S_1 passing through P_1. From the interpretation of $L_k(u)$ in terms of the distribution of the unit mass at P_1 over S_1 (see I, §9) follows that $I(r)$ is continuous at $r = PP_1$, but that its derivative is discontinuous there by the amount $[h_k'(\omega_p'))$, the value of h_k' at $P_1]/4\pi$. A convenient way of proving this and other similar results is by spreading out the concentrated (point) mass over a finite region, thus representing $1/r$ as the limit of a function with a continuous Laplacian.

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