

# EQUIVALENCE OF PAIRS OF MATRICES\*

BY

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1. **Introduction.** Two pairs of matrices,  $[A_1, A_2]$  and  $[B_1, B_2]$ , with elements in a commutative field  $F$ , are said to be *equivalent*† if and only if there exist two non-singular matrices  $P$  and  $Q$ , with elements in  $F$ , such that  $A_1 = PB_1Q$  and  $A_2 = PB_2Q$ .

The totality of pairs of matrices may be separated into different classes in such a way that all pairs in one class are equivalent to one another while pairs in different classes are not equivalent. The problem which arises naturally is to determine a set of invariants which will characterize the pairs in each class and to select from each class a *canonical* pair defined uniquely in terms of these invariants. A *rational* solution of the problem is one which is carried out completely in the field  $F$ ; the invariants and canonical pairs obtained in such a solution will be *rational*.

The *rank* of a pair  $[A_1, A_2]$  is the maximum rank of the matrices of the *matrix pencil*  $A = A_1x_1 + A_2x_2$ , where  $x_1$  and  $x_2$  are indeterminates in  $F$ . A matrix pencil is said to be *non-singular* if it is square and of rank equal to its order (otherwise it is called *singular*), and it is said to be *regular* if the rank of either one of its coefficients is the same as the rank of the pencil.

Non-singular matrix pencils were first classified by Weierstrass‡ who constructed an irrational canonical form defined by means of the elementary divisors of the pencil. Frobenius§ later gave a rational treatment of the non-singular case. Kronecker|| treated the singular case and gave an irrational canonical form. Muth¶ gave a full account of the theory of pairs of bilinear forms as it stood at the turn of the century. De Séguier\*\* seems to have been the first to give a rational treatment of the singular case. More recently, it has received the attention of Dickson,†† Turnbull and Aitken,‡‡ Wedderburn,§§

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\* Presented to the Society, March 27, 1937; received by the editors May 25, 1937.

† C. C. MacDuffee, *The Theory of Matrices*, Berlin, 1933, p. 48.

‡ K. Weierstrass, Monatsberichte, Preussische Akademie der Wissenschaften, 1868, pp. 310–338.

§ G. Frobenius, Sitzungsberichte, Preussische Akademie der Wissenschaften, 1894, pp. 31–44.

|| L. Kronecker, Sitzungsberichte, Preussische Akademie der Wissenschaften, 1890, pp. 1225–1237.

¶ P. Muth, *Theorie und Anwendung der Elementarteiler*, Leipzig, 1899.

\*\* J. A. de Séguier, Bulletin de la Société Mathématique de France, vol. 36 (1908), pp. 20–40.

†† L. E. Dickson, these Transactions, vol. 29 (1927), pp. 239–253.

‡‡ Turnbull and Aitken, *Canonical Matrices*, Glasgow, 1932, chap. 9.

§§ J. H. M. Wedderburn, *Lectures on Matrices*, American Mathematical Society Colloquium Publications, vol. 17, 1934, chap. 4.

Turnbull,\* Ledermann,† Williamson,‡ and others.

In this paper the problem of constructing a rational canonical form in the singular case is reduced to the consideration of the non-singular case. The proofs are completely rational, quite elementary, and relatively short. The canonical form which is obtained is defined essentially in terms of the set of invariants shown by Williamson‡ to characterize the classes of equivalent matrices. The method of proof is very similar to that used by Ingraham§ in his treatment of the equivalence of singular pencils of Hermitian matrices.

**2. Preliminary remarks.** Consider a singular pencil  $A = A_1x_1 + A_2x_2$  of rank  $\rho(A) = r$  and order  $[\theta, \theta']$ . Set  $R_1 = A_1t_{11} + A_2t_{21}$  and  $R_2 = A_1t_{12} + A_2t_{22}$ , where the  $t_{ij}$  are quantities of  $F$  such that  $|t_{ij}| \neq 0$ . If  $R = R_1x_1 + R_2x_2$  and  $T = \|t_{ij}\|$ , then the relations above may be written  $R = AT$ , and the pencils  $A$  and  $R$  are said to be *transformable*. If  $B$  is a second matrix pencil, it follows easily that  $A$  is equivalent to  $B$  ( $A \sim B$ ) if and only if  $AT \sim BT$ . In particular, there exist two quantities  $t_{11}^0$  and  $t_{21}^0$  of  $F$  not both zero and such that  $\rho(A_1t_{11}^0 + A_2t_{21}^0) = r$ , and in this case the pencil  $R = AT$  is said to be regular.

If it is desired only to obtain necessary and sufficient conditions for the equivalence of two pencils, then there is no loss of generality in considering only regular pencils. However, if a canonical form in the most strict sense is required, it is necessary to start with the original pencils rather than their regular transforms, as has been pointed out by Ledermann.¶ Canonical forms will be constructed only for regular pencils, but the invariants used will be shown to afford a satisfactory classification for all pencils. It is felt that this solves the important part of the problem.

**3. Rational canonical form for regular matrix pencils.** Constant non-singular matrices  $P$  and  $Q$  exist¶¶ such that

$$PR_1Q = e = \begin{vmatrix} 1_r & 0 \\ 0 & 0 \end{vmatrix};$$

hence  $R \sim ex_1 + PR_2Qx_2 = ex_1 + ax_2 = R_0$ . If we set

$$R_0 = \begin{vmatrix} 1_r & 0 \\ 0 & 0 \end{vmatrix} x_1 + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} x_2,$$

\* H. W. Turnbull, Proceedings of the Edinburgh Mathematical Society, (2), vol. 4 (1935) pp. 67-76.

† W. Ledermann, *ibid.*, (2), vol. 4 (1935), pp. 92-105.

‡ J. Williamson, *ibid.*, (2), vol. 4 (1936), pp. 224-231.

§ Ingraham and Wegner, these Transactions, vol. 38 (1935), pp. 145-162.

¶ Loc. cit.

¶¶ MacDuffee, loc. cit., p. 43.

it follows immediately that  $a_{22}=0$ , for otherwise the rank of  $R_0$  would be greater than  $r$ , which is impossible since  $R_0$  has been assumed to be regular.

Since  $1_r x_1 + a_{11} x_2$  is non-singular, the rows of  $(a_{21} \ 0)$  must be linearly dependent on the rows of  $(1_r x + a_{11} \ a_{12})$ ; thus there exists a matrix  $X_{21}$  such that  $X_{21}(1_r x + a_{11}) = a_{21}$  and  $X_{21} a_{12} = 0$ . Necessarily then,  $X_{21} = a_{21}(1_r x + a_{11})^{-1}$ , and  $a_{21}(1_r x + a_{11})^{-1} a_{12} = 0$ . For  $x$  sufficiently large

$$(1_r x + a_{11})^{-1} = \frac{1}{x} \sum_{k=0}^{\infty} \left( \frac{-a_{11}}{x} \right)^k;$$

hence  $a_{21} a_{11}^k a_{12} = 0$  for  $k=0, 1, 2, \dots$ .

Conversely, if  $a_{21} a_{11}^k a_{12} = 0$  for  $k=0, 1, 2, \dots$ , then  $a_{21}(1_r x + a_{11})^{-1} a_{12} = 0$ . Hence, if  $X_{21} = a_{21}(1_r x + a_{11})^{-1}$ , it follows that  $X_{21}(1_r x + a_{11}) = a_{21}$  and  $X_{21} a_{12} = 0$ , so that the last  $\theta - r$  rows of  $ex + a$  are dependent on the first  $r$  rows. This proves the following lemma:

**LEMMA A.** *The rank of the matrix pencil*

$$\left\| \begin{array}{cc} 1_r & 0 \\ 0 & 0 \end{array} \right\| x_1 + \left\| \begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right\| x_2$$

is  $r$  if and only if  $a_{22} = 0$  and  $a_{21} a_{11}^k a_{12} = 0$  for  $k=0, 1, 2, \dots$ .

Lemma A holds true only if the coefficient field  $F$  is commutative and has characteristic zero. For a field of characteristic  $p \neq 0$  it is necessary to alter the treatment slightly. The present author has shown (*Annals of Mathematics*, (2), vol. 36 (1935), p. 865) that the matrix pencil of Lemma A is equivalent to a pencil

$$\left\| \begin{array}{ccc} 1_r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right\| x_1 + \left\| \begin{array}{ccc} b_{11} & 0 & b_{13} \\ 0 & 1_s & 0 \\ b_{31} & 0 & 0 \end{array} \right\| x_2,$$

where  $s = \rho(a_{22})$  and  $b_{31} b_{11}^k b_{13} = 0$  for  $k=0, 1, 2, \dots$ . It follows easily that there is no loss of generality in considering pencils satisfying the conditions of Lemma A; and in this way the proofs given in this paper may be extended to include pencils with coefficients in an arbitrary field. This more general method of proof has been used by the present author in a recent paper (*Strict equivalence of matrix pencils*, presented to the Society December 29, 1937, but not yet published) treating the problem of equivalence of matrix pencils, singular and non-singular.

We now proceed with the construction of a canonical form for the regular pencil  $S = ex + a$ . If  $\alpha$  and  $\beta$  are non-singular matrices of orders  $\theta$  and  $\theta'$  with elements in  $F$ , then  $\alpha\beta = e$  if

$$\alpha = \left\| \begin{array}{cc} \alpha_{11} & \alpha_{12} \\ 0 & \alpha_{22} \end{array} \right\| \quad \text{and} \quad \beta = \left\| \begin{array}{cc} \alpha_{11}^{-1} & 0 \\ \beta_{21} & \beta_{22} \end{array} \right\|,$$

where  $\alpha_{22}$  and  $\beta_{22}$  are non-singular, and so

$$\alpha\beta = \left\| \begin{array}{cc} \alpha_{11}a_{11}\alpha_{11}^{-1} + \alpha_{12}a_{21}\alpha_{11}^{-1} + \alpha_{11}a_{12}\beta_{21} & \alpha_{11}a_{12}\beta_{22} \\ \alpha_{22}a_{21}\alpha_{11}^{-1} & 0 \end{array} \right\|.$$

Now if the rank of  $a_{21}$  is  $r_1$ ,  $\alpha_{22}$  and  $\alpha_{11}$  may be chosen so that

$$\alpha_{22}a_{21}\alpha_{11}^{-1} = \left\| \begin{array}{cc} 1_{r_1} & 0 \\ 0 & 0 \end{array} \right\|,$$

and it follows from Lemma A that the first  $r_1$  rows of  $\alpha_{11}a_{12}\beta_{22}$  must be zero. Furthermore, if the rank of  $a_{12}$  is  $c_1$ , it is clear that  $\beta_{22}$  may be chosen so that

$$\alpha_{11}a_{12}\beta_{22} = \left\| \begin{array}{cc} 0 & 0 \\ 0 & 1_{c_1} \end{array} \right\|.$$

With this choice of  $\alpha_{ij}$  and  $\beta_{kk}$  the pencil  $T = \alpha S \beta$  takes the form

$$T = \left\| \begin{array}{cccccc} 1_{r_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1_h & 0 & 0 & 0 & 0 \\ 0 & 0 & 1_{c_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0_{r_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0_{jk} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0_{c_1} \end{array} \right\| x + \left\| \begin{array}{cccccc} x_1 & a_1 & a_4 & 0 & 0 & 0 \\ x_2 & a_2 & a_3 & 0 & 0 & 0 \\ x_3 & x_4 & x_5 & 0 & 0 & 1_{c_1} \\ 1_{r_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right\|,$$

where  $h = r - r_1 - c_1$ ,  $j = \theta - r - r_1 - c_1$ , and  $k = \theta' - r - r_1 - c_1$ . Finally  $\alpha_{12}$  and  $\beta_{21}$  may clearly be chosen so that the  $x_i$  are all zero; then

$$T = \left\| \begin{array}{cccccc} x & a_1 & a_4 & 0 & 0 & 0 \\ 0 & x + a_2 & a_3 & 0 & 0 & 0 \\ 0 & 0 & x & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right\|.$$

The rank of  $T$  is  $r$ , since  $\alpha$  and  $\beta$  were chosen non-singular, therefore

$$\left\| \begin{array}{cc} a_1 & a_4 \\ x + a_2 & a_3 \end{array} \right\|$$

is regular and of rank  $r - r_1 - c_1$ . Hence, by Lemma A,  $a_4 = 0$  and  $a_1 a_2^k a_3 = 0$  for  $k = 0, 1, 2, \dots$ .

Consider a second regular pencil  $U = U_1 x_1 + U_2 x_2$  which is equivalent to  $R$ . The rank of  $U_1$  is necessarily  $r$ , therefore, constant non-singular matrices  $P_0$  and  $Q_0$  exist such that  $P_0 U_1 Q_0 = e$ . Then if

$$V = P_0 U Q_0 = e x_1 + b x_2 = e x_1 + \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} x_2,$$

it follows from Lemma A that  $b_{22} = 0$ . Since  $V \sim S$  there exist constant non-singular matrices  $x$  and  $y$  such that  $xV = Sy$ , which equation is equivalent to the relations  $xe = ey$  and  $xb = ay$ . From  $xe = ey$  it follows that  $x_{11} = y_{11}$ ,  $x_{21} = 0$ ,  $y_{12} = 0$ , and that  $x_{11}$ ,  $x_{22}$ , and  $y_{22}$  are non-singular. Then from  $xb = ay$  it follows that  $x_{22} b_{21} = a_{21} x_{11}$  and  $x_{11} b_{12} = a_{12} y_{22}$ . Since  $x_{22}$ ,  $x_{11}$ , and  $y_{22}$  are necessarily non-singular this shows that the ranks of  $b_{21}$  and  $b_{12}$  are the same as the ranks  $r_1$  and  $c_1$  of  $a_{21}$  and  $a_{12}$ .  $r_1$  will be called the first "row invariant subrank" and  $c_1$  the first "column invariant subrank" of  $R$  or of any pencil equivalent to  $R$ . It follows that constant non-singular matrices  $\alpha_0$  and  $\beta_0$  can be chosen so that  $W = \alpha_0 V \beta_0$  takes a form analogous to that of  $T$  but with  $a_k$  replaced by  $b_k$ .

The pencils  $U$  and  $R$  are equivalent if and only if there exist constant non-singular matrices  $p$  and  $q$  such that  $pT = Wq$ . From  $pe = eq$  it is clear that  $p$  and  $q$  must be of the forms

$$p = \begin{vmatrix} z & * \\ 0 & * \end{vmatrix} \quad \text{and} \quad q = \begin{vmatrix} z & 0 \\ * & * \end{vmatrix},$$

hence  $pT = Wq$  may be replaced by the equation

$$\begin{vmatrix} p_{11} & p_{12} & p_{13} & p_{14} & p_{15} & p_{16} \\ p_{21} & p_{22} & p_{23} & p_{24} & p_{25} & p_{26} \\ p_{31} & p_{32} & p_{33} & p_{34} & p_{35} & p_{36} \\ 0 & 0 & 0 & p_{44} & p_{45} & p_{46} \\ 0 & 0 & 0 & p_{64} & p_{65} & p_{66} \\ 0 & 0 & 0 & p_{64} & p_{65} & p_{66} \end{vmatrix} \begin{vmatrix} 0 & a_1 & 0 & 0 & 0 & 0 \\ 0 & a_2 & a_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 0 & b_1 & 0 & 0 & 0 & 0 \\ 0 & b_2 & b_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{vmatrix} \begin{vmatrix} p_{11} & p_{12} & p_{13} & 0 & 0 & 0 \\ p_{21} & p_{22} & p_{23} & 0 & 0 & 0 \\ p_{31} & p_{32} & p_{33} & 0 & 0 & 0 \\ q_{41} & q_{42} & q_{43} & q_{44} & q_{45} & q_{46} \\ q_{61} & q_{62} & q_{63} & q_{64} & q_{65} & q_{66} \\ q_{61} & q_{62} & q_{63} & q_{64} & q_{65} & q_{66} \end{vmatrix}.$$

This equation is equivalent to the set of relations:

$$\begin{aligned} p_{54} &= p_{64} = p_{12} = p_{13} = q_{64} = q_{65} = p_{23} = 0, \\ p_{11} &= p_{44}, \quad p_{33} = q_{66}, \quad p_{34} = q_{61}, \\ p_{14} &= b_1 p_{21}, \quad p_{32} a_3 = q_{63}, \quad p_{12} a_3 = b_1 p_{23}, \\ p_{24} &= b_2 p_{21} + b_3 p_{31}, \quad p_{11} a_1 + p_{12} a_2 = b_1 p_{22}, \\ p_{31} a_1 + p_{32} a_2 &= q_{62}, \quad p_{22} a_3 = b_2 p_{23} + b_3 p_{33}, \\ p_{21} a_1 + p_{22} a_2 &= b_2 p_{22} + b_3 p_{32}. \end{aligned}$$

It follows easily that these equations have a solution such that  $p$  is non-singular if and only if there exist constant non-singular matrices  $p_{11}$ ,  $p_{22}$ , and  $p_{33}$  which satisfy the relations:

$$\begin{aligned} p_{11}a_1 &= b_1p_{22}, & p_{22}a_3 &= b_3p_{33}, \\ p_{21}a_1 + p_{22}a_2 &= b_2p_{22} + b_3p_{32}. \end{aligned}$$

These equations may be rewritten in the form

$$\left\| \begin{array}{cc} p_{22} & p_{21} \\ 0 & p_{11} \end{array} \right\| \left\| \begin{array}{cc} a_2 & a_3 \\ a_1 & 0 \end{array} \right\| = \left\| \begin{array}{cc} b_2 & b_3 \\ b_1 & 0 \end{array} \right\| \left\| \begin{array}{cc} p_{22} & 0 \\ p_{32} & p_{33} \end{array} \right\|,$$

and this is simply the condition for the equivalence of the two regular pencils

$$T^1 = \left\| \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right\| x + \left\| \begin{array}{cc} a_2 & a_3 \\ a_1 & 0 \end{array} \right\|, \quad W^1 = \left\| \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right\| x + \left\| \begin{array}{cc} b_2 & b_3 \\ b_1 & 0 \end{array} \right\|.$$

The pencils  $T^1$  and  $W^1$  will be called "first kernels" of the pencils  $R$  and  $U$ . Thus the problem of classifying singular pencils of rank  $r$  has been reduced to that of classifying singular pencils of rank  $r-r_1-c_1$ , or else to that of classifying non-singular pencils if  $a_1$  and  $a_3$  happen to be zero.

If  $r_{j+1}$  and  $c_{j+1}$  are the first invariant subranks of  $T^j$ , and  $T^{j+1}$  is a first kernel of  $T^j$  for  $j=1, 2, 3, \dots, n$ , and if  $T^{n+1}$  is non-singular or zero; then  $r_j$  and  $c_j$  for  $j=1, 2, \dots, n+1$  will be called the "invariant subranks" of  $R$ , and  $T^{n+1}$  a "kernel" of  $R$ . This proves the following:

**THEOREM 1.** *Two regular matrix pencils are equivalent if and only if they have identical sets of invariant subranks and equivalent kernels.*

It is clear that the construction which leads to  $T$  can be extended until a rational canonical form for  $R$  is obtained. This canonical form would display the invariant subranks and invariant factors of  $R$ . The invariant factors of  $R$  are clearly the same as those of any kernel of  $R$  except for  $\sum_{j=1}^{n+1}(r_j+c_j)$  units which would appear in the normal form of  $R$  but not in the normal form of any kernel of  $R$ . This demonstrates the corollary:

**COROLLARY 1.1.** *Two regular matrix pencils are equivalent if and only if they have the same invariant subranks and the same elementary divisors.*

**4. Transformable matrix pencils.** Let  $A = A_1x_1 + A_2x_2$  be an arbitrary matrix pencil, and define matrices  $M_k(A)$  and  $N_k(A)$ , for  $k=1, 2, 3, \dots$ , by the relations

$$M_1(A) = \left\| \begin{array}{cc} A_1 & A_2 \end{array} \right\|, \quad M_2(A) = \left\| \begin{array}{ccc} A_1 & A_2 & 0 \\ 0 & A_1 & A_2 \end{array} \right\|, \dots,$$

$$N_1(A) = \left\| \begin{matrix} A_1 \\ A_2 \end{matrix} \right\|, \quad N_2(A) = \left\| \begin{matrix} A_1 & 0 \\ A_2 & A_1 \\ 0 & A_2 \end{matrix} \right\|, \dots$$

It is convenient to denote by  $m_k(A)$  and  $n_k(A)$  the ranks of  $M_k(A)$  and  $N_k(A)$  and to call  $m_k(A)$  the "row singularities" and  $n_k(A)$  the "column singularities" of  $A$ .

It is obvious that equivalent matric pencils have the same row and column singularities. We now proceed to prove the following:

**THEOREM 2.** *Transformable matric pencils have the same singularities.*

**Proof.\*** Consider a matric pencil  $A = A_1x_1 + A_2x_2$  and a non-singular transformation of indeterminates  $x = tx'$ , or more explicitly

$$x_1 = t_{11}x'_1 + t_{12}x'_2, \quad x_2 = t_{21}x'_1 + t_{22}x'_2.$$

Under this transformation, the pencil  $A$  is carried into the pencil

$$A' = At = (t_{11}A_1 + t_{21}A_2)x'_1 + (t_{12}A_1 + t_{22}A_2)x'_2 = A'_1x'_1 + A'_2x'_2,$$

and the theorem states that  $m_k(A) = m_k(A')$  and  $n_k(A) = n_k(A')$ . The first of these equalities will be demonstrated by constructing non-singular matrices  $T^k$  such that

$$M_k(A)T^k = T^{k-1}M_k(A') \quad \text{for } k = 1, 2, 3, \dots,$$

and the second can be shown by a similar construction.

If  $u_0, u_1, u_2, \dots, u_k$  are  $k+1$  indeterminates, then the identity

$$(1) \quad \begin{aligned} &u_0x_1^k + C_{k,1}u_1x_1^{k-1}x_2 + C_{k,2}u_2x_1^{k-2}x_2^2 + \dots + u_kx_2^k \\ &\equiv u'_0x_1'^k + C_{k,1}u'_1x_1'^{k-1}x_2' + C_{k,2}u'_2x_1'^{k-2}x_2'^2 + \dots + u'_kx_2'^k \end{aligned}$$

defines  $u'_0, u'_1, \dots, u'_k$  as linear combinations of  $u_0, u_1, \dots, u_k$ , and these may be written in either of the forms

$$u'_i = \sum_{j=0}^k T_{ij}^k u_j \quad \text{or} \quad u' = T^k u.$$

Now if  $T^k = \|T_{ij}^k\|$ , then  $t \rightarrow T^k$  is a representation of the full linear group of all non-singular matrices of order two, and hence  $T^k$  is non-singular since  $t$  is.

If (1) is differentiated with respect to  $x'_1$  and  $x'_2$ , there results:

$$(2a) \quad \begin{aligned} &(u_0t_{11} + u_1t_{21})x_1^{k-1} + C_{k-1,1}(u_1t_{11} + u_2t_{21})x_1^{k-2}x_2 + \dots + (u_{k-1}t_{11} + u_k t_{21})x_2^{k-1} \\ &\equiv u'_0x_1'^{k-1} + C_{k-1,1}u'_1x_1'^{k-2}x_2' + \dots + u'_{k-1}x_2'^{k-1}, \end{aligned}$$

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\* I am indebted to Dr. A. H. Clifford for this proof.

$$(2b) \quad (u_0t_{12} + u_1t_{22})x_1^{k-1} + C_{k-1,1}(u_1t_{12} + u_2t_{22})x_1^{k-2}x_2 + \dots + (u_{k-1}t_{12} + u_kt_{22})x_2^{k-1} \\ \equiv u'_1 x_1^{k-1} + C_{k-1,1}u'_2 x_1^{k-2}x'_2 + \dots + u'_k x_2^{k-1}.$$

If (2a) is multiplied by  $A_1$ , and (2b) by  $A_2$ , and the resulting equations are added, it follows that

$$(u_0A'_1 + u_1A'_2)x_1^{k-1} + C_{k-1,1}(u_1A'_1 + u_2A'_2)x_1^{k-2}x_2 + \dots + (u_{k-1}A'_1 + u_kA'_2)x_2^{k-1} \\ \equiv (u'_0A_1 + u'_1A_2)x_1^{k-1} + C_{k-1,1}(u'_1A_1 + u'_2A_2)x_1^{k-2}x'_2 + \dots + (u'_{k-1}A_1 + u'_kA_2)x_2^{k-1},$$

and from this identity that

$$u'_i A_1 + u'_{i+1} A_2 \equiv \sum_{j=0}^{k-1} T_{ij}^{k-1} (u_j A'_1 + u_{j+1} A'_2) \quad \text{for } i = 0, 1, \dots, k - 1.$$

These identities may be written in the form

$$M_k(A)u' \equiv T^{k-1}M_k(A')u$$

or, since  $u' = T^k u$ , in the form

$$M_k(A)T^k u \equiv T^{k-1}M_k(A')u.$$

The indeterminate vector  $u$  may be cancelled in this equation and so

$$M_k(A)T^k = T^{k-1}M_k(A')$$

as was to be shown.

It is convenient, at this point, to state the following:

LEMMA B.\* *The invariant factors of transformable matrix pencils are connected by the same transformation of the indeterminates  $x_1$  and  $x_2$  as the pencils themselves.*

5. **Equivalence of general matrix pencils.** Williamson† has shown that the minimal numbers of a matrix pencil can be expressed in terms of its singularities, from which follows the theorem:

THEOREM 3. *Two matrix pencils are equivalent if and only if they have the same singularities and invariant factors.*

This theorem may also be proved with the help of Theorem 2 and Lemma B by showing that the invariant subranks of a regular matrix pencil can be expressed in terms of its singularities. This will now be done for the row subranks, and an analogous treatment of the column subranks would complete the proof. There is no loss of generality if the pencil is taken to be in canonical form.

\* See MacDuffee, loc. cit.

† Loc. cit.



Consider the regular canonical pencil  $W = e_0x + a$  of rank  $r$ , with row sub-ranks  $r_k$  and column subranks  $c_k$ . Set

$$E_k = \left\| \begin{array}{ccc} 1_{r_k} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right\|, \quad F_k = \left\| \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1_{c_k} \end{array} \right\|, \quad \text{and} \quad a = \left\| \begin{array}{cc} E_1 & 0 \\ A_1 & F_1 \end{array} \right\|.$$

Then square matrices  $A_k$  may be defined by the relations

$$A_k = \left\| \begin{array}{ccc} 0 & E_k & 0 \\ 0 & A_k & F_k \\ 0 & 0 & 0 \end{array} \right\| \quad \text{for} \quad k = 1, 2, \dots, \phi,$$

where  $A_{\phi+1}$  is the canonical kernel of  $S$  and  $E_k A_k^j F_k = 0$  for  $j = 1, 2, \dots$ . Of course  $r_k$  is the rank of  $E_k$ , and  $c_k$  is the rank of  $F_k$ .

Now, by definition,

$$m_2(W) = \rho \left\| \begin{array}{ccc} e_0 & a & 0 \\ 0 & e_0 & a \end{array} \right\| = \rho \left\| \begin{array}{cccccc} 0 & 0 & E_1 & 0 & 0 & 0 \\ 1 & 0 & A_1 & F_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & E_1 & 0 \\ 0 & 0 & 1 & 0 & A_1 & F_1 \end{array} \right\| = r + \rho \left\| \begin{array}{ccc} E_1 & 0 & 0 \\ 0 & E_1 & 0 \\ 1 & A_1 & F_1 \end{array} \right\|.$$

If the first column of this matrix is multiplied by  $-F_1$  and added to the last column, and then the third row is multiplied by  $-E_1$  and added to the first row; since  $E_1 F_1 = 0$ , it follows that

$$m_2(W) = r + \rho \left\| \begin{array}{cc} 0 & -E_1 A_1 \\ 0 & E_1 \\ 1 & A_1 \end{array} \right\| = 2r + \rho \left\| \begin{array}{c} E_1 \\ E_1 A_1 \end{array} \right\|.$$

In similar fashion, it is easily shown that

$$m_k(W) = kr + \rho \left\| \begin{array}{c} E_1 \\ E_1 A_1 \\ E_1 A_1^2 \\ \dots \\ E_1 A_1^{k-1} \end{array} \right\| \quad \text{for} \quad k = 1, 2, 3, \dots$$

Since  $E_2 A_2^j F_2 = 0$ , it follows that

$$A_1^j = \left\| \begin{array}{ccc} 0 & E_2 A_2^{j-1} & 0 \\ 0 & A_2^j & A_2^{j-1} F_2 \\ 0 & 0 & 0 \end{array} \right\|,$$

and hence that

$$\rho \left\| \begin{array}{c} E_1 \\ E_1 A_1 \\ \dots \\ E_1 A_1^{k-1} \end{array} \right\| = r_1 + \rho \left\| \begin{array}{c} E_2 \\ E_2 A_2 \\ \dots \\ E_2 A_2^{k-1} \end{array} \right\| ;$$

and a simple induction now shows that

$$m_k(W) = kr + \sum_{j=1}^{k-1} r_j.$$

This equation provides the necessary relationship between the row singularities and subranks of  $W$  and the proof of Theorem 3 is complete. Of course,

$$r_k = m_k - r - \sum_{j=1}^{k-1} m_j \quad \text{and} \quad c_k = n_k - r - \sum_{j=1}^{k-1} n_j$$

are the inverse equations which express the subranks in terms of the singularities.

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