DEFINITELY SELF-ADJOINT BOUNDARY VALUE PROBLEMS*

BY
GILBERT A. BLISS

1. **Introduction.** The boundary value problem to be considered in this paper is that of finding a constant λ and a set of functions $y_i(x)$ $(a \leqq x \leqq b;\ i=1, \cdots, n)$ satisfying differential equations and boundary conditions of the form

$$(1.1) \qquad y_i' = [A_{i\alpha}(x) + \lambda B_{i\alpha}(x)]y, \qquad M_{i\alpha}y_\alpha(a) + N_{i\alpha}y_\alpha(b) = 0,$$

in which the matrix $\|M_{i\alpha}N_{i\alpha}\|$ is a matrix of real constants of rank n. Repeated subscripts indicate summation, as in tensor analysis, and it will be understood that all subscripts have the range $1, \cdots, n$ unless otherwise explicitly specified. The system

$$(1.2) \qquad z_i' = - z_\alpha[A_{\alpha i} + \lambda B_{\alpha i}], \qquad z_\alpha(a)P_{\alpha i} + z_\alpha(b)Q_{\alpha i} = 0$$

is by definition *adjoint* to (1.1) if the matrix of constants $\|P_{\alpha i}Q_{\alpha i}\|$ satisfies the conditions

$$M_{i\alpha}P_{\alpha k} - N_{i\alpha}Q_{\alpha k} = 0 \qquad (i, k = 1, \cdots, n).$$

The system which is given in (1.1) is said to be *self-adjoint* provided that it is equivalent to its adjoint system (1.2) by a non-singular transformation $z_i = T_{i\alpha}(x)y_\alpha$.

This definition of self-adjoint boundary value problems and a further definition of so-called definite self-adjointness were given by the author in a paper published in 1926† which will be designated in the text below by the Roman numeral I. In that paper it was stated that the boundary value problems arising from the calculus of variations are all definitely self-adjoint. This statement is true for non-singular problems of the calculus of variations without side conditions, the only ones whose boundary value problems had been studied up to that time so far as is known to the writer. It is not true, however, for problems of the calculus of variations such as those of Mayer, Lagrange, and Bolza whose boundary value problems are self-adjoint but not definitely self-adjoint according to the definition given in I. One of the earliest

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† Bliss, *A boundary value problem for a system of ordinary differential equations of the first order*, these Transactions, vol. 28 (1926), pp. 561–584.

formulations of a case of this more complicated kind was that of Cope for the problem of Mayer with variable end points.*

In the following pages a modification of the earlier definition of definite self-adjointness will be given which seems to be applicable to all of the boundary value problems so far studied arising from problems of the calculus of variations involving simple integrals. The new definition involves a property analogous to the normality of a minimizing arc for a problem of Bolza, and is weaker than the older definition in the sense that it imposes fewer restrictions. It will be shown, however, that for a definitely self-adjoint boundary value problem as here defined most of the properties deduced in the paper I cited above are still valid. For example, the characteristic numbers are all real and have indices equal to their multiplicities, and the expansion theorems proved in the paper I also hold. It is not possible to show that the number of characteristic numbers is always infinite. Examples will be cited showing that this is in fact not the case. When the set of characteristic numbers is finite the class of functions for which the expansion theorems hold is of course severely limited. The boundary value problems arising from the calculus of variations are a special type of definitely self-adjoint problems which have an infinity of characteristic numbers, as has been shown by several writers.†

In the paragraphs below frequent use is made of the results and proofs of the paper I to which reference has been made above.

2. **The definition of definite self-adjointness and its first consequences.** It is understood that $A_{ik}(x), B_{ik}(x)$ are real, single-valued and continuous on $a \leqq x \leqq b$. The definition fundamental for this paper is then the following:

DEFINITION. A boundary value problem (1.1) is said to be *definitely self-adjoint* if it is self-adjoint and has the further properties:

(1) the matrix of functions $S_{ik}(x) = T_{\alpha i}(x) B_{\alpha k}(x)$ is symmetric at each value x on the interval ab;

(2) the quadratic form $S_{\alpha\beta}(x)\xi_\alpha\xi_\beta$ is non-negative at each value x on ab;

(3) the set $y_i(x) \equiv 0$ is the only set of functions which satisfies on ab the conditions

$$(2.1) \qquad y_i' = A_{i\alpha} y_\alpha, \qquad M_{i\alpha} y_\alpha(a) + N_{i\alpha} y_\alpha(b) = 0, \qquad S_{\alpha\beta} y_\alpha y_\beta = 0.$$

* Cope, *An analogue of Jacobi's condition for the problem of Mayer with variable end-points*, Dissertation, University of Chicago, 1927. For a synopsis see *Abstracts of Theses*, The University of Chicago, vol. 6 (1927–1928), pp. 15–21. See also American Journal of Mathematics, vol. 59 (1937), pp. 655–672.

† Hu, *The problem of Bolza and its accessory boundary value problem*, *Contributions to the Calculus of Variations* 1931–1932, The University of Chicago Press, p. 400; Morse, *Sufficient conditions in the problem of Lagrange with variable end conditions*, American Journal of Mathematics, vol. 53 (1931), pp. 517–546, especially §16.

The property (3) is analogous to normality in the calculus of variations, as will be shown in a later section. Since the quadratic form with matrix S_{ik} is non-negative, it follows that every set of values y_i which satisfy the equation $S_{\alpha\beta}y_\alpha y_\beta=0$ must also satisfy $S_{i\alpha}y_\alpha=0$ and consequently the equations $B_{i\alpha}y_\alpha=0$, since the determinant $|T_{ik}|$ is different from zero. In the conditions (2.1) we can therefore replace the last equation by $B_{i\alpha}y_\alpha=0$ if desirable.

Let $Y_{ik}(x, \lambda)$ be the elements of a matrix whose columns are n linearly independent solutions of the differential equations in (1.1), and let $s_i(y)$ represent the first member of the second equation (1.1). Then the *characteristic numbers* of the boundary value problem are the roots of the determinant given by

$$D(\lambda) = | s_i[Y_k(x, \lambda)]|$$

in which the symbol $s_i[Y_k(x, \lambda)]$ represents the value of $s_i(y)$ formed for the kth column of the matrix of elements Y_{ik}. The *index* of a root λ_0 of $D(\lambda)$ is by definition the number r when $n-r$ is the rank of $D(\lambda_0)$, and the *multiplicity* of λ_0 is its multiplicity as a root of $D(\lambda)$.

THEOREM 2.1. *For a definitely self-adjoint boundary value problem every root of the determinant $D(\lambda)$ is real, and the independent characteristic solutions of the boundary value problem corresponding to such a root may be chosen real.*

For suppose that $y_i=y_{i1}+(-1)^{1/2}y_{i2}$ were a solution of the boundary value problem, not identically zero and corresponding to an imaginary root $\lambda=\lambda_1+(-1)^{1/2}\lambda_2$ of $D(\lambda)$. Then the conjugate imaginary set $\bar{y}_i=y_{i1}-(-1)^{1/2}y_{i2}$ would be a solution corresponding to the root $\bar{\lambda}=\lambda_1-(-1)^{1/2}\lambda_2$. According to I, Theorem 8, we would have

$$S_{\alpha\beta}y_\alpha\bar{y}_\beta = S_{\alpha\beta}y_{\alpha 1}y_{\beta 1} + S_{\alpha\beta}y_{\alpha 2}y_{\beta 2} = 0.$$

This would imply a contradiction since by a remark made above the equations $B_{i\alpha}y_{\alpha 1}=B_{i\alpha}y_{\alpha 2}=0$ would be consequences of the last equation, and one verifies readily by substitution in (1.1) that the functions $y_{i1}(x)$ and $y_{i2}(x)$ would satisfy the equations (2.1) and hence be identically zero.

THEOREM 2.2. *For a definitely self-adjoint boundary value problem the index of every root of $D(\lambda)$ is equal to its multiplicity.*

The proof is identical with that of I, Theorem 10, down to the last equation on page 572 which would again imply $B_{i\alpha}y_{\alpha 1}\equiv 0$ and $y_{i1}\equiv 0$, as in the paragraph above preceding Theorem 2.2, and this would be a contradiction since the functions y_{i1} in the proof are not identically zero.

For the new definition of definite self-adjointness the Theorem 11 of the paper I will be replaced by the following theorem which is analogous to a theorem of Hu* for boundary value problems of the calculus of variations:

THEOREM 2.3. *If for a set of functions $f_i(x)$ continuous on the interval ab the condition*

$$\int_a^b S_{\alpha\beta} y_\alpha f_\beta dx = 0 \tag{2.2}$$

is satisfied by every solution $y_i(x)$ of a definitely self-adjoint boundary value problem, then it is also satisfied by every set of functions $y_i(x)$ satisfying the following equations

$$y_i' = A_{i\alpha} y_\alpha + B_{i\alpha} g_\alpha, \qquad s_i(y) = 0 \tag{2.3}$$

with functions $g_i(x)$ continuous on the interval ab.

The condition (2.2) for all solutions $y_i(x)$ of the boundary value problem implies, as in the proof of I, Theorem 11, that the non-homogeneous system

$$y_i' = (A_{i\alpha} + \lambda B_{i\alpha}) y_\alpha + B_{i\alpha} f_\alpha, \qquad s_i(y) = 0$$

has a solution $y_i(x, \lambda)$ expressible by power series

$$y_i(x, \lambda) = u_{i0}(x) + u_{i1}(x)\lambda + u_{i2}(x)\lambda^2 + \cdots$$

whose coefficients $u_{i\mu}(x)$ ($\mu = 0, 1, 2, \cdots$) have continuous derivatives on the interval ab, and which converge uniformly for values x, λ satisfying conditions of the form $a \leqq x \leqq b$, $|\lambda| \leqq \rho$. From the proof of I, Theorem 11, it follows that

$$u_{i0}' = A_{i\alpha} u_{\alpha 0} + B_{i\alpha} f_\alpha, \qquad s_i(u_0) = 0, \tag{2.4}$$

and also that

$$W_0 = \int_a^b S_{\alpha\beta} u_{\alpha 0} u_{\beta 0} dx = 0.$$

From the last equation and the properties (1)–(3) we deduce the identities $B_{i\alpha} u_{\alpha 0} \equiv 0$. Consider now a set of functions $y_i(x)$ which satisfy equations of the form (2.3). From (2.3) and the equations (19), (20) of I it follows that the corresponding functions $z_i = T_{i\alpha} y_\alpha$ satisfy the equations

$$z_i' = -z_\alpha A_{\alpha i} - B_{\alpha i} T_{\alpha\beta} g_\beta, \qquad t_i(z) = 0, \tag{2.5}$$

where $t_i(z)$ is a symbol for the first member of the second equation (1.2). From (2.4) and (2.5) and the identities $B_{i\alpha} u_{\alpha 0} \equiv 0$, we have

$$z_\alpha u_{\alpha 0}' + z_\alpha' u_{\alpha 0} = S_{\alpha\beta} y_\alpha f_\beta,$$

* Hu, loc. cit., Theorem 7.3, p. 396.

and hence with the help of equation (7) of I we find that every set of functions y_i which satisfy the equations (2.3) will also satisfy (2.2), as was to be demonstrated.

COROLLARY 2.1. *If the determinant* $|B_{ik}(x)|$ *is different from zero on the interval ab, then* $f_i \equiv 0$ *is the only set of functions which satisfy the condition* (2.2) *with every solution* $y_i(x)$ *of a definitely self-adjoint boundary value problem.*

This follows from the equations (2.4) and the identities $u_{\alpha 0} \equiv 0$ which are consequences of the identities $B_{i\alpha}u_{\alpha 0} \equiv 0$ when the determinant $|B_{ik}|$ is nowhere zero.

COROLLARY 2.2. *If the functions* f_i *satisfy equations of the form*

$$f_i' = A_{i\alpha}f_\alpha + B_{i\alpha}g_\alpha, \qquad s_i(f) = 0$$

with functions $g_i(x)$ *continuous on the interval ab, and if they satisfy the condition* (2.2) *with every solution of a definitely self-adjoint boundary value problem, then they also satisfy the identities* $B_{i\alpha}f_\alpha \equiv 0$.

This follows readily from Theorem 2.3 when we note that the functions $y_i = f_i$ satisfy equations of the form (2.3), and therefore from (2.2) that

$$\int_a^b S_{\alpha\beta}f_\alpha f_\beta dx = 0.$$

By reasoning similar to that used a number of times above it follows then that $B_{i\alpha}f_\alpha \equiv 0$.

3. **The expansion theorems.** Since the roots of the power series $D(\lambda)$ form a finite or infinite denumerable set, and since the number of linearly independent solutions $y_i(x)$ of the boundary value problem associated with each root is equal to the multiplicity of the root, it follows that the solutions and their corresponding characteristic numbers can be enumerated and denoted by symbols $y_{i\nu}(x)$, λ_ν $(\nu = 1, 2, \cdots)$. Furthermore these solutions can be normed and orthogonalized by well known processes so that

$$\int_a^b S_{\alpha\beta}y_{\alpha\mu}y_{\beta\nu}dx = \delta_{\mu\nu}, \tag{3.1}$$

where $\delta_{\mu\mu} = 1$, $\delta_{\mu\nu} = 0$ if $\mu \neq \nu$. For an arbitrary set of functions $f_i(x)$ continuous on the interval ab the constants c_ν may be defined by the equations

$$c_\nu = \int_a^b S_{\alpha\beta}(\xi)y_{\alpha\nu}(\xi)f_\beta(\xi)d\xi. \tag{3.2}$$

The fundamental Theorem 13 of the paper I needs a different proof with the new definition of definite self-adjointness. It may be written in the following form:

THEOREM 3.1. *For every set of functions $f_i(x)$ satisfying equations of the form*

$$f_i' = A_{i\alpha} f_\alpha + B_{i\alpha} g_\alpha, \qquad s_i(f) = 0 \tag{3.3}$$

with functions $g_i(x)$ continuous on the interval ab, the series

$$\phi_i = \sum_\nu c_\nu y_{i\nu}(x) \tag{3.4}$$

converge uniformly on the interval ab, and $B_{i\alpha}(f_\alpha - \phi_\alpha) \equiv 0$.

The sums ϕ_i may contain only a finite number of terms if the set of characteristic numbers λ_ν is finite. But the uniform convergence of these series can in every case be proved as in I, §6. To prove the identities $B_{i\alpha}(f_\alpha - \phi_\alpha) \equiv 0$ we note first that for every ν

$$\int_a^b S_{\alpha\beta}(f_\alpha - \phi_\alpha) y_{\beta\nu} dx = 0 \tag{3.5}$$

because of the equations (3.1) and (3.2). From Theorem 2.3 it follows therefore that

$$\int_a^b S_{\alpha\beta}(f_\alpha - \phi_\alpha) f_\beta dx = 0 \tag{3.6}$$

since the functions f_i by hypothesis satisfy the equations (3.3) which are of the form (2.3). Furthermore

$$\int_a^b S_{\alpha\beta}(f_\alpha - \phi_\alpha) \phi_\beta dx = 0 \tag{3.7}$$

because of the form of the functions (3.4) and the relations (3.5). By subtracting (3.7) from (3.6) we find that

$$\int_a^b S_{\alpha\beta}(f_\alpha - \phi_\alpha)(f_\beta - \phi_\beta) dx = 0;$$

hence, by the usual argument, we obtain the desired identities.

COROLLARY 3.1. *If the determinant $|B_{ik}(x)|$ is nowhere zero on the interval ab, then for every set of functions $f_i(x)$ having continuous derivatives on that interval and satisfying the boundary conditions $s_i(f) = 0$ the sums* (3.4) *converge uniformly and are equal to the functions f_i on the interval ab.*

The corollary is identical with Corollary 1 of paper I and is proved in the same way.

COROLLARY 3.2. *If the functions* $f_i(x)$ *satisfy equations of the form* (3.3), *and if furthermore the functions* $g_i(x)$ *in those equations are solutions of a similar system*

$$g_i' = A_{i\alpha}g_\alpha + B_{i\alpha}h_\alpha, \qquad s_i(g) = 0$$

with functions $h_i(x)$ *continuous on the interval ab, then the series* (3.4) *converge uniformly and are equal to the functions* $f_i(x)$.

This is Corollary 2 of I, page 576, but its proof needs emendation for the new definition of definite self-adjointness. We use the notations

$$d_\nu = \int_a^b S_{\alpha\beta}(\xi)y_{\alpha\nu}(\xi)g_\beta(\xi)d\xi, \qquad \psi_i = \sum_\nu d_\nu y_{i\nu}.$$

The equation

$$\phi_i' = A_{i\alpha}\phi_\alpha + B_{i\alpha}\psi_\alpha \tag{3.8}$$

is equation (39) of I and is proved in the same way. From (3.3), (3.8), and the fact that $s_i(y_\nu)=0$ it follows that

$$f_i' - \phi_i' = A_{i\alpha}(f_\alpha - \phi_\alpha) + B_{i\alpha}(g_\alpha - \psi_\alpha), \qquad s_i(f-\phi) = 0. \tag{3.9}$$

The last term in the first equation (3.9) vanishes identically since the equations $B_{i\alpha}(g_\alpha - \psi_\alpha) \equiv 0$ are consequences of Theorem 3.1 applied to the functions g_i in place of the f_i. The similar identities $B_{i\alpha}(f_\alpha - \phi_\alpha) \equiv 0$ for the functions f_i, from Theorem 3.1, imply that $(f_\alpha - \phi_\alpha)S_{\alpha\beta}(f_\beta - \phi_\beta) \equiv 0$ and hence from equations (3.9) and the property (3) in the definition of definite self-adjointness that $f_i - \phi_i \equiv 0$.

4. **The boundary value problem associated with a problem of Bolza.** The second variation of the problem of Bolza may be taken in the form

$$J_2(\xi, \eta) = 2\gamma[\xi_1, \eta(x_1), \xi_2, \eta(x_2)] + \int_{x_1}^{x_2} 2\omega(x, \eta, \eta')dx$$

in which 2γ is a homogeneous quadratic form in its $2n+2$ arguments ξ_1, $\eta_i(x_1)$, ξ_2, $\eta_i(x_2)$ $(i=1, \cdots, n)$, and 2ω is a homogeneous quadratic form in the $2n$ variables $\eta_i(x)$, $\eta_i'(x)$ with coefficients functions of x.

An accessory minimum problem associated with this second variation is that of finding in a class of sets ξ_1, ξ_2, $\eta_i(x)$, satisfying conditions of the form

$$\Phi_\beta(x, \eta, \eta') = 0 \qquad (\beta = 1, \cdots, m < n),$$

$$\Psi_\mu[\xi_1, \eta(x_1), \xi_2, \eta(x_2)] = 0 \qquad (\mu = 1, \cdots, p \leqq 2n+2),$$

$$\int_{x_1}^{x_2} \eta_i\eta_i dx = 1,$$

one which minimizes $J_2(\xi, \eta)$. The functions Φ_β are homogeneous and linear in the $2n$ variables η_i, η_i' with coefficients functions of x, and the functions Ψ_μ are p homogeneous linear independent expressions with constant coefficients in their $2n+2$ arguments.

The differential equations and end conditions for a minimizing set ξ_1, ξ_2, $\eta_i(x)$, for the accessory problem may be expressed with the help of the notations

$$\Omega = \omega + \mu_\beta \Phi_\beta, \qquad \zeta_i = \Omega_{\eta_i'}$$

as follows:*

$$d\Omega_{\eta_i'}/dx = \Omega_{\eta_i} - \lambda\eta_i, \qquad \Phi_\beta = 0, \tag{4.1}$$

$$\begin{aligned} \gamma_1 + \epsilon_\mu \Psi_{\mu,1} &= 0, \\ -\zeta_{i1} + \gamma_{i1} + \epsilon_\mu \Psi_{\mu,i1} &= 0, \\ \gamma_2 + \epsilon_\mu \Psi_{\mu,2} &= 0, \\ \zeta_{i2} + \gamma_{i2} + \epsilon_\mu \Psi_{\mu,i2} &= 0, \\ \Psi_\mu &= 0. \end{aligned} \tag{4.2}$$

The notations η_{is}, ζ_{is} $(s=1, 2)$ represent the values $\eta_i(x_s)$, $\zeta_i(x_s)$ $(s=1, 2)$. The subscripts 1, 2, $i1$, $i2$ attached to γ and Ψ_μ indicate partial derivatives with respect to ξ_1, ξ_2, η_{i1}, η_{i2}, respectively. It is to be shown that the equations (4.1) and (4.2) are equivalent to a boundary value problem of the type studied in the preceding sections.

The accessory minimum problem is said to be *non-singular* if the determinant of coefficients of the variables η_i, μ_β in the first members of the equations

$$\begin{aligned} \Omega_{\eta_i'}(x, \eta, \eta', \mu) &= \zeta_i, \\ \Phi_\beta(x, \eta, \eta') &= 0 \end{aligned} \tag{4.3}$$

is everywhere different from zero on the interval x_1x_2. It is said to satisfy the *non-tangency condition* if the equations $\Psi_\mu = 0$ have no non-vanishing solution ξ_1, ξ_2, $\eta_i(x_1)$, $\eta_i(x_2)$ with $\eta_i(x_1) = \eta_i(x_2) = 0$, or, in other words, if the matrix of coefficients of ξ_1 and ξ_2 in the functions Ψ_μ is of rank 2. Finally the accessory problem is said to be *normal* if the only solution ξ_1, ξ_2, $\eta_i(x)$, $\mu_\beta(x)$, ϵ_μ of equations (4.1) and (4.2) with $\eta_i(x) \equiv 0$ on the interval x_1x_2 is the one whose elements all vanish identically.†

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* Bliss, *The problem of Bolza in the calculus of variations*, mimeographed lecture notes, The University of Chicago, 1935, p. 73, equations (14.1), and p. 76, equation (14.8).

† These definitions are customary ones. See, for example, Bliss, *The problem of Bolza in the calculus of variations*, loc. cit., pp. 34, 82, and §§9, 10. In the two sections cited the notion of normality is analyzed in considerable detail.

The differential equations (4.1) can be expressed in terms of the so-called canonical variables x, η_i, ζ_i related to the variables x, η_i, η_i', μ_β by means of the equations (4.3). If the accessory minimum problem is non-singular, these equations have solutions

$$\eta_i' = \Pi_i(x, \eta, \zeta), \qquad \mu_\beta = \mathrm{M}_\beta(x, \eta, \zeta)$$

which are linear in the variables η_i, ζ_i. In terms of the homogeneous quadratic form in η_i, ζ_i defined by the equation

$$\begin{aligned} 2\mathcal{H} &= 2\zeta_i\Pi_i - 2\Omega(x, \eta, \Pi, \mathrm{M}) \\ &= U_{ij}(x)\eta_i\eta_j + 2V_{ij}(x)\eta_i\zeta_j + W_{ij}(x)\zeta_i\zeta_j \end{aligned}$$

the differential equations (4.1) take the well known canonical form

$$\begin{aligned} d\eta_i/dx &= \mathcal{H}_{\zeta_i} = V_{ji}\eta_j + W_{ij}\zeta_j, \\ d\zeta_i/dx &= -\mathcal{H}_{\eta_i} - \lambda\eta_i = -U_{ij}\eta_j - V_{ij}\zeta_j - \lambda\eta_i. \end{aligned} \tag{4.4}$$

The matrices of elements U_{ij} and W_{ij} are, of course, symmetric.

The end conditions (4.2) can also be transformed into a more convenient form. Let

$$\xi_1, \quad \xi_2, \quad \eta_{i1}, \quad \eta_{i2}, \quad \zeta_{i1}, \quad \zeta_{i2}, \quad \epsilon_\mu, \tag{4.5}$$

$$\alpha_1, \quad \alpha_2, \quad a_{i1}, \quad a_{i2}, \quad b_{i1}, \quad b_{i2}, \quad \theta_\mu \tag{4.6}$$

be two sets satisfying those conditions. If we multiply the first four equations in (4.2), respectively, by α_1, a_{i1}, α_2, a_{i2} and add, and then subtract the similar sum with the two solutions interchanged, it follows from the fifth equation (4.2) and well known properties of quadratic forms, that

$$b_{i1}\eta_{i1} - a_{i1}\zeta_{i1} - b_{i2}\eta_{i2} + a_{i2}\zeta_{i2} = 0. \tag{4.7}$$

Consider now $2n$ linearly independent solutions of equations (4.2)

$$\begin{matrix} \xi_{i,1}, & \xi_{i,2}, & \eta_{i,k1}, & \eta_{i,k2}, & \zeta_{i,k1}, & \zeta_{i,k2}, & \epsilon_{i,\mu}, \\ \alpha_{i,1}, & \alpha_{i,2}, & a_{i,k1}, & a_{i,k2}, & b_{i,k1}, & b_{i,k2}, & \theta_{i,\mu} \end{matrix} \tag{4.8}$$

for $i=1, \cdots, n$. If the accessory minimum problem satisfies the non-tangency condition, the elements η, ζ, a, b in the sets (4.8) form a $2n\times 4n$-dimensional matrix which is of rank $2n$. Otherwise there would be a solution (4.5) of equations (4.2) with elements η, ζ all zero, formed by taking a linear combination of the $2n$ solutions (4.8) with constant coefficients not all zero. The elements ξ_1, ξ_2 of this solution would also vanish, on account of the fifth of equations (4.2) and the non-tangency condition. The elements ϵ_μ would then also vanish because of the first four of equations (4.2) and the independence of the functions Ψ_μ. This would contradict the independence of the $2n$ solutions (4.8). It is evident then that the $2n$ equations

$$\begin{aligned} \zeta_{i,k1}\eta_{k1} - \eta_{i,k1}\zeta_{k1} - \zeta_{i,k2}\eta_{k2} + \eta_{i,k2}\zeta_{k2} &= 0, \\ b_{i,k1}\eta_{k1} - a_{i,k1}\zeta_{k1} - b_{i,k2}\eta_{k2} + a_{i,k2}\zeta_{k2} &= 0, \end{aligned} \tag{4.9}$$

related to the set (4.8) as (4.7) is to (4.6), are linearly independent. They are linear combinations of the equations (4.2) and are equivalent to this latter system in the sense that with every set of values η_{i1}, η_{i2}, ζ_{i1}, ζ_{i2} satisfying equations (4.9) there is associated a unique solution of equations (4.2) whose other elements ξ_1, ξ_2, ϵ_μ are determined by the first, third, and fifth of equations (4.2). The equations (4.4) and (4.9) define a boundary value problem for the $2n$ functions $\eta_i(x)$, $\zeta_i(x)$ analogous to that characterized by the equations (1.1) for the functions $y_i(x)$.

THEOREM 4.1. *For a problem of Bolza having a non-singular normal accessory minimum problem satisfying the non-tangency condition the boundary value problem defined by equations* (4.4) *and* (4.9) *is definitely self-adjoint according to the definition in* §2 *above.*

To prove this theorem we note first that necessary and sufficient conditions for the system (1.1) to be self-adjoint, taken from equations (19) and (20) of the paper I, can be expressed by use of matrix notation in the form

$$TA + \bar{A}T + T' = 0, \quad TB + \bar{B}T = 0, \quad MT^{-1}(a)\bar{M} = NT^{-1}(b)\bar{N}, \tag{4.10}$$

where the bars indicate transposed matrices and T' is the matrix of derivatives of the elements of T. For the boundary value problem defined by equations (4.4) and (4.9) the matrices involved are the $2n$-dimensional matrices

$$A = \begin{pmatrix} V_{ji} & W_{ij} \\ -U_{ij} & -V_{ij} \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ \delta_{ij} & 0 \end{pmatrix},$$

$$M = \begin{pmatrix} \zeta_{i,k1} & -\eta_{i,k1} \\ b_{i,k1} & -a_{i,k1} \end{pmatrix}, \qquad N = \begin{pmatrix} -\zeta_{i,k2} & \eta_{i,k2} \\ -b_{i,k2} & a_{i,k2} \end{pmatrix}.$$

These satisfy the equations (4.10) with the special transformation matrix

$$T = \begin{pmatrix} 0 & \delta_{ik} \\ -\delta_{ik} & 0 \end{pmatrix}.$$

In proving the first equation (4.10) use is made of the symmetry of the matrices U and W, and in proving the third equation relation (4.7) for the various pairs of the solutions (4.8) is needed. The matrix $S=\bar{T}B$ of §2 above is

$$S = \begin{pmatrix} 0 & -\delta_{ij} \\ \delta_{ij} & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ -\delta_{jk} & 0 \end{pmatrix} = \begin{pmatrix} \delta_{ik} & 0 \\ 0 & 0 \end{pmatrix}.$$

Evidently this matrix is symmetric and its quadratic form is non-negative. The only functions $\eta_i(x)$, $\zeta_i(x)$ which make this quadratic form vanish identically have the form $\eta_i(x)\equiv 0$, $\zeta_i(x)$, and no set of functions of this type can satisfy equations (4.4) with $\lambda=0$ and the end conditions (4.9). Otherwise there would be a related solution ξ_1, ξ_2, $\eta_i(x)$, $\mu_\beta(x)$, ϵ_μ of the equations (4.1) and (4.2) with $\eta_i(x)\equiv 0$ on the interval x_1x_2, which is impossible when the accessory minimum problem is normal. Thus all of the conditions (1), (2), (3) of the definition of definite self-adjointness in §2 are satisfied by the boundary value problem associated with equations (4.4) and (4.9), as stated in Theorem 4.1.

The assumption of the non-tangency condition can be omitted, as has recently been suggested to me by W. T. Reid, if the formulation of the accessory minimum problem is slightly modified. The constants ξ_1 and ξ_2 in this problem can be replaced by the values $\eta_{n+1}(x_1)$, $\eta_{n+2}(x_2)$ of two functions $\eta_{n+1}(x)$, $\eta_{n+2}(x)$ subjected to differential equations

$$\eta_{n+1}' = \eta_{n+2}' = 0$$

which are to be adjoined to the equations $\Phi_\beta=0$. In the norming integral in the second paragraph of this section the integrand is to be replaced by the sum of the squares of all of the variables $\eta_\sigma(x)$ $(\sigma=1,\cdots,n+2)$. One verifies readily then that for the new problem the end conditions contain only equations of the form of the second, fourth, and last of the equations (4.2) and the construction of the end conditions (4.9) does not involve the non-tangency condition.

5. **Transformations and examples.** If a definitely self-adjoint boundary value problem of the form (1.1) for a set of functions $y_i(x)$ is transformed into one for functions $u_i(x)$ by a non-singular transformation $y_i=U_{ik}(x)u_k$, the property of definite self-adjointness will be preserved. This can be verified by means of the following useful and easily derived transformation formulas, in which the subscript 1 designates the matrices associated with the transformed problem:

$$(5.1)\qquad \begin{aligned} A_1 &= U^{-1}AU - U^{-1}U', & B_1 &= U^{-1}BU, \\ M_1 &= MU(a), & N_1 &= NU(b), \\ P_1 &= U^{-1}(a)P, & Q_1 &= U^{-1}(b)Q, \\ T_1 &= \bar{U}TU, & S_1 &= \bar{U}SU. \end{aligned}$$

It is understood that in these formulas a bar indicates a transposed matrix and a prime a matrix of derivatives. With the help of these relations one can readily deduce normal forms for definitely self-adjoint boundary value problems when the equations involve only two functions y_1 and y_2 and the rank of the matrix $B(x)$, and consequently of $S(x)$, is constant on the interval ab.*

Consider first the case when the determinant of $B(x)$ is everywhere different from zero on the interval ab and the matrix $S(x)$ therefore positive definite. From the second equation (4.10) and the symmetry of $S=\bar{T}B$ it follows that

$$0 = TB + \bar{B}T = (T + \bar{T})B, \qquad 0 = T + \bar{T},$$

so that T is skew-symmetric. There exists a transformation U taking S into the identity matrix, and we have then

$$T = \begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix}, \qquad S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 1/t \\ -1/t & 0 \end{pmatrix}$$

with the help of the relation $S=\bar{T}B$. The transformation

$$U = \begin{pmatrix} 1 & 0 \\ 0 & 1/t \end{pmatrix}$$

now transforms these matrices into the forms shown in (5.3) below, with $s=1/t$. One can readily verify the fact that the most general transformation leaving T and S in (5.3) invariant has the form

$$U = \begin{pmatrix} u_{11} & -s^2u_{21} \\ u_{21} & u_{11} \end{pmatrix}, \qquad u_{11}^2 + s^2u_{21}^2 = 1, \tag{5.2}$$

and that it will also leave B in (5.3) invariant under the transformation (5.1). The lower left-hand element of A_1 after such a transformation when set equal to zero, and the derivative of the last equation above, have the forms

$$u_{21}u_{11}' - u_{11}u_{21}' + \cdots = 0,$$
$$u_{11}u_{11}' + s^2u_{21}u_{21}' + ss'u_{21}^2 = 0,$$

where the dots indicate terms not containing derivatives of the elements u_{ik}. If u_{11} and u_{21} are determined by these differential equations with initial values at a single point satisfying the second equation (5.2), they will satisfy that equation identically. The first equation (4.10) shows that $a_{22}=-a_{11}$, and we have the following theorem:

* For a more complete classification see Bamforth, *A classification of boundary value problems for a system of ordinary differential equations of the second order*, Dissertation, University of Chicago, 1927.

THEOREM 5.1. *When $n=2$ every definitely self-adjoint boundary value problem with $|B|\neq 0$ on the interval ab is transformable into a problem with matrices of the form*

$$(5.3)\qquad \begin{aligned} A &= \begin{pmatrix} a_{11} & a_{12} \\ 0 & -a_{11} \end{pmatrix}, & B &= \begin{pmatrix} 0 & s^2 \\ -1 & 0 \end{pmatrix}, \\ T &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & S &= \begin{pmatrix} 1 & 0 \\ 0 & s^2 \end{pmatrix} \end{aligned}$$

with $s(x)\neq 0$ on ab and $|M|=|N|$. Conversely, every problem with these properties is definitely self-adjoint. Such a problem has always an infinity of characteristic numbers.

The relation (20) of I shows that $|M|=|N|$. It is evident that the functions f_i described in Corollary 3.1 above could not all be expansible as there stated if there were only a finite set of characteristic numbers and functions.

The case when the rank of the matrix $B(x)$ is unity everywhere on the interval ab gives rise to a number of normal forms of definitely self-adjoint problems. The matrix $S(x)$ is then transformable into

$$S = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

From the formula $S=\overline{T}B$ and this form of S it can readily be seen that the matrix B and the most general transformation U leaving S invariant have the forms

$$B = \begin{pmatrix} b_{11} & 0 \\ b_{21} & 0 \end{pmatrix}, \qquad U = \begin{pmatrix} \pm 1 & 0 \\ u_{21} & u_{22} \end{pmatrix}.$$

Since B has rank unity the elements b_{11}, b_{21} do not vanish simultaneously, and a transformation U with leading element $+1$ can be chosen so that $u_{22}=-b_{21}+u_{21}b_{11}\neq 0$. Such a transformation will take B into the form

$$(5.4)\qquad B = \begin{pmatrix} b & 0 \\ -1 & 0 \end{pmatrix}$$

by means of the second of the formulas (5.1). The conditions $TB+\overline{B}T=0$, $S=\overline{T}B$, $|T|\neq 0$ then imply that b vanishes identically, and that T has the form

$$(5.5)\qquad T = \begin{pmatrix} t & 1 \\ -1 & 0 \end{pmatrix}.$$

With the help of the first equation (19) of I we find that

$$(5.6)\quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & -a_{11} \end{pmatrix}, \qquad t = c\exp\left[-\int_a^x 2a_{11}dx\right], \quad a_{12}t = 0.$$

The most general transformation U leaving invariant the matrices S and (5.4) with $b=0$ is found to be

$$U = \begin{pmatrix} \pm 1 & 0 \\ u & \pm 1 \end{pmatrix},$$

and this will also leave T invariant. By means of such a transformation we can make the lower left-hand element in A vanish identically, by a method similar to that used above. The following theorem can then be established without serious difficulty:

THEOREM 5.2. *When $n=2$ every definitely self-adjoint boundary value problem with matrix $B(x)$ identically of rank one on the interval ab is transformable into a problem with matrices of the form*

$$(5.7)\qquad A = \begin{pmatrix} a_{11} & a_{12} \\ 0 & -a_{11} \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \qquad S = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

When $a_{12}\neq 0$ the only transformation matrix possible is

$$(5.8)\qquad T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and the problem is definitely self-adjoint if and only if the end conditions have matrices $M=(m_{ik})$, $N=(n_{ik})$ with $|M|=|N|$.

When $a_{12}\equiv 0$ the possible transformation matrices have the form

$$T = \begin{pmatrix} t & 1 \\ -1 & 0 \end{pmatrix}, \qquad t = c\exp\left[-\int_a^x 2a_{11}dx\right]$$

where c is a constant. The problem is definitely self-adjoint with a matrix T having $c\neq 0$ if and only if the matrices M and N of the end conditions have equal determinants and satisfy the conditions

$$m_{12} = n_{12}\phi, \quad m_{22} = n_{22}\phi, \quad \phi = \exp\left[-\int_a^b a_{11}dx\right], \qquad (m_{12}, m_{22}) \neq (0, 0).$$

The problem is definitely self-adjoint with a matrix T having $c=0$ if and only if M and N have equal determinants and

$$(m_{12} + n_{12}\phi,\ m_{22} + n_{22}\phi) \neq (0, 0).$$

To prove the second statement of the theorem we note that the last two equations (5.6) imply $t\equiv 0$ when $a_{12}\neq 0$, and that equation (20) of I is satisfied only when $|M| = |N|$. If these conditions are fulfilled, the problem is self-adjoint. It is also definitely self-adjoint, according to the definition of §3 above, since the equations

$$(5.9) \qquad y_1' = a_{11}y_1 + a_{12}y_2, \qquad y_2' = -a_{11}y_2, \qquad Sy = (y_1, 0) = (0, 0)$$

imply that y_2 must vanish identically when $a_{12} \neq 0$.

For the case when $a_{12} \equiv 0$ the arguments of the preceding paragraphs show that the only possible transformation matrix is (5.5) with t satisfying the conditions in (5.6). The third equation (4.10) implies $|M| = |N|$ and

$$m_{12}^2 - n_{12}^2\phi^2 = m_{12}m_{22} - n_{12}n_{22}\phi^2 = m_{22}^2 - n_{22}^2\phi^2 = 0.$$

For definite self-adjointness the solution

$$y_1 = 0, \qquad y_2 = y_2(a) \exp\left[-\int_a^x a_{11}dx\right]$$

of equations (5.9) must vanish identically if it satisfies also the end conditions of the problem, that is, if

$$(5.10) \qquad (m_{12} + n_{12}\phi)y_2(a) = 0, \qquad (m_{22} + n_{22}\phi)y_2(b) = 0.$$

This will be true if and only if the coefficients of $y_2(a)$ in the last two equations are not both zero. The statements in the theorem now follow readily from equations (5.9) and (5.10)

One can construct without difficulty definitely self-adjoint boundary value problems which have only a finite number of characteristic numbers. For example, the problem with the matrices

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \qquad M = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad N = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

is definitely self-adjoint with the matrix (5.8) and has the determinant $D(\lambda) = 1$, and hence has no characteristic numbers. The problem with the same matrices A, B and end-matrices

$$M = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \qquad N = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$$

is definitely self-adjoint and has $D(\lambda) = 2 - \lambda(b-a)$. It has a single characteristic number $\lambda = 2/(b-a)$. When

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad N = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

the problem is self-adjoint but not definitely so, and the determinant $D(\lambda)$ vanishes identically. These examples are transforms into the normal forms described above of some equally simple ones communicated to me by Professor W. T. Reid. They show that the property of definite self-adjointness does not imply an infinity of characteristic numbers.

The boundary value problems arising from problems of Bolza in the plane are all of the first type described in Theorem 5.2 and have a_{12} everywhere different from zero. Theorem 3.1 shows that in this case every function $f_1(x)$ with a continuous second derivative on the interval ab is expansible in the form (3.4), provided only that it satisfies the conditions (3.3) with some functions f_2 and g_1 at $x=a$ and $x=b$. It is evident that such problems must have an infinity of characteristic numbers since otherwise such expansions would not be possible in all cases.

University of Chicago,
Chicago, Ill.