

# IRREDUCIBLE SYSTEMS OF ALGEBRAIC DIFFERENTIAL EQUATIONS\*

BY

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## INTRODUCTION

Let  $\mathcal{F}$  be a domain of rationality, and let  $y_1, \dots, y_n$  be a set of indeterminates. Then the set of prime ideals in the ring of polynomials  $\mathcal{F}[y_1, \dots, y_n]$  satisfies a divisor-chain condition for decreasing sequences as well as for increasing sequences. That is, a sequence of prime ideals  $\Sigma_1, \Sigma_2, \dots$  in  $\mathcal{F}[y_1, \dots, y_n]$  must be of finite length not only if  $\Sigma_{i+1}$  properly includes  $\Sigma_i$  for every  $i$ , but also if  $\Sigma_{i+1}$  is properly included in  $\Sigma_i$  for every  $i$ .†

However, if the domain of rationality  $\mathcal{F}$  is a set of functions meromorphic in an open region  $\mathfrak{A}$ , and if  $\mathcal{F}$  is closed to differentiation (in other words, if  $\mathcal{F}$  is a *field*, in the terminology of algebraic differential equations‡), and if  $\mathcal{F}\{y_1, \dots, y_n\}$  is the differential-ring consisting of all forms in  $y_1, \dots, y_n$  with coefficients in  $\mathcal{F}$ , then the set of prime differential-ideals in  $\mathcal{F}\{y_1, \dots, y_n\}$  satisfies a divisor-chain condition for increasing sequences,§ but does not satisfy such a condition for decreasing sequences. That is, we can have an infinite sequence  $\Sigma_1, \Sigma_2, \dots$  of prime differential-ideals such that  $\Sigma_{i+1}$  is properly included in  $\Sigma_i$ . In the set-theoretic sense the sequence  $\Sigma_1, \Sigma_2, \dots$  converges to a limiting set  $\Sigma$  which is the intersection of the  $\Sigma_i$ . If  $\mathcal{M}_i$  is the manifold of  $\Sigma_i$ , then the sequence  $\mathcal{M}_1, \mathcal{M}_2, \dots$  is a monotonically increasing sequence converging in the set-theoretic sense to a set  $\mathcal{N}$  which is the union of the  $\mathcal{M}_i$ . However, while the limiting set  $\Sigma$  is a prime differential-ideal, the limiting set  $\mathcal{N}$  not only is not the manifold of  $\Sigma$ , but is not a manifold at all. We are concerned in this paper with the relation between  $\mathcal{N}$  and the manifold  $\mathcal{M}$  of  $\Sigma$ .

In the terminology of ADE, what we are considering is an infinite sequence  $\Sigma_1, \Sigma_2, \dots$  of closed irreducible systems in  $y_1, \dots, y_n$  such that  $\mathcal{M}_i$ , the manifold of  $\Sigma_i$ , is a proper part of the manifold of  $\Sigma_{i+1}$ , ( $i = 1, 2, \dots$ ).

\* Presented to the Society, February 26, 1938, under the title *Sequences of systems of algebraic differential equations*; received by the editors April 12, 1938.

† Cf. Van der Waerden, *Moderne Algebra*, vol. 2, pp. 25, 63. The set of all ideals in  $\mathcal{F}[y_1, \dots, y_n]$  satisfies a divisor-chain condition for increasing sequences, but not for decreasing sequences.

‡ See, for example, J. F. Ritt, *Differential Equations from the Algebraic Standpoint*, American Mathematical Society Colloquium Publications, vol. 14, New York, 1932. We shall refer to this book by the letters ADE, and we shall use the terminology of ADE without further reference.

§ As a consequence of Ritt's theorem on the completeness of infinite systems, ADE, §7. The set of all differential-ideals in  $\mathcal{F}\{y_1, \dots, y_n\}$  does not satisfy a divisor-chain condition for increasing sequences (by ADE, §11).

The set  $\mathcal{N}$  is the union  $\mathcal{M}_1 + \mathcal{M}_2 + \dots$ , and  $\mathcal{M}$  is the manifold of the system  $\Sigma$  consisting of all forms  $F$  such that  $F$  is in every  $\Sigma_i$ .<sup>\*</sup> Not only does  $\mathcal{M}$  contain solutions which do not appear in  $\mathcal{N}$ , but there is even a sense in which we may say that  $\mathcal{M}$  is of *higher dimensionality* than  $\mathcal{N}$ . This is expressed in the statement that  $\Sigma$  has more arbitrary unknowns than  $\Sigma_i$ , ( $i=1, 2, \dots$ ) (Theorem 3). On the other hand, we shall see that  $\mathcal{M}$  may be described as the set of all ordered sets of  $n$  analytic functions which can be approximated in a certain manner by solutions in  $\mathcal{N}$  (Theorem 4).

Approximability, as we shall define it, will not imply the familiar uniform approximability in a region. Indeed, for certain sequences  $\Sigma_1, \Sigma_2, \dots$ , every solution of  $\Sigma$  which is not in  $\mathcal{N}$  possesses no region of analyticity in which it may be uniformly approximated by solutions of the  $\Sigma_i$ . On the other hand, there exist sequences  $\Sigma_1, \Sigma_2, \dots$  such that every solution of  $\Sigma$  has a region of analyticity in which it can be approximated uniformly by solutions of the  $\Sigma_i$ .

As a converse to Theorem 3, we have the theorem that for every closed irreducible system  $\Sigma$  with a non-empty set of arbitrary unknowns there is a sequence  $\Sigma_1, \Sigma_2, \dots$  of closed irreducible systems such that  $\Sigma_{i+1}$  holds  $\Sigma_i$ ,  $\Sigma$  is the set of forms common to the  $\Sigma_i$ , and  $\Sigma_i$  has fewer arbitrary unknowns than  $\Sigma$ , ( $i=1, 2, \dots$ ). In fact,  $\Sigma_i$  may be taken to have no arbitrary unknowns.

In studying the sequences  $\Sigma_1, \Sigma_2, \dots$  we use several preliminary theorems which are demonstrated in Part I of this paper. These theorems are extensions of results obtained by Ritt. Theorem 1 deals with the possibility of approximating a solution of a prime algebraic system by solutions which do not annul a specified simple form. Theorem 2 has to do with an analogous question for differential equations.

Lemmas 1 and 2 of Part II are devoted to the study of the degree of freedom which one enjoys in assigning initial conditions to a solution of a prime algebraic system.

### PART I. APPROXIMATION THEOREMS

1. The following theorem is due to Ritt: *Let  $\Sigma$  be an indecomposable system of simple forms in  $y_1, \dots, y_n$ . Let  $B$  be any simple form which does not hold  $\Sigma$ . Given any solution of  $\Sigma$ , analytic in an open region  $\mathfrak{A}_1$ , there is an open region  $\mathfrak{A}'$ , contained in  $\mathfrak{A}_1$ , in which the given solution can be approximated uniformly, with arbitrary closeness, by solutions of  $\Sigma$  for which  $B$  is distinct from 0 throughout  $\mathfrak{A}'$ .* †

<sup>\*</sup> We shall see that  $\Sigma$  is closed and irreducible (Theorem 3, below).

† ADE, §64.

We shall use the following modification of Ritt's result:

**THEOREM 1.** *Let  $\Sigma$  be an indecomposable system of simple forms in  $y_1, \dots, y_n$ . Let  $B$  be any simple form which does not hold  $\Sigma$ . Given any solution of  $\Sigma$ , analytic in an open region  $\mathfrak{A}_1$ , there is an open region  $\mathfrak{M}$  in  $\mathfrak{A}_1$ , such that  $\mathfrak{A}_1 - \mathfrak{M}$  is isolated in  $\mathfrak{A}_1$ ,\* and such that for every bounded simply-connected open region  $\mathfrak{C}$  which lies with its boundary in  $\mathfrak{M}$ , there exists a sequence of solutions of  $\Sigma$ , analytic in  $\mathfrak{C}$ , for each of which  $B$  is distinct from zero throughout  $\mathfrak{C}$ , the sequence converging to the given solution in  $\mathfrak{C}$ , uniformly in every closed subset of  $\mathfrak{C}$ .*

Following the procedure in ADE, §64, we introduce the system  $\Sigma_1$  in  $z_1, \dots, z_n$  with which are associated the simple forms  $R, G, D, E_{ij}$ , ( $i=q+1, \dots, n; j=0, 1, \dots, g-1$ ), as in §§59-61. We denote by  $C_i$  some simple form of  $\Sigma_1$  which is of degree  $m$  in the  $z_j$ , ( $j=1, 2, \dots, q, q+i$ ), and of degree  $m$  in  $z_{q+i}$ , the coefficient of  $(z_{q+i})^m$  being unity ( $i=1, \dots, p$ ). With  $\Sigma_1$  and  $B$  are associated the simple forms  $C, N, X$ , and  $Y$  of §64. We let  $H = XR + YN$ , and we may and do assume that  $X$  and  $Y$  are so chosen that  $H$  is divisible by  $DG$ . By  $\xi_1, \dots, \xi_n$  we understand that solution of  $\Sigma_1$  which corresponds, under the transformation of §57, to the given solution of  $\Sigma$ . Proceeding from this definition of  $H$  and the  $\xi_i$  as in §63, we introduce constants  $b_1, \dots, b_q$  such that  $H$ , under the substitution  $z_i = \xi_i + b_i$ , ( $i=1, \dots, q$ ), becomes a function of  $x$  not identically zero. Then  $H$ , under the substitution  $z_i = \xi_i + b_i h$ , ( $i=1, \dots, q$ ), becomes a polynomial

$$\alpha_r h^r + \alpha_{r+1} h^{r+1} + \dots + \alpha_s h^s,$$

where  $r \geq 1$ , the  $\alpha_i$  are functions of  $x$ , meromorphic in  $\mathfrak{A}_1$ , and  $\alpha_r(x) \neq 0$ .

Let  $\Pi$  be the set of simple forms  $C, N, X, Y, R, G, D, E_{ij}, C_1, \dots, C_p$ . Let  $\mathfrak{M}$  be the set of points of  $\mathfrak{A}_1$  at which the coefficients in  $\Pi$  are analytic and at which the function  $\alpha_r$  is different from zero. Evidently  $\mathfrak{A}_1 - \mathfrak{M}$  is isolated in  $\mathfrak{A}_1$ . The functions  $\alpha_i$  are analytic in  $\mathfrak{M}$ .

Let  $\mathfrak{C}$  be a bounded simply-connected open region which lies with its boundary in  $\mathfrak{M}$ . Since  $\mathfrak{C}$  is at a positive distance from the boundary of  $\mathfrak{M}$ , the function  $\alpha_r$  is bounded away from zero in  $\mathfrak{C}$ , and the functions  $\alpha_{r+1}, \dots, \alpha_s$  are bounded in  $\mathfrak{C}$ . This implies that for every sufficiently small nonzero constant  $h$  the polynomial  $\alpha_r h^r + \dots + \alpha_s h^s$  vanishes nowhere in  $\mathfrak{C}$ . Therefore in the considerations of ADE, §63, we may take  $\mathfrak{A}_2 = \mathfrak{C}$ . Moreover, we may take  $\mathfrak{A}_3 = \mathfrak{C}$ , since the functions  $\xi_1, \dots, \xi_q$  and the coefficients in  $C_1, \dots, C_p$  are bounded in  $\mathfrak{C}$ .

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\* We shall say that a subset  $\mathfrak{S}$  of an open region  $\mathfrak{R}$  is *isolated in  $\mathfrak{R}$*  if  $\mathfrak{S}$  is empty, or if  $\mathfrak{S}$  is a non-empty set which has no limit points in  $\mathfrak{R}$ .

Let  $\mathfrak{D}$  be a region which lies with its boundary in  $\mathfrak{C}$ . Taking  $\mathfrak{A}' = \mathfrak{D}$ , and following the procedure of ADE, §63, we determine a sequence of solutions

$$\zeta_{1,i}, \dots, \zeta_{n,i}, \quad i = 1, 2, \dots,$$

of  $\Sigma_1$  for each of which  $H$  is distinct from zero throughout  $\mathfrak{C}$ , the sequence converging to  $\xi_1, \dots, \xi_n$  uniformly in  $\mathfrak{D}$ . Moreover, the sequence is so constructed that there is a positive number  $d'$  for which the inequalities

$$|\zeta_{ji}| < d', \quad j = 1, \dots, n; i = 1, 2, \dots,$$

are valid in  $\mathfrak{C}$ . Hence by Vitali's theorem\* the sequence  $\zeta_{1,i}, \dots, \zeta_{n,i}$  converges to  $\xi_1, \dots, \xi_n$  in  $\mathfrak{C}$ , uniformly in every closed subset of  $\mathfrak{C}$ . Corresponding to this sequence is a sequence of solutions of  $\Sigma$  for each of which  $B$  is distinct from zero throughout  $\mathfrak{C}$ , the sequence converging to the given solution in  $\mathfrak{C}$ , uniformly in every closed subset of  $\mathfrak{C}$ .

2. We use Theorem 1 to prove the following lemma:

LEMMA. Let  $\Sigma$  be a non-trivial closed irreducible system in  $y_1, \dots, y_n$ , and let  $B$  be a form which does not hold  $\Sigma$ . Given any positive integer  $m$ , and any solution  $\bar{y}_1(x), \dots, \bar{y}_n(x)$  of  $\Sigma$ , analytic in an open region  $\mathfrak{B}$ , let  $\mathfrak{B}_m$  be the set of all points  $x_0$  in  $\mathfrak{B}$  such that for every  $\epsilon > 0$  there is a solution  $y_1(x), \dots, y_n(x)$  of  $\Sigma$ , analytic at  $x_0$ , for which  $\mathfrak{B}$  is different from zero at  $x_0$ , and

$$|y_{ij}(x_0) - \bar{y}_{ij}(x_0)| < \epsilon, \quad i = 1, \dots, n; j = 0, 1, \dots, m. \dagger$$

Then  $\mathfrak{B} - \mathfrak{B}_m$  is isolated in  $\mathfrak{B}$ .

Let  $u_1, \dots, u_q$  be a set of arbitrary unknowns for  $\Sigma$ , let  $y_1, \dots, y_p$  be the remaining unknowns in  $\Sigma, \ddagger$  and let

$$(1) \quad A_1, \dots, A_p$$

be a basic set for  $\Sigma$  with the unknowns ordered  $u_1, \dots, u_q; y_1, \dots, y_p$ .

Let

$$(2) \quad \bar{u}_1, \dots, \bar{u}_q; \bar{y}_1, \dots, \bar{y}_p$$

be a solution of  $\Sigma$ , analytic in an open region  $\mathfrak{B}$ .

Following the procedure in ADE, §73, without change, we determine the prime algebraic system  $\Omega$ . Corresponding to (2) is a solution  $\bar{u}_{ik}, \bar{y}_{ik}$  of  $\Omega$ , analytic in  $\mathfrak{B}$ . In accordance with Theorem 1 of this paper there is an open region  $\mathfrak{M}$  in  $\mathfrak{B}$ , whose complement in  $\mathfrak{B}$  is isolated in  $\mathfrak{B}$ , such that, in every open region which with its boundary is included in a bounded simply-con-

\* Montel, *Les Familles Normales de Fonctions Analytiques*, p. 30.

† The second subscript is an index of differentiation.

‡ We renumber the unknowns if necessary.

nected subregion  $\mathfrak{C}$  of  $\mathfrak{M}$ , the solution  $\bar{u}_{ik}, \bar{y}_{jk}$  of  $\Omega$  can be approximated uniformly by solutions of  $\Omega$  for which  $BS_1 \cdots S_p$  is distinct from zero throughout  $\mathfrak{C}$ . In particular, if  $x_0$  is a point of  $\mathfrak{M}$ , then for every  $\epsilon > 0$  there is a solution  $u_{ik}, y_{jk}$  of  $\Omega$ , analytic at  $x_0$ , for which  $BS_1 \cdots S_p$  is different from zero at  $x_0$  and

$$(3) \quad \begin{aligned} |u_{ik}(x_0) - \bar{u}_{ik}(x_0)| < \epsilon, \quad |y_{jk}(x_0) - \bar{y}_{jk}(x_0)| < \epsilon, \\ i = 1, \dots, q; j = 1, \dots, p; k = 0, 1, \dots, m. \end{aligned}$$

Now the  $u_{ik}(x_0), y_{jk}(x_0)$  in (3) furnish initial conditions for a normal solution of the set of differential forms (1). Hence for every point  $x_0$  in  $\mathfrak{B}$  and every  $\epsilon > 0$  there is a solution  $u_i, y_j$  of  $\Sigma$ , analytic at  $x_0$ , which satisfies (3), and which gives  $BS_1 \cdots S_p$  a nonzero value at  $x_0$ . Thus  $\mathfrak{M}$  is included in  $\mathfrak{B}_m$ , and therefore  $\mathfrak{B} - \mathfrak{B}_m$  is isolated in  $\mathfrak{B}$ .

3. We use this lemma to prove the following theorem:

**THEOREM 2.** *Let  $\Sigma$  be a non-trivial closed irreducible system in  $y_1, \dots, y_n$ , and let  $B$  be a form which does not hold  $\Sigma$ . Then the open region\*  $\mathfrak{A}$  contains a subset  $\mathfrak{B}$  whose complement in  $\mathfrak{A}$  is at most denumerably infinite, such that for every point  $x_0$  in  $\mathfrak{B}$ , every solution  $\bar{y}_1, \dots, \bar{y}_n$  of  $\Sigma$ , analytic at  $x_0$ , every positive integer  $m$ , and every positive number  $\epsilon$  there is a solution  $y_1, \dots, y_n$  of  $\Sigma$ , analytic at  $x_0$ , for which  $B$  is different from zero at  $x_0$  and*

$$(4) \quad |y_{ij}(x_0) - \bar{y}_{ij}(x_0)| < \epsilon, \quad i = 1, 2, \dots, n; j = 0, 1, \dots, m.$$

We shall use the following notation: If  $\bar{y}_1, \dots, \bar{y}_n$  is a solution of  $\Sigma$ , analytic in an open region  $\mathfrak{B}$ , then by  $\mathfrak{B}(\bar{y}_1, \dots, \bar{y}_n; \mathfrak{B})$  we shall mean the set of points  $x_0$  in  $\mathfrak{B}$  such that for every positive integer  $m$  and every  $\epsilon > 0$  there is a solution  $y_1, \dots, y_n$  of  $\Sigma$ , analytic at  $x_0$ , for which  $B$  is different from zero at  $x_0$  and (4) holds. Now for every choice of  $\bar{y}_1, \dots, \bar{y}_n$  and  $\mathfrak{B}$  the set  $\mathfrak{B}(\bar{y}_1, \dots, \bar{y}_n; \mathfrak{B})$  has a complement in  $\mathfrak{B}$  which is at most denumerably infinite. For let  $\bar{y}_1, \dots, \bar{y}_n$  be a solution of  $\Sigma$ , analytic in  $\mathfrak{B}$ . Then for every positive integer  $m$  let  $\mathfrak{B}_m$  be the set of points  $x_0$  in  $\mathfrak{B}$  such that for every  $\epsilon > 0$  there is a solution  $y_1, \dots, y_n$  of  $\Sigma$ , analytic at  $x_0$ , for which  $\mathfrak{B}$  is different from zero at  $x_0$  and (4) holds. It is easy to see that  $\mathfrak{B}(\bar{y}_1, \dots, \bar{y}_n; \mathfrak{B})$  is identical with the intersection of the  $\mathfrak{B}_i, (i = 1, 2, \dots)$ . By the lemma just proved,  $\mathfrak{B} - \mathfrak{B}_i, (i = 1, 2, \dots)$ , is at most denumerably infinite. Consequently  $\mathfrak{B} - \mathfrak{B}(\bar{y}_1, \dots, \bar{y}_n; \mathfrak{B})$  is at most denumerably infinite.†

\* We recall that  $\mathfrak{A}$  is the open region in which are defined the functions belonging to  $\mathfrak{F}$ , the underlying field of coefficients.

† This statement is an extension of a theorem of Ritt, according to which, for every solution  $\bar{y}_1, \dots, \bar{y}_n$  of  $\Sigma$ , analytic in  $\mathfrak{B}$ , the set  $\mathfrak{B}(\bar{y}_1, \dots, \bar{y}_n; \mathfrak{B})$  is *dense* in  $\mathfrak{B}$ . (ADE, §74, and Ritt, *On the singular solutions of algebraic differential equations*, Annals of Mathematics, vol. 37 (1936), note 18.)

Let  $\mathfrak{R}_1, \mathfrak{R}_2, \dots$  be the set of those circles contained in  $\mathfrak{A}$  whose centers have rational coordinates and whose radii are rational. Let  $\mathfrak{S}_i, (i=1, 2, \dots)$ , be the set of all solutions of  $\Sigma$  which are analytic in the closed envelope of  $\mathfrak{R}_i$ . We consider  $\mathfrak{S}_i$  to be a metric space, the distance between two solutions  $y_1, \dots, y_n$  and  $z_1, \dots, z_n$  in  $\mathfrak{S}_i$  being given by

$$(5) \quad \delta_i((y_1, \dots, y_n), (z_1, \dots, z_n)) = \max ( |y_1 - z_1| + |y_2 - z_2| + \dots + |y_n - z_n| ),$$

where the maximum is take over the closed envelope of  $\mathfrak{R}_i$ . Then  $\mathfrak{S}_i$  is a separable space.\* For  $\mathfrak{S}_i$  is a subset of the separable space  $\mathcal{A}_i$  consisting of all ordered sets of  $n$  functions  $y_1, \dots, y_n$  analytic in the closed envelope of  $\mathfrak{R}_i$ , the distance between two elements  $y_1, \dots, y_n$  and  $z_1, \dots, z_n$  of  $\mathcal{A}_i$  being given by (5);  $\mathcal{A}_i$  is separable because the subset of  $\mathcal{A}_i$  consisting of all ordered sets of  $n$  polynomials with rational complex numbers for coefficients is dense in  $\mathcal{A}_i$  and denumerable.

Let  $\mathfrak{C}_i$  be a denumerable dense subset of  $\mathfrak{S}_i, (i=1, 2, \dots)$ . Let  $\mathfrak{Q}$  be the set-theoretic sum

$$\sum (\mathfrak{R}_i - \mathfrak{P}(\bar{y}_1, \dots, \bar{y}_n; \mathfrak{R}_i)),$$

where  $i$  ranges over the positive integers, and for each  $i$  the solution  $\bar{y}_1, \dots, \bar{y}_n$  ranges over  $\mathfrak{C}_i$ . Then  $\mathfrak{Q}$  is at most denumerably infinite. We define  $\mathfrak{P}$  as the complement of  $\mathfrak{Q}$  in  $\mathfrak{A}$ . Let  $x_0$  be a point of  $\mathfrak{P}, \bar{y}_1, \dots, \bar{y}_n$  a solution of  $\Sigma$ , analytic at  $x_0, m$  a positive integer, and  $\epsilon$  a positive number. Then there is a  $\mathfrak{R}_i$  containing  $x_0$  such that  $\bar{y}_1, \dots, \bar{y}_n$  is analytic in the closed envelope of  $\mathfrak{R}_i$ . There exists a solution  $\bar{y}_1, \dots, \bar{y}_n$  belonging to  $\mathfrak{C}_i$  such that

$$(6) \quad | \bar{y}_{ij}(x_0) - \bar{y}_{ij}(x_0) | < \epsilon/2, \quad i = 1, \dots, n; j = 0, 1, \dots, m,$$

since there is a sequence of solutions in  $\mathfrak{C}_i$  convergent to  $\bar{y}_1, \dots, \bar{y}_n$  uniformly in  $\mathfrak{R}_i$ . Since  $x_0$  is in  $\mathfrak{P}(\bar{y}_1, \dots, \bar{y}_n; \mathfrak{R}_i)$ , there is a solution  $y_1, \dots, y_n$  of  $\Sigma$ , analytic at  $x_0$ , for which  $\mathfrak{B}$  is different from zero at  $x_0$  and

$$(7) \quad | y_{ij}(x_0) - \bar{y}_{ij}(x_0) | < \epsilon/2, \quad i = 1, \dots, n; j = 0, 1, \dots, m.$$

By (6) and (7) we have (4). Since  $\mathfrak{A} - \mathfrak{P}$  is at most denumerably infinite, we have our theorem.

## PART II. SEQUENCES OF IRREDUCIBLE SYSTEMS

4. We state first the following lemma:

LEMMA 1. *Let  $\Sigma$  be a prime system in the unknowns  $u_1, \dots, u_q; y_1, \dots, y_p$ , where  $u_1, \dots, u_q$  is a set of unconditioned unknowns for  $\Sigma$ , and let*

\* Of course  $\mathfrak{S}_i$  may be an empty or finite set.

$$(8) \quad A_1, \dots, A_p$$

be a basic set for  $\Sigma$ , with  $A_i$  introducing  $y_i$ , ( $i=1, \dots, p$ ). Let  $F$  be a simple form which does not hold  $\Sigma$ . Then there exists a nonzero simple form  $G$  in  $u_1, \dots, u_q$  which is a linear combination of the simple forms  $A_1, \dots, A_p, F$ .

Let  $J$  be the product of the initials in (8). Let  $K$  be a nonzero simple form in  $u_1, \dots, u_q$  such that every solution of (8) which annuls  $J$  is a solution of  $K$ .\* Let  $\Phi$  be the system of simple forms  $A_1, \dots, A_p, F$ , and let  $\Phi$  be equivalent to the prime systems  $\Pi_1, \dots, \Pi_r; \Lambda_1, \dots, \Lambda_s$  where every  $\Pi_i$  is held by  $K$  and no  $\Lambda_i$  is held by  $K$ . It is easy to see that each  $\Lambda_i$  is held by  $\Sigma + F$  and is therefore of lower dimensionality than  $\Sigma$ ; that is, has a nonzero simple form  $G_i$  in  $u_1, \dots, u_q$ .

Set  $H = G_1 \dots G_s K$ . Then  $H$  is a nonzero simple form in  $u_1, \dots, u_q$  which holds  $\Phi$ . Consequently, there is a positive integer  $\sigma$  such that  $H^\sigma$  is a linear combination of the simple forms of  $\Phi$ . We evidently may take  $H^\sigma$  for the simple form  $G$  whose existence is to be demonstrated.

5. We can now prove the following lemma:

LEMMA 2. Let  $\Lambda$  be a prime system in the unknowns  $v_1, \dots, v_t; z_1, \dots, z_s$ , where  $v_1, \dots, v_t$  is a set of unconditioned unknowns for  $\Lambda$ , and let  $C_1, \dots, C_s$  be a basic set for  $\Lambda$ , with  $C_i$  introducing  $z_i$ , ( $i=1, \dots, s$ ). Let  $T_i$  be the separant of  $C_i$ , and let  $F$  be any form which does not hold  $\Lambda$ . Then there is a set  $\mathfrak{R}$  in  $\mathfrak{A}$  with the following properties:

(i)  $\mathfrak{A} - \mathfrak{R}$  is isolated in  $\mathfrak{A}$ .

(ii) If  $\sigma$  is any integer with  $1 \leq \sigma \leq s$ , if  $x_0$  is any point of  $\mathfrak{R}$ , and if  $a_1, \dots, a_t; b_1, \dots, b_s$  is a set of complex numbers such that  $C_i = 0$  and  $T_i \neq 0$ , ( $i=1, \dots, \sigma$ ), when  $x = x_0$ ,  $v_j = a_j$ , and  $z_k = b_k$ , ( $j=1, \dots, t; k=1, \dots, \sigma$ ), then for every  $\delta > 0$  there is a solution  $v_1(x), \dots, v_t(x); z_1(x), \dots, z_s(x)$  of  $\Lambda$ , analytic at  $x_0$ , satisfying the inequalities

$$|v_j(x_0) - a_j| < \delta, \quad |z_k(x_0) - b_k| < \delta, \quad j = 1, \dots, t; k = 1, \dots, \sigma,$$

and giving  $F$  a nonzero value at  $x_0$ .†

Let  $I_i$  be the initial of  $C_i$ , ( $i=1, \dots, s$ ), and let  $k$  be any integer with  $1 \leq k \leq s-1$ . Now  $C_1, \dots, C_k$  is a basic set for a prime system  $\Lambda_k$  which has  $v_1, \dots, v_t$  for a set of unconditioned unknowns.‡  $I_{k+1}$  does not hold  $\Lambda_k$ , since it does not hold  $\Lambda$ . By Lemma 1 there is an identity

$$(9) \quad G_k = E_{k,1}C_1 + E_{k,2}C_2 + \dots + E_{k,k}C_k + E_{k,k+1}I_{k+1},$$

\* Ritt, *Systems of algebraic differential equations*, Annals of Mathematics, (2), vol. 36 (1935), §8.

† We shall apply this lemma only for the case  $\sigma=1$ .

‡ Cf. ADE, §45.

where  $G_k$  is a nonzero simple form in  $v_1, \dots, v_t$ . Likewise, since  $T_1 \dots T_s F$  does not hold  $\Lambda$ , there is an identity

$$(10) \quad G = F_1 C_1 + F_2 C_2 + \dots + F_s C_s + F_{s+1} T_1 \dots T_s F,$$

where  $G$  is a nonzero simple form in  $v_1, \dots, v_t$ . Set  $L = I_1 G G_1 \dots G_{s-1}$ . Then  $L$  is a nonzero simple form in  $v_1, \dots, v_t$ . We now present the set  $\mathfrak{N}$ . Let  $\mathfrak{N}$  consist of all points in  $\mathfrak{A}$  at which the coefficients of the simple forms appearing in the identities (10) and (9), ( $k=1, \dots, s-1$ ), are analytic, and at which  $L$  has one or several nonzero coefficients. Evidently  $\mathfrak{A} - \mathfrak{N}$  is isolated in  $\mathfrak{A}$ . Now let  $\sigma$  be any positive integer with  $1 \leq \sigma \leq s$ , let  $x_0$  be any point of  $\mathfrak{N}$ , assume  $\delta > 0$ , and let  $a_1, \dots, a_i; b_1, \dots, b_\sigma$  be complex numbers such that  $C_i = 0$  and  $T_i \neq 0$ , ( $i=1, \dots, \sigma$ ), when  $x = x_0, v_j = a_j, z_k = b_k$ , ( $j=1, \dots, t; k=1, \dots, \sigma$ ). Since  $T_1 \dots T_\sigma$  is equal to the Jacobian  $\partial(C_1, \dots, C_\sigma) / \partial(z_1, \dots, z_\sigma)$ , there is a unique set of functions  $f_1(x, v_1, \dots, v_t), \dots, f_\sigma(x, v_1, \dots, v_t)$ , analytic near  $(x_0, a_1, \dots, a_t)$  such that  $b_i = f_i(x_0, a_1, \dots, a_t)$ , ( $i=1, \dots, \sigma$ ), and such that the substitution of  $f_i(x, v_1, \dots, v_t)$  for  $z_i$  in  $C_1, \dots, C_\sigma$  yields  $\sigma$  functions of  $x, v_1, \dots, v_t$  each of which is identically zero.

Let  $\mathfrak{R}$  be a neighborhood of  $(x_0, a_1, \dots, a_t)$  in which every  $f_i(x, v_1, \dots, v_t)$  is analytic ( $i=1, \dots, \sigma$ ) such that for every point  $(x_1, c_1, \dots, c_t)$  in  $\mathfrak{R}$  the relations

$$|c_j - a_j| < \delta, \quad |f_i(x_0, c_1, \dots, c_t) - b_i| < \delta, \quad i = 1, \dots, \sigma,$$

are valid. Let  $c_1, \dots, c_t$  be chosen so that  $(x_0, c_1, \dots, c_t)$  is a point of  $\mathfrak{R}$  at which  $L$  is not zero. Such a point exists because  $L$  has one or several coefficients different from zero at  $x_0$ . Let  $d_i = f_i(x_0, c_1, \dots, c_t)$ , ( $i=1, \dots, \sigma$ ). Then the substitution

$$(11) \quad x = x_0, v_j = c_j, z_i = d_i, \quad i = 1, \dots, \sigma; j = 1, \dots, t,$$

annuls  $C_1, \dots, C_\sigma$  but not  $G_\sigma$ , and therefore does not annul  $I_{\sigma+1}$  (by (9), with  $k=\sigma$ ). Therefore the polynomial in  $z_{\sigma+1}$  obtained from  $C_{\sigma+1}$  by the substitution (11) has at least one root  $d_{\sigma+1}$ . The substitution

$$(12) \quad x = x_0, v_j = c_j, z_i = d_i, \quad i = 1, \dots, \sigma + 1; j = 1, \dots, t,$$

annuls  $C_1, \dots, C_{\sigma+1}$  but not  $G_{\sigma+1}$ , and therefore does not annul  $I_{\sigma+2}$  (by (9), with  $k=\sigma+1$ ). Hence the polynomial in  $z_{\sigma+2}$  obtained from  $C_{\sigma+2}$  by the substitution (12) has at least one root  $d_{\sigma+2}$ .

Continuing in this manner, we obtain a set of values  $c_1, \dots, c_t; d_1, \dots, d_s$  such that the substitution

$$(13) \quad x = x_0, v_j = c_j, z_i = d_i, \quad i = 1, \dots, s; j = 1, \dots, t,$$

annuls  $C_1, \dots, C_s$  but not  $G$ , and therefore does not annul  $T_1 \dots T_s F$  (by



(10)). Since  $T_1 \cdots T_s = \partial(C_1, \dots, C_s) / \partial(z_1, \dots, z_s)$ , there is a unique set of functions  $\zeta_1(x, v_1, \dots, v_t), \dots, \zeta_s(x, v_1, \dots, v_t)$ , analytic near  $(x_0, c_1, \dots, c_t)$ , such that  $d_i = \zeta_i(x_0, c_1, \dots, c_t)$ , ( $i=1, \dots, s$ ), and such that the substitution  $z_i = \zeta_i(x, v_1, \dots, v_t)$ , ( $i=1, \dots, s$ ), transforms  $C_1, \dots, C_s$  into  $s$  functions of  $x, v_1, \dots, v_t$ , each of which is identically zero.

Let  $v_j(x) = c_j$ ,  $z_i(x) = \zeta_i(x, c_1, \dots, c_t)$ , ( $j=1, \dots, t$ ;  $i=1, \dots, s$ ). Then  $v_j(x), z_i(x)$  is evidently a solution of  $\Lambda$ , analytic at  $x_0$ , for which  $F$  is different from zero at  $x_0$ , and

$$|v_j(x_0) - a_j| < \delta, \quad |z_k(x_0) - b_k| < \delta, \quad j = 1, \dots, t; \quad k = 1, \dots, \sigma.$$

6. We now prove the following theorem:

**THEOREM 3.** *Let*

$$(14) \quad \Sigma_1, \Sigma_2, \dots$$

*be a sequence of closed irreducible systems in the unknowns  $y_1, \dots, y_n$  such that the manifold of  $\Sigma_i$  is a proper part of the manifold of  $\Sigma_{i+1}$ , ( $i=1, 2, \dots$ ). Let  $\Sigma$  be the set of all forms  $F$  such that  $F$  is in every system  $\Sigma_i$ , ( $i=1, 2, \dots$ ). Then  $\Sigma$  is a closed irreducible system having more arbitrary unknowns than  $\Sigma_i$ , ( $i=1, 2, \dots$ ).\**

$\Sigma$  is obviously closed.  $\Sigma$  is irreducible because if  $GH$  holds  $\Sigma$ , then either  $G$  holds an infinite set of the  $\Sigma_i$ , hence all the  $\Sigma_i$ , or  $H$  does; so either  $G$  is in  $\Sigma$  or  $H$  is.

Now let  $\Sigma_i$  be any system in (14). Evidently, if there is a set of unknowns in which  $\Sigma_i$  has no nonzero form, then  $\Sigma$  has no nonzero form in the unknowns of that set. Hence  $\Sigma$  has at least as many arbitrary unknowns as  $\Sigma_i$ . Now suppose that there is an  $m$  such that  $\Sigma_m$  has the same number of arbitrary unknowns as  $\Sigma$ . Then there is a set of unknowns  $y_{i_1}, y_{i_2}, \dots, y_{i_q}$  which is a set of arbitrary unknowns for  $\Sigma$ , and which is also a set of arbitrary unknowns for  $\Sigma_m$ . Now  $\Sigma_j$ , ( $j \geq m$ ), has no nonzero form in  $y_{i_1}, \dots, y_{i_q}$  because  $\Sigma_m$  has no such form. But  $\Sigma_j$  has not more than  $q$  arbitrary unknowns, because  $\Sigma$  has  $q$  arbitrary unknowns. Hence  $y_{i_1}, \dots, y_{i_q}$  is a set of arbitrary unknowns for every  $\Sigma_j$ , ( $j \geq m$ ). Taking  $y_{i_1}, \dots, y_{i_q}$  as a set of arbitrary unknowns for  $\Sigma$  and for each  $\Sigma_j$ , ( $j \geq m$ ), we introduce a resolvent for  $\Sigma$  and for each  $\Sigma_j$  (adjoining  $x$  to  $\mathfrak{F}$  if necessary) and we let  $\rho, \rho_j$  be the orders of the resolvents of  $\Sigma, \Sigma_j$ , respectively ( $j \geq m$ ). By a theorem of E. Gourin,† since  $\Sigma$  has the same

\* For a closed irreducible system  $\Lambda$ , the number of unknowns in a set of arbitrary unknowns for  $\Lambda$  is independent of the manner in which the set is chosen (ADE, §30). This number we call the *number of arbitrary unknowns in  $\Lambda$* .

† E. Gourin, *On irreducible systems of algebraic differential equations*, Bulletin of the American Mathematical Society, vol. 39 (1933), pp. 593-595.

set of arbitrary unknowns as each of the  $\Sigma_i$ , we have  $\rho_m < \rho_{m+1} < \dots < \rho$ , which is clearly impossible.

COROLLARY.  $\Sigma$  has a non-empty set of arbitrary unknowns.

7. We now make the following definition:

DEFINITION. Let  $n$  be any positive integer. Let  $f_1(x), \dots, f_n(x)$  be an ordered set of  $n$  functions analytic in an open region  $\mathfrak{B}$ . Let  $\mathcal{N}$  be a set each of whose elements is an ordered set of  $n$  functions which have a region of analyticity in common. Then if  $m$  is a positive integer, we shall say that a point  $x_0$  of  $\mathfrak{B}$  is a point of  $m$ th order contact between the set  $f_1(x), \dots, f_n(x)$  and the set of sets  $\mathcal{N}$  if for every  $\epsilon > 0$  there is a set  $y_1(x), \dots, y_n(x)$  in  $\mathcal{N}$ , analytic at  $x_0$ , such that

$$|y_{ij}(x_0) - f_{ij}(x_0)| < \epsilon, \quad i = 1, \dots, n; j = 0, 1, \dots, m.$$

If  $x_0$  is a point of  $m$ th order contact between  $f_1(x), \dots, f_n(x)$  and  $\mathcal{N}$  for every  $m$ , then we shall say simply that  $x_0$  is a point of contact between  $f_1(x), \dots, f_n(x)$  and  $\mathcal{N}$ .

8. We can now state the following theorem:

THEOREM 4. Let  $\Sigma_1, \Sigma_2, \dots$ , and  $\Sigma$  be as in the hypothesis of Theorem 3. Let  $\mathcal{M}_i$  be the manifold of  $\Sigma_i$ , ( $i = 1, 2, \dots$ ), and let  $\mathcal{N}$  be the set-theoretic sum  $\mathcal{M}_1 + \mathcal{M}_2 + \dots$ . If  $f_1(x), \dots, f_n(x)$  is an ordered set of  $n$  functions analytic in an open region  $\mathfrak{B}$ , then a necessary and sufficient condition for  $f_1(x), \dots, f_n(x)$  to be a solution of  $\Sigma$  is that  $\mathfrak{B}$  contain a point of contact between  $f_1, \dots, f_n$  and  $\mathcal{N}$ .

Sufficiency proof. Let

$$(15) \quad f_1(x), \dots, f_n(x)$$

be an ordered set of  $n$  functions analytic in an open region  $\mathfrak{B}$  which contains a point  $x_0$  of contact between (15) and  $\mathcal{N}$ . We shall prove that (15) is a solution of  $\Sigma$ .

If  $H$  is a form in  $\Sigma$  whose coefficients are analytic at  $x_0$ , then  $H$ , considered as a function of  $x$  and the letters appearing in  $H$ , is continuous when  $x$  is near  $x_0$ .

Let  $m$  be a positive integer greater than the order of  $H$  in  $y_i$ , ( $i = 1, \dots, n$ ). Assume  $\epsilon > 0$ . Let

$$(16) \quad y_1(x), \dots, y_n(x)$$

be a solution in  $\mathcal{N}$ , analytic at  $x_0$ , such that

$$(17) \quad |f_{ij}(x_0) - y_{ij}(x_0)| < \epsilon, \quad i = 1, \dots, n; j = 0, 1, \dots, m.$$

---

\* The second subscript is an index of differentiation.

When we substitute (15) and (16) in  $H$ , we obtain functions  $h(x)$  and  $k(x)$ , respectively, which are analytic near  $x_0$ . Evidently  $k(x) \equiv 0$ , since (16) is in  $\mathcal{N}$  and is therefore a solution of  $\Sigma$ . In particular  $k(x_0) = 0$ . Now  $\epsilon$  is arbitrarily small; consequently, the relations (17) and the continuity of  $H$  imply that  $|k(x_0) - h(x_0)|$  is arbitrarily small. Thus  $|h(x_0)|$  is arbitrarily small, so that  $h(x_0) = 0$ . Now  $h'(x_0)$  may be obtained by substituting (15) in  $H'$  and putting  $x = x_0$ .\* Hence, by the preceding argument,  $h'(x_0) = 0$ . Continuing in this manner we prove that every derivative of  $h(x)$  vanishes at  $x_0$ . This means  $h(x) \equiv 0$ . Hence (15) is a solution of  $H$ , where  $H$  is any form of  $\Sigma$  whose coefficients are analytic at  $x_0$ . But if  $G$  is any form of  $\Sigma$ , the product of  $G$  by a suitable nonzero function  $\psi(x)$  in  $\mathcal{F}$  is a form  $H$  of  $\Sigma$  with coefficients analytic at  $x_0$ .†

Since (15) annuls  $H$ , it annuls  $G$ . This proves that (15) is a solution of  $\Sigma$ .‡

**Necessity proof.** The necessity of the condition is implied by Theorem 6, below.

9. The next theorem is as follows:

**THEOREM 5.** *Let the notations  $\Sigma_i$ ,  $\mathcal{M}_i$ , and  $\Sigma$  have the same significance as in the hypothesis of Theorem 4. Then there exists a function  $b = b(m)$ , defined on the positive integers, and assuming positive integral values, such that for every solution  $y_1(x), \dots, y_n(x)$  of  $\Sigma$ , analytic in an open region  $\mathfrak{B}$ , and every positive integer  $m$ , the set of points of  $m$ th order contact between  $y_1(x), \dots, y_n(x)$  and  $\mathcal{M}_{b(m)}$  is a set whose complement in  $\mathfrak{B}$  is isolated in  $\mathfrak{B}$ .*

Let  $u_1, \dots, u_q$  be a set of arbitrary unknowns for  $\Sigma$ . If  $\Sigma$  is non-trivial, we introduce a resolvent for  $\Sigma$  with a new unknown  $\omega$  satisfying  $\omega - Q = 0$ .§ If  $\Sigma$  is trivial (that is, if  $\Sigma$  has no nonzero forms) we introduce a new unknown  $\omega$  satisfying  $\omega - Q = 0$ , where  $Q \equiv 0$ .

In either case let  $\Omega$  be the set of forms holding  $\Sigma + (\omega - Q)$ , and let  $\Omega_i$  be the set of forms holding  $\Sigma_i + (\omega - Q)$ , ( $i = 1, 2, \dots$ ). Then  $\Omega_{i+1}$  holds  $\Omega_i$ ,  $\Omega$  is closed and irreducible,|| and  $\Omega$  is the set of all forms  $F$  such that  $F$  is in every system  $\Omega_i$ , ( $i = 1, 2, \dots$ ).

Let  $y_1, \dots, y_p$  be the unknowns in  $\Omega$  other than  $u_1, \dots, u_q; \omega$ .¶ Let the unknowns be ordered  $u_1, \dots, u_q; \omega; y_1, \dots, y_p$ , and let

$$(18) \quad R, A_1, \dots, A_p$$

\* Superscripts indicate differentiation.

† For if  $G$  has a coefficient  $\phi(x)$  with a pole at  $x_0$ , then the reciprocal of  $\phi(x)$  will have a zero at  $x_0$ , and a suitable power of that reciprocal will serve as  $\psi(x)$ .

‡ This proof is similar to a proof given by Ritt for a different theorem, ADE, §72.

§ ADE, §§25-29.

||  $\Omega_i$  is also closed and irreducible.

¶ We renumber the unknowns if necessary.

be a corresponding basic set for  $\Omega$ . Then  $A_i$  is of zero order in  $y_i$  and is linear in  $y_{\alpha}$ , ( $i=1, \dots, p$ ).\*

Let  $h$  be the order of  $R$  in  $\omega$ . We assert that if  $a$  is any positive integer, then there is an integer  $b$  depending upon  $a$  such that the system  $\Omega_b$  has no nonzero form in the letters

$$(19) \quad u_{\alpha\beta}, \omega_\gamma, \quad \alpha = 1, \dots, q; \beta = 0, 1, \dots, a; \gamma = 0, 1, \dots, h - 1.$$

For let us assume that this assertion is false. Then there is an  $a$  such that every  $\Omega_i$  has a nonzero form in the letters (19). From each  $\Omega_i$  let a nonzero form  $F_i$  in the letters (19) be selected which is of minimum rank. Without loss of generality we may assume that  $F_i$  is algebraically irreducible ( $i=1, 2, \dots$ ). Since  $\Omega_2$  must have solutions, each  $F_i$ , ( $i>2$ ), involves unknowns.

Since the totality of letters involved in the  $F_i$  is a finite set, there is an infinite subset of the  $F_i$  such that if  $F_k$  and  $F_l$  are two forms in the subset, then  $F_k$  and  $F_l$  have the same order in  $u_\alpha$ , ( $\alpha=1, \dots, q$ ), and the same order in  $\omega$ . We assert that the quotient of any two forms in this subset is a (nonzero) function in  $\mathcal{F}$ . For if  $F_k$  and  $F_l$ , ( $k < l$ ), are relatively prime, then the resultant  $G$  of  $F_k$  and  $F_l$ , with respect to the highest letter in  $F_k$  and  $F_l$ , is a nonzero form free of that letter. Then  $\Omega_k$  has the form  $G$  in the letters (19).† But  $G$  is lower than  $F_k$ . This contradiction with the minimal property of  $F_k$  proves that there is an infinite set of the  $F_i$ , each of which is the product of a fixed  $F_k$  by a nonzero function in  $\mathcal{F}$ . Then this  $F_k$  is in all the  $\Omega_i$ , and therefore in  $\Omega$ , although it is lower than  $R$ . This contradiction proves that for every  $a$  there is a  $b$  such that  $\Omega_b$  has no nonzero form in the letters (19).

Now let  $S, S_i$  be the separant of  $R, A_i$ , ( $i=1, \dots, p$ ), respectively, and define  $K_1 \equiv SS_1 \dots S_p$ . Discarding a finite set of the  $\Omega_i$ , if necessary, we assume that  $K_1$  holds no  $\Omega_i$ , since  $K_1$  is not in  $\Omega$ .

Let  $g$  be a positive integer greater than the maximum order of each form of (18) in each unknown.

Let  $m$  be any positive integer, to be fixed throughout the remainder of this proof. Let  $a = m + g$ , and let  $b = b(m)$  be the smallest positive integer such that  $\Omega_b$  has no nonzero form in the letters (19).

We take any set of arbitrary unknowns for  $\Omega_b$ , order the remaining unknowns in any fashion, and let

$$(20) \quad B_1, B_2, \dots, B_r$$

be a corresponding basic set for  $\Omega_b$ .

\* Since either  $R$  is a resolvent for  $\Sigma$ , or  $\Sigma$  is trivial. In the latter case (18) is simply the form  $R$ .

† Since  $G$  is a linear combination of  $F_k$  and  $F_l$ , each of which is in  $\Omega_k$ .

Let  $\tau, \tau_i, \sigma_j$  be the orders of the highest derivatives of  $\omega, y_i, u_j$ , ( $i = 1, \dots, p$ ;  $j = 1, \dots, q$ ), respectively, appearing in (20).\* Let  $K_2$  be the product of the separants in (20). Let  $\Lambda$  be the set of all simple forms that vanish for all solutions of the system

$$(21) \quad B_{ij}, \quad i = 1, \dots, r; j = 0, 1, \dots, a, \dagger$$

for which  $K_2 \neq 0$ , where the forms (21) are to be considered as simple forms in the unknowns

$$(22) \quad \begin{matrix} u_{ij}, \omega_k, \gamma_{\mu\nu}, & j = 0, 1, \dots, a + \sigma_i; i = 1, \dots, q; \\ k = 0, 1, \dots, a + \tau; & \nu = 0, 1, \dots, a + \tau_\mu; \mu = 1, \dots, p. \end{matrix}$$

It is easy to see that  $\Lambda$  is prime, that every simple form which holds  $\Lambda$ , when considered as a form in the unknowns  $u_1, \dots, u_q; \omega; y_1, \dots, y_p$ , and their derivatives, will hold  $\Omega_b$ , and that every form of  $\Omega_b$  in the letters (22), when considered as a simple form in those letters, will hold  $\Lambda$ .‡

Since every form in  $\Lambda$  is in  $\Omega_b$ , there is no nonzero form in  $\Lambda$  in the letters (19). Renaming the letter (22), let

$$(23) \quad v_1, \dots, v_t$$

be a set of unconditioned unknowns for  $\Lambda$ , and let

$$(24) \quad z_1, \dots, z_s$$

be the other unknowns in  $\Lambda$ . We may and do choose (23), (24) so that (23) includes (19) and also so that  $z_1 = \omega_h$ , the latter being possible because  $R$  is in  $\Lambda$ .

With the unknowns ordered  $v_1, \dots, v_t; z_1, \dots, z_s$ , let

$$(25) \quad C_1, C_2, \dots, C_s$$

be a basic set for  $\Lambda$ . Then  $R$  can be taken for  $C_1$ . For if  $F$  were a simple form of  $\Lambda$  in the unknowns  $v_1, \dots, v_t; \omega_h$ , of lower degree than  $R$  in  $\omega_h$ , then the resultant of  $R$  and  $F$  with respect to  $\omega_h$  would be a nonzero simple form of  $\Lambda$  in the letters (23),§ although (23) is a set of unconditioned unknowns. We shall assume that  $C_1 = R$ .

\* If  $\omega, y_i$ , or  $u_j$  does not appear in (20), then  $\tau, \tau_i$ , or  $\sigma_j$ , respectively, is to be taken as zero.

† The second subscript is an index of differentiation.

‡ If  $F$  holds  $\Lambda$ , then  $K_2 F$  vanishes for every solution of  $\Omega_b$ , since such a solution either annuls  $K_2$  or yields a solution of (21) for which  $K_2 \neq 0$ ; hence  $F$  holds  $\Omega_b$ , since  $K_2$  does not. Conversely, if  $F$  holds  $\Omega_b$ , then  $F$  vanishes for every solution of (21) for which  $K_2 \neq 0$ , since such a solution provides, at every point where the coefficients in (20) are analytic and  $K_2 \neq 0$ , initial conditions for a normal solution of (20). If  $FG$  holds  $\Lambda$ , then it holds  $\Omega_b$ ; so either  $F$  holds  $\Omega_b$ , hence  $\Lambda$ , or  $G$  does. Thus  $\Lambda$  is indecomposable.  $\Lambda$  is obviously simply closed.

§  $R$  is algebraically irreducible as a polynomial in  $\omega_h$ , in the field  $\mathcal{F}(v_1, \dots, v_t)$ . Cf. ADE, §§65, 45.

We note that  $K_2$  does not hold  $\Lambda$ .

Taking  $K_2$  for the form  $F$  in the hypothesis of Lemma 2, we let  $\mathfrak{N}$  be the corresponding point set in  $\mathfrak{X}$  with the properties (i), (ii) of the lemma.

Let  $\mathfrak{N}_m$  be the set of points in  $\mathfrak{N}$  at which the coefficients of the forms in (18) and (20) are analytic. Evidently  $\mathfrak{X} - \mathfrak{N}_m$  is isolated in  $\mathfrak{X}$ .

Let  $x_0$  be a point of  $\mathfrak{N}_m$ , and let

$$(26) \quad \bar{u}_1(x), \dots, \bar{u}_q(x); \bar{\omega}(x); \bar{y}_1(x), \dots, \bar{y}_p(x)$$

be a normal solution of (18), analytic at  $x_0$  and giving  $K_1$  a nonzero value at  $x_0$ . We shall prove that  $x_0$  is a point of  $m$ th order contact between (26) and the manifold of  $\Omega_b$ . Assume

$$(27) \quad R_k \equiv S\omega_{h+k} + V_k, \quad A_{ij} \equiv S_i y_{ij} + T_{ij}, \\ k = 1, 2, \dots, m; i = 1, 2, \dots, p; j = 0, 1, \dots, m.*$$

Then  $V_k$  is of order less than  $h+k$  in  $\omega$ , and  $T_{ij}$  is of order less than  $j$  in  $y_i$ . Evidently the equations

$$(28) \quad R_k = 0, A_{ij} = 0, \quad k = 1, 2, \dots, m; i = 1, \dots, p; j = 0, 1, \dots, m,$$

define  $\omega_{h+k}, y_{ij}$  recursively as functions of  $x, \omega_h$ , and the letters (19), continuous near  $(x_0, \bar{u}_{\alpha\beta}(x_0), \bar{\omega}_\gamma(x_0), \bar{\omega}_h(x_0)), (\alpha=1, \dots, q; \beta=0, 1, \dots, a; \gamma=0, 1, \dots, h-1)$ .†

Assume  $\epsilon > 0$ . Then there is a  $\delta > 0$  such that if  $\bar{u}_1(x), \dots, \bar{u}_q(x); \bar{\omega}(x); \bar{y}_1(x), \dots, \bar{y}_p(x)$  is a solution of  $\Omega_b$  with

$$(29) \quad |\bar{u}_{\alpha\beta}(x_0) - \bar{u}_{\alpha\beta}(x_0)| < \delta, \quad |\bar{\omega}_\gamma(x_0) - \bar{\omega}_\gamma(x_0)| < \delta, \quad |\bar{\omega}_h(x_0) - \bar{\omega}_h(x_0)| < \delta,$$

then

$$(30) \quad |\bar{u}_{jk}(x_0) - \bar{u}_{jk}(x_0)| < \epsilon, \quad |\bar{\omega}_k(x_0) - \bar{\omega}_k(x_0)| < \epsilon, \\ |\bar{y}_{ik}(x_0) - \bar{y}_{ik}(x_0)| < \epsilon, \\ j = 1, \dots, q; i = 1, \dots, p; k = 0, 1, \dots, m.$$

This results from the fact that every solution of  $\Omega$ , and therefore also every solution of  $\Omega_b$ , must satisfy the equations (28).

Now each  $v_j, (j=1, \dots, t)$ , corresponds to one of the letters (22); let  $a_j$  be the value at  $x_0$  assigned to that letter by (26). Let  $b_1 = \bar{\omega}_h(x_0)$ .

Then  $C_1, (C_1=R)$ , vanishes under the substitution  $x=x_0, v_j=a_j, \omega_h=b_1, (j=1, \dots, t)$ , and  $S$  does not; thus, since  $x_0$  is in  $\mathfrak{N}$ , there is a solution  $v_j(x)$ ,

\*  $R_k$  is the  $k$ th derivative of  $R$ , and  $A_{ij}$  is the  $j$ th derivative of  $A_i$ .

† Henceforth, whenever  $\alpha, \beta, \gamma$  appear as subscripts, we shall understand that their ranges are the ones given here.

$z_i(x)$ , ( $i=1, \dots, s; j=1, \dots, t$ ), of  $\Lambda$ , analytic at  $x_0$ , for which  $|v_j(x_0) - a_j| < \delta$ ,  $|z_1(x_0) - b_1| < \delta$ , ( $j=1, \dots, t$ ), and for which  $K_2$  is different from zero at  $x_0$ .

Evidently this solution of  $\Lambda$  provides initial conditions at  $x_0$  for a normal solution  $\bar{u}_1, \dots, \bar{y}_p$  of (20) which satisfies (29). The inequalities (30) are valid for this solution of  $\Omega_b$ . Therefore  $x_0$  is a point of  $m$ th order contact between (26) and the manifold of  $\Omega_b$ .

Now let

$$(31) \quad u_1(x), \dots, u_q(x); y_1(x), \dots, y_p(x)$$

be a solution of  $\Sigma$ , analytic in an open region  $\mathfrak{B}$ . Corresponding to (31) is a solution

$$(32) \quad u_1(x), \dots, u_q(x); \omega(x); y_1(x), \dots, y_p(x)$$

of  $\Omega$ , analytic in  $\mathfrak{B}$ . According to the lemma of §2 there is a set  $\mathfrak{B}_m$  whose complement in  $\mathfrak{B}$  is isolated in  $\mathfrak{B}$ , such that every point  $x_0$  in  $\mathfrak{B}_m$  is a point of  $m$ th order contact between (32) and the set of those solutions of  $\Omega$  which give  $K_1$  a nonzero value at  $x_0$ . Let  $\mathfrak{R}_m = \mathfrak{B}_m \cdot \mathfrak{N}_m$ . Then for every  $\epsilon > 0$  and every point  $x_0$  in  $\mathfrak{R}_m$  there is a solution

$$(33) \quad \bar{u}_1(x), \dots, \bar{u}_q(x); \bar{\omega}(x); \bar{y}_1(x), \dots, \bar{y}_p(x)$$

of  $\Omega$ , analytic at  $x_0$ , for which

$$(34) \quad \begin{aligned} |\bar{u}_{jk}(x_0) - u_{jk}(x_0)| &< \epsilon, & |\bar{\omega}_k(x_0) - \omega_k(x_0)| &< \epsilon, \\ |\bar{y}_{ik}(x_0) - y_{ik}(x_0)| &< \epsilon, \\ i &= 1, \dots, p; j = 1, \dots, q; k = 0, 1, \dots, m, \end{aligned}$$

and for which  $K_1 \neq 0$  at  $x_0$ ; and then there is a solution

$$(35) \quad \bar{u}_1(x), \dots, \bar{u}_q(x); \bar{\omega}(x); \bar{y}_1(x), \dots, \bar{y}_p(x)$$

of  $\Omega_b$  for which (30) holds. By (30) and (34) we have in particular

$$(36) \quad \begin{aligned} |\bar{u}_{jk}(x_0) - u_{jk}(x_0)| &< 2\epsilon, & |\bar{y}_{ik}(x_0) - y_{ik}(x_0)| &< 2\epsilon, \\ i &= 1, \dots, q; j = 1, \dots, p; k = 0, 1, \dots, m. \end{aligned}$$

Consequently every point of  $\mathfrak{R}_m$  is a point of  $m$ th order contact between (31) and  $\mathfrak{N}_b$ . Since  $\mathfrak{B} - \mathfrak{R}_m$  is evidently isolated in  $\mathfrak{B}$ , we have our theorem.

10. As a consequence of Theorem 5 we have the following theorem:

**THEOREM 6.** *Let the notations  $\Sigma_i, \mathfrak{N}_i, \Sigma, \mathfrak{N}$  have the same significance as in Theorem 4. Then the open region  $\mathfrak{A}$  contains a subset  $\mathfrak{B}$  whose complement in  $\mathfrak{A}$  is at most denumerably infinite, such that if  $x_0$  is a point in  $\mathfrak{B}$ , and  $\bar{y}_1, \dots, \bar{y}_n$  is a solution of  $\Sigma$ , analytic at  $x_0$ , then  $x_0$  is a point of contact between  $\bar{y}_1, \dots, \bar{y}_n$  and  $\mathfrak{N}$ .*

We prove Theorem 6 by using Theorem 5 in the same manner in which the lemma of §2 was used in proving Theorem 2. We simply replace the concept of a solution of  $\Sigma$  for which  $B$  is different from zero at  $x_0$ , by that of a solution in  $\mathcal{N}$ .

11. Extending this result in a special case, we assert that when  $\mathcal{F}$  consists purely of constants, then for every solution  $y_1(x), \dots, y_n(x)$  of  $\Sigma$ , analytic in an open region  $\mathfrak{B}$ , every point of  $\mathfrak{B}$  is a point of contact between  $y_1(x), \dots, y_n(x)$  and  $\mathcal{N}$ .

For every point  $x_1$  in  $\mathfrak{B}$ , every positive integer  $m$ , every  $\epsilon > 0$ , and every  $\delta > 0$ , there is a solution  $\bar{y}_1(x), \dots, \bar{y}_n(x)$  in  $\mathcal{N}$ , analytic at a point  $x_0$  in  $\mathfrak{B}$ , with  $|x_1 - x_0| < \delta$ , and with

$$|\bar{y}_{ij}(x_0) - y_{ij}(x_0)| < \epsilon, \quad i = 1, \dots, n; j = 0, 1, \dots, m.$$

We assume that  $\delta$  is sufficiently small so that

$$|y_{ij}(x_1) - y_{ij}(x_0)| < \epsilon, \quad i = 1, \dots, n; j = 0, 1, \dots, m.$$

Then  $|\bar{y}_{ij}(x_0) - y_{ij}(x_1)| < 2\epsilon$ , and this may be written  $|z_{ij}(x_1) - y_{ij}(x_1)| < 2\epsilon$ , where

$$z_i(x) \equiv \bar{y}_i(x + x_0 - x_1), \quad i = 1, \dots, n; j = 0, 1, \dots, m.$$

But  $z_1(x), \dots, z_n(x)$  is in  $\mathcal{N}$ , since  $\mathcal{F}$  consists purely of constants. Therefore  $x_1$  is a point of contact between  $y_1(x), \dots, y_n(x)$  and  $\mathcal{N}$ .

PART III. EXAMPLES

12. We give an example of a sequence  $\Sigma_1, \Sigma_2, \dots$  and a solution  $y_1(x), \dots, y_n(x)$  of  $\Sigma$ , analytic in an open region  $\mathfrak{B}$ , such that the set of those points in  $\mathfrak{B}$  which are not points of contact between  $y_1(x), \dots, y_n(x)$  and  $\mathcal{N}$  is dense in  $\mathfrak{B}$ . From §11 we know that in such an example there must be functions in  $\mathcal{F}$  which are not constants.

Let  $\mathcal{F}$  consist of all rational functions of  $x$ . Let

$$(37) \quad a_1, a_2, \dots$$

be a sequence of points dense in the complex plane. Let  $\Sigma_1, \Sigma_2, \dots$  be the closed systems in one unknown  $y$  such that the manifold of  $\Sigma_n, (n = 1, 2, \dots)$ , is the family of functions

$$(38) \quad y = \frac{c_1}{x - a_1} + \frac{c_2}{(x - a_1)^2(x - a_2)} + \dots + \frac{c_n}{(x - a_1)^n(x - a_2)^{n-1} \dots (x - a_n)},$$

where the  $c_i$  are arbitrary constants.



Then it is easy to see that the manifold of  $\Sigma_n$  is identical with the manifold of a linear differential equation in  $y$ , with coefficients in  $\mathcal{Y}$ . This equation affords a basic set for  $\Sigma_n$ ; therefore  $\Sigma_n$  is irreducible.\* Obviously the manifold of  $\Sigma_n$  is a proper part of the manifold of  $\Sigma_{n+1}$ .

Now  $\Sigma$  must have a non-empty set of arbitrary unknowns, as we have seen, so that  $y$  is a set of arbitrary unknowns for  $\Sigma$ . In other words, there is no nonzero form in  $\Sigma$ ; so every analytic function is a solution of  $\Sigma$ . Let  $\mathcal{N}$  be the union of the manifolds of the  $\Sigma_i$ ; that is, let  $\mathcal{N}$  be the union of the families of functions (38), ( $n=1, 2, \dots$ ). Let  $f(x)$  be a function which is analytic in an open region  $\mathfrak{B}$  and which is not in  $\mathcal{N}$ . Then the set of those points of the sequence (37) that lie in  $\mathfrak{B}$  is dense in  $\mathfrak{B}$ . No point in (37) is a point of contact between  $f(x)$  and  $\mathcal{N}$ , since for every positive integer  $l$  the only functions in  $\mathcal{N}$  which are analytic at the point  $a_l$  are the functions which are in the family (38) when  $n=l-1$ . Hence the complement in  $\mathfrak{B}$  of the set of points of contact between  $f(x)$  and  $\mathcal{N}$  is dense in  $\mathfrak{B}$ .

We note that there is no open subregion  $\mathfrak{B}_1$  of  $\mathfrak{B}$  in which  $f(x)$  may be approximated uniformly, with arbitrary closeness, by a solution in  $\mathcal{N}$ . For if such an open region  $\mathfrak{B}_1$  exists, then every point of  $\mathfrak{B}_1$  is a point of contact between  $f(x)$  and  $\mathcal{N}$ .

13. The phenomenon exemplified in the preceding section is in marked contrast with that appearing in the following example:

Let  $\Sigma_k$  be the closed irreducible† system in the unknown  $y$  with a basic set  $y_k$ .‡

$\Sigma$  is trivial as in the preceding example. Here, however, if  $f(x)$  is any function, analytic in an open region  $\mathfrak{B}$ , then every point of  $\mathfrak{B}$  has a neighborhood in which  $f(x)$  may be uniformly approximated by solutions of the  $\Sigma_i$ . In short, every polynomial is a solution of some system  $\Sigma_i$ .

14. In the example of §12, for certain solutions of  $\Sigma$  there existed no region in which uniform approximation by solutions of the  $\Sigma_i$  was possible. Each solution of  $\Sigma$  having this property had the additional property that every subregion of its domain of analyticity contained a point which was not a point of contact between the given solution and the union of the manifolds of the  $\Sigma_i$ . This second property of course implies the first. We shall prove that the converse is not true. That is, we shall exhibit a sequence  $\Sigma_1, \Sigma_2, \dots$  and a solution of the corresponding system  $\Sigma$ , such that every point in the domain of analyticity of the given solution is a point of contact between the solution and the union of the manifolds of the  $\Sigma_i$ , while on the other hand, there is

\* Cf. ADE, §§65, 45.

† The existence of such a  $\Sigma_k$  follows from two theorems of Ritt, ADE, §§65, 45.

‡ Subscripts indicate differentiation.

no open region in which the solution can be uniformly approximated by solutions of the  $\Sigma_i$ .

Let

$$(39) \quad \alpha_1, \alpha_2, \dots$$

be a sequence of complex constants. In terms of the sequence (39) we define a sequence of operators

$$(40) \quad \theta_1, \theta_2, \dots$$

as follows:

$$(41) \quad \theta_i g(x) = g(x)(g'(x) + \alpha_i g(x)), \theta_i A = A(A' + \alpha_i A), \quad i = 1, 2, \dots,$$

for every analytic function  $g(x)$  and every form  $A$ . Set  $\phi_k = \theta_k \theta_{k-1} \dots \theta_2 \theta_1$ , ( $k = 1, 2, \dots$ ). Let

$$(42) \quad \Sigma_1, \Sigma_2, \dots$$

be the sequence of closed systems such that the manifold of  $\Sigma_n$ , ( $n = 1, 2, \dots$ ), is the family of all functions  $y(x)$  which satisfy the equation

$$(43) \quad \frac{d}{dx} (\phi_n y(x)) = 0.$$

Set  $A_k \equiv \phi_k y$ , set  $B_k \equiv A_k'$ , and let  $S_k$  be the separant of  $B_k$ , ( $k = 1, 2, \dots$ ). Evidently the manifold of  $B_{i+1}$  includes the manifold of  $B_i$ , so that  $\Sigma_{i+1}$  holds  $\Sigma_i$ , ( $i = 1, 2, \dots$ ). Since  $S_i S_{i+1}$  does not hold  $B_i$ , the general solution of  $B_{i+1}$  includes the general solution of  $B_i$ .

We shall prove that  $\Sigma_n$  is irreducible. This is equivalent to proving that the manifold of  $B_n$  is identical with the general solution of  $B_n$ . It suffices to prove that the manifold of  $S_n$  is in the general solution of  $B_n$ . This last is easy to see when  $n = 1$ . We assume that it is true when  $n = r$ . Then the manifold of  $B_r$  is identical with the general solution of  $B_r$ . Now  $S_{r+1} = A_r S_r$ . Hence every solution of  $S_{r+1}$  is in the general solution of  $B_r$ , and consequently in the general solution of  $B_{r+1}$ .

Thus  $\Sigma_n$ , ( $n = 1, 2, \dots$ ), is irreducible. As in §12, the system  $\Sigma$  is the trivial system of which every analytic function is a solution.

Let  $\mathcal{M}_i$  be the manifold of  $\Sigma_i$ , ( $i = 1, 2, \dots$ ), and let  $\mathcal{N} = \mathcal{M}_1 + \mathcal{M}_2 + \dots$ .

Let  $y(x)$  be an analytic function such that for some  $k$  the function  $\phi_k y(x)$  is not identically zero, and has a zero at a point  $x_0$ . Then  $y(x)$  is not in  $\mathcal{N}$ . For suppose  $y(x)$  is in  $\mathcal{N}$ . There exists an  $n$  such that  $y(x)$  satisfies (43). We may and do assume that  $n > k$ . Let  $x_0$  be a zero of order  $\sigma$  for  $\phi_k y(x)$ . Then it is easy to see that  $x_0$  is a zero of order  $2^{n-k}(\sigma - 1) + 1$  for  $\phi_n y(x)$ . But this is

impossible, since  $\phi_n y(x)$  is a constant, by equation (43). Hence  $y(x)$  is not in  $\mathcal{N}$ .

Let  $f(x)$  be a polynomial of positive degree. Then for every choice of a sequence (39) the function  $\phi_k f(x)$  is a polynomial of positive degree ( $k=1, 2, \dots$ ). Let  $\mathfrak{R}_1, \mathfrak{R}_2, \dots$  be a sequence of open sets such that for every open region  $\mathfrak{B}$  there is a  $k$  such that  $\mathfrak{R}_k$  is included in  $\mathfrak{B}$ . We shall choose a sequence (39) in such a way that for every  $k$  the function  $\phi_k f(x)$  has a zero in  $\mathfrak{R}_k$ .

We take a point  $b_1$  in  $\mathfrak{R}_1$  such that  $f(b_1) \neq 0$ , and define  $\alpha_1 = -f'(b_1)/f(b_1)$ . Then  $\phi_1 f(x)$  has the zero  $b_1$  in  $\mathfrak{R}_1$ . When  $\alpha_1, \alpha_2, \dots, \alpha_{n-1}$  have been chosen so that  $\phi_k f(x)$  has a zero  $b_k$  in  $\mathfrak{R}_k$ , ( $k=1, 2, \dots, n-1$ ), we take a point  $b_n$  in  $\mathfrak{R}_n$  at which  $\phi_{n-1} f(x)$  is not zero, and, letting  $g(x) \equiv \phi_{n-1} f(x)$ , we define  $\alpha_n = -g'(b_n)/g(b_n)$ . Then  $\phi_n f(x)$  has the zero  $b_n$  in  $\mathfrak{R}_n$ . Proceeding in this manner we determine a sequence of points

$$(44) \quad b_1, b_2, \dots,$$

dense in the complex plane, and a choice of the sequence (39), such that the point  $b_k$  is a zero of  $\phi_k f(x)$ , ( $k=1, 2, \dots$ ). We now retain this choice of (39).

Now suppose that there exists an open region  $\mathfrak{B}_1$  in which  $f(x)$  can be approximated uniformly by solutions of the  $\Sigma_i$ . Let  $y_1(x), y_2(x), \dots$  be a sequence of functions in  $\mathcal{N}$ , converging to  $f(x)$  uniformly in  $\mathfrak{B}_1$ . Then for each  $k$  the sequence  $\phi_k y_1(x), \phi_k y_2(x), \dots$  converges to  $\phi_k f(x)$  uniformly in  $\mathfrak{B}_1$ . Let  $k$  be such that  $\phi_k f(x)$  has a zero in  $\mathfrak{B}_1$ .\* Then there is an  $m$  such that  $\phi_k y_m(x)$  is not identically zero and has a zero in  $\mathfrak{B}_1$ . We have seen that this implies that  $y_m(x)$  is not in  $\mathcal{N}$ . This contradiction implies that there is no open region in which  $f(x)$  can be approximated uniformly by solutions of the  $\Sigma_i$ . On the other hand, every point of the complex plane is a point of contact between  $f(x)$  and  $\mathcal{N}$ , since  $\mathcal{F}$  is a field of constants.

Another such example can be constructed as follows: Let  $\theta$  be the operator such that

$$\theta g(x) = g(x)[2(g'(x))^3 - 3g(x)g'(x)g''(x) + (g(x))^2g'''(x)],$$

for every analytic function  $g(x)$ . Let  $\Sigma_1, \Sigma_2, \dots$  be the sequence of closed systems such that the manifold of  $\Sigma_n$ , ( $n=1, 2, \dots$ ), is the family of all functions  $y(x)$  which satisfy the equation

$$(45) \quad \frac{d}{dx} (\theta^n y(x)) = 0.$$

Arguments similar to those used in the preceding example show that  $\Sigma_i$  is

\* We are using here the fact that the sequence (44) is dense in the complex plane.

irreducible and is held by  $\Sigma_{i+1}$ , ( $i=1, 2, \dots$ ), and also show that if  $y(x)$  is a function such that  $\theta^k y(x)$  has a simple zero for some  $k$ , then  $y(x)$  is not in  $\mathcal{N}$ .

Let  $\omega_1, \omega_2$  be any two complex numbers whose ratio is not real, and let  $\sigma(x)$  be the corresponding  $\sigma$ -function of Weierstrass.\* Using a functional equation of  $\sigma(x)$ ,† we find that

$$\theta^k \sigma(x) = 2^{3k(k-1)/2} \sigma(2^k x), \quad k = 1, 2, \dots$$

Every circle which is of diameter greater than  $|\omega_1| + |\omega_2|$  contains a simple zero of  $\sigma(x)$ , so that every circle which is of diameter greater than  $(|\omega_1| + |\omega_2|)/2^k$  contains a simple zero of  $\theta^k \sigma(x)$ .

Thus if  $\sigma(x)$  can be approximated uniformly in an open region by solutions of the  $\Sigma_i$ , there is a  $k$  such that for some function  $y(x)$  in  $\mathcal{N}$  the function  $\theta^k y(x)$  has a simple zero at some point in that open region. This contradiction proves that there is no open region in which  $\sigma(x)$  can be approximated uniformly by solutions of the  $\Sigma_i$ .

15. Let  $\Sigma_1, \Sigma_2, \dots$  be a sequence of closed irreducible systems in  $y_1, \dots, y_n$  such that the manifold of  $\Sigma_i$  is a proper part of the manifold of  $\Sigma_{i+1}$ , ( $i=1, 2, \dots$ ). Let  $\Sigma$  be the set of all forms  $F$  such that  $F$  holds every  $\Sigma_i$ . Discarding a finite set of the  $\Sigma_i$  if necessary, we assume that there is a fixed set of unknowns  $y_1, \dots, y_q$  which is a set of arbitrary unknowns for every  $\Sigma_i$ .‡ Of course  $q < n$ . Let  $A_{i, q+1}, A_{i, q+2}, \dots, A_{i, n}$  be a corresponding basic set for  $\Sigma_i$ , with  $A_{ij}$  introducing  $y_j$ , and let  $\gamma_{ij}$  be the order of  $A_{ij}$  in  $y_j$ , ( $j=q+1, \dots, n; i=1, 2, \dots$ ). Let  $r$  be the number of arbitrary unknowns in  $\Sigma$ . Then  $q < r \leq m$ . If  $\Sigma$  is not trivial, that is, if  $r < n$ , let  $A_{r+1}, A_{r+2}, \dots, A_n$  be a basic set for  $\Sigma$ , with  $A_k$  introducing  $y_k$ ,§ and let  $\gamma_k$  be the order of  $A_k$  in  $y_k$ , ( $k=r+1, \dots, n$ ). We consider the question of whether the values of  $r$ , and (when  $\gamma_k$  exist) of the  $\gamma_k$ , are determined uniquely by  $n, q$ , and the  $\gamma_{ij}$ . We know that the answer is affirmative when  $n-q=1$ , since in this case  $n-r=0$  and  $\Sigma$  is trivial. We shall indicate by examples in §§15, 16 that the answer is negative when  $n-q \geq 2$ .

Let  $\sigma$  be any nonnegative integer, and let  $\Sigma_n$ , ( $n=1, 2, \dots$ ), be the closed irreducible system in  $u, y$  with a basic set  $A_n, B_n$  where

$$A_n \equiv u_n, \quad B_n \equiv y - \sum_{i=1}^{n-1} i^\sigma u_i. ||$$

Since the order of  $A_n$  in  $u$  becomes infinite with  $n$ ,  $\Sigma$  cannot have a non-

\* Hurwitz and Courant, *Funktionentheorie*, p. 179.

† Ibid., p. 184.

‡ We renumber the unknowns, if necessary.

§ We renumber the unknowns again, if necessary.

|| For the existence of such a  $\Sigma_n$  cf. ADE, §§65, 45.

zero form in  $u$  alone. We shall prove that  $\Sigma$  has a nonzero form in  $u$  and  $y$ , so that  $u$  is a set of arbitrary unknowns for  $\Sigma$ . Moreover, we shall prove that the nonzero forms in  $\Sigma$  of lowest order in  $y$  are of order  $\sigma + 1$  in  $y$ . Set

$$C_{nj} \equiv y_j - \sum_{i=1}^{n-j-1} i^\sigma u_{i+j}, \quad j = 0, 1, \dots, \sigma + 1; n = 1, 2, \dots .$$

Evidently  $C_{nj}$  is a form in  $\Sigma_n$  since it is a linear combination of derivatives of  $A_n$  and  $B_n$ . Set

$$D_n \equiv \sum_{j=0}^{\sigma+1} (-1)^j {}_{\sigma+1}C_j C_{nj} . *$$

It is easy to see that

$$D_n \equiv \sum_{j=0}^{\sigma+1} (-1)^j {}_{\sigma+1}C_j \left[ y_j - \sum_{k=j+1}^{\sigma+2} (k-j)^\sigma u_k \right] . \dagger$$

Since  $D_n$  is independent of  $n$ ,  $D_n$  is in  $\Sigma_j$  for every  $j$ . Therefore  $\Sigma$  contains the nonzero form  $D_n$  which is of order  $\sigma + 1$  in  $y$ . Let  $m$  be any positive integer greater than  $\sigma - 1$ , and let  $a_0, a_1, \dots, a_m; b_0, b_1, \dots, b_\sigma$  be any set of complex numbers. Let  $n = m + \sigma + 2$ . We shall prove that  $\Sigma_n$  has a solution  $u(x), y(x)$  such that  $u_i(0) = a_i, y_k(0) = b_k, (i = 0, 1, \dots, m; k = 0, 1, \dots, \sigma)$ .

It is easily seen that such a solution exists if the system of linear equations

$$(46) \quad b_k = \sum_{i=1}^{m-k} i^\sigma a_{i+k} + \sum_{\alpha=m+1}^{n-1} (\alpha - k)^\sigma u_\alpha, \quad k = 0, 1, \dots, \sigma,$$

has a solution in  $u_\alpha, (\alpha = m + 1, \dots, n - 1)$ .

The system (46) will have a solution if the determinant  $d(m, \sigma)$  is not zero, where the element of  $d(m, \sigma)$  in the  $i$ th row and  $j$ th column is  $(m + 1 + j - i)^\sigma, (i, j = 1, \dots, \sigma + 1)$ . By repeated subtraction of adjacent columns, and then of adjacent rows, in  $d(m, \sigma)$ , it can be shown that  $d(m, \sigma) = (\sigma!)^{\sigma+1} \neq 0$ .

Hence  $\Sigma$  has a solution  $u(x), y(x)$  such that

$$u_i(0) = a_i, y_k(0) = b_k, \quad i = 0, 1, \dots, m; k = 0, 1, \dots, \sigma,$$

where  $m$  is any positive integer and the  $a_i$  and  $b_k$  are arbitrary. This obviously implies that  $\Sigma$  has no nonzero form whose order in  $y$  is less than  $\sigma + 1$ . Evidently if  $u$  is taken as a set of arbitrary unknowns for  $\Sigma$ , then  $\sigma + 1$  is the order in  $y$  of a basic set for  $\Sigma$ .

\* Note that  ${}_{\sigma+1}C_j$  is a binomial coefficient, not a form.

† This reduces quickly to proving the identity  $\sum_{j=0}^{\sigma+1} (-1)^j {}_{\sigma+1}C_j (k-j)^\sigma = 0$ , which holds for all complex  $k$ , since the left-hand member is the  $\sigma$ th derivative at the origin of  $e^{kx}(1 - e^{-x})^{\sigma+1}$ .

16. Let  $\Sigma_n$ , ( $n=1, 2, \dots$ ), be the closed irreducible system in  $u, y$  with a basic set  $A_n, B_n$  where  $A_n \equiv u_n, B_n \equiv y - \sum_{i=1}^{n-1} i! u_i$ . We shall prove that for every positive integer  $m$  and every set of complex numbers  $a_0, a_1, \dots, a_m; b_0, b_1, \dots, b_m$  the system  $\Sigma_{2m+2}$  has a solution  $u, y$  in which  $u_i(0) = a_i, y_i(0) = b_i, (i, j=0, 1, \dots, m)$ . It suffices to prove that the system of linear equations

$$(47) \quad b_k = \sum_{i=1}^{m-k} i! a_{i+k} + \sum_{\alpha=m+1}^{2m+1} (\alpha - k)! u_\alpha, \quad k = 0, 1, \dots, m,$$

has a solution in  $u_\alpha, (\alpha=m+1, \dots, 2m+1)$ . Such a solution exists because the determinant  $d_m$  of the coefficients is not zero, where the element of  $d_m$  in the  $i$ th row and  $j$ th column is  $(m+1+j-i)!, (i, j=1, \dots, m+1)$ . The value of  $d_m$  is easily seen to be  $(-1)^{m(m+1)/2} (1!2! \dots m!)^2 (m+1)!$ .

Consequently, for every set of complex numbers  $a_0, a_1, \dots, a_m; b_0, b_1, \dots, b_m, \Sigma$  has a solution  $u(x), y(x)$  with  $u_j(0) = a_j, y_i(0) = b_i, (i, j=0, 1, \dots, m)$ . This means that  $\Sigma$  has no nonzero form. That is,  $\Sigma$  is the trivial system having  $u, y$  as a set of arbitrary unknowns.

17. Let  $\Sigma$  be any closed irreducible system in  $u_1, \dots, u_q; y_1, \dots, y_p$ , where  $u_1, \dots, u_q, (q \geq 1)$ , is a set of arbitrary unknowns. Let  $s$  be any integer with  $0 \leq s < q$ . We assert that there is a sequence

$$(48) \quad \Sigma_1, \Sigma_2, \dots$$

of closed irreducible systems such that  $\Sigma_i$  has  $s$  arbitrary unknowns, such that the manifold of  $\Sigma_i$  is a proper part of the manifold of  $\Sigma_{i+1}, (i=1, 2, \dots)$ , and such that  $\Sigma$  is the totality of forms common to the  $\Sigma_i, (i=1, 2, \dots)$ .

For example, we may construct a sequence (48) as follows:

Let  $A_1, A_2, \dots, A_p$  be a basic set for  $\Sigma$ , with  $A_i$  introducing  $y_i$ . Let  $m$  be a positive integer greater than the order of  $A_i$  in  $u_j, (i=1, \dots, p; j=1, \dots, q-s)$ . Then for every positive integer  $n$  the ascending set

$$(49) \quad u_{1,m+n}, u_{2,m+n}, \dots, u_{q-s,m+n}, A_1, \dots, A_p$$

is the basic set of a closed irreducible system. This is a simple consequence of Ritt's theorems characterizing the basic set of a closed irreducible system.\* Let  $\Sigma_n$  be the closed irreducible system having (49) for a basic set ( $n=1, 2, \dots$ ). It is easy to see that with this definition of  $\Sigma_n$  the sequence (48) has the desired properties.

\* ADE, §§65, 45.