ON THE SEMI-CONTINUITY OF DOUBLE INTEGRALS IN PARAMETRIC FORM

BY

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INTRODUCTION

The purpose of the present investigation is to extend the scope of the important and comprehensive results of McShane on the semi-continuity of double integrals in parametric form(1). The class of surfaces considered by McShane is very general, but we shall see that the conditions placed by him upon his surfaces can be greatly relaxed without hurting the validity of any one of his theorems. Our condition (see 1.19) is however not only less restrictive but also possesses a certain degree of finality. Indeed, we shall be able to prove that *the class of surfaces for which our condition holds is precisely the class of surfaces which admit of a representation where the Lebesgue area of the surface is given by the usual integral formula* (see 3.15, 3.19).

The methods used in this paper are based on previous work by McShane in the calculus of variations and by the author on the area of surfaces. Since this work is scattered in a number of papers in various periodicals, it seemed advisable to attempt at this time a somewhat self-contained and systematic presentation, as well as to carry out various simplifications of detail suggested by a comparative study of the literature. In particular, *the theory of the Lebesgue area of surfaces will not be presupposed. On the contrary, we shall find that the most advanced results of that theory are simple corollaries of our results on general double integrals.*

CHAPTER 1. PRELIMINARIES(2)

1.1. We shall be concerned with integrands of the form $f(x^1, x^2, x^3, X^1, X^2, X^3)$, defined for all values of the six independent variables $x^1, x^2, x^3, X^1, X^2, X^3$. We shall follow the condensed notations used in McShane [1, 2]. Accordingly, we shall use x to refer to the triple x^1, x^2, x^3 and X to refer to the triple X^1, X^2, X^3. It will be convenient for us to interpret x as a point with coordinates x^1, x^2, x^3 and X as a vector with components X^1, X^2, X^3. We shall use $\|X\|$ to denote the length of the vector X. If $X\neq 0$, then ${}_uX$ will denote the unit vector $X/\|X\|$. We shall write $f(x, X)$ for $f(x^1, x^2, x^3, X^1, X^2, X^3)$.

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(1) See McShane [1, 2]. Numbers in square brackets refer to the references at the end of this paper.

(2) For the convenience of the reader, we have collected in this chapter the notations, definitions and lemmas we need. *The reader is requested to turn to this chapter whenever in doubt about the meaning of a term or symbol.*

1.2. We shall say that the integrand f is *admissible* if the following conditions are satisfied.

(1) f is continuous for all values of the six independent variables.

(2) f is positively homogeneous of degree one with respect to X. That is,

$$f(x, tX) = tf(x, X) \qquad \text{for } t \geqq 0.$$

(3) f has continuous partial derivatives of the first and second order for $X \neq 0$.

REMARK. As a consequence of (2), we have

$$f(x^1, x^2, x^3, 0, 0, 0) = f(x, 0) = 0.$$

1.3. Let us put

$$f_i = \begin{cases} \dfrac{\partial f}{\partial X_i} & \text{if } X \neq 0, \\ 0 & \text{if } X = 0, \end{cases} \qquad i = 1, 2, 3.$$

We have then, by (2) in 1.2, the identity(3)

$$f(x, X) = X^\alpha f_\alpha(x, X).$$

We define, as usual,

$$E(x^1, x^2, x^3, X^1, X^2, X^3, \overline{X}^1, \overline{X}^2, \overline{X}^3) = E(x, X, \overline{X}) = f(x, \overline{X}) - \overline{X}^\alpha f_\alpha(x, X).$$

1.4. For an admissible f the following facts are easily established. If A is a bounded closed set in x-space, then there exists a constant $M > 0$ such that, for $x \in A$ and for every X

$$|f(x, X)| \leqq M\|X\|;$$

and, for every $x \in A$ and for every pair of vectors $X \neq 0$, $\overline{X} \neq 0$,

$$|E(x, X, \overline{X})| \leqq M\|\overline{X}\| \cdot \|{}_u\overline{X} - {}_uX\|^2 (4).$$

1.5. LEMMA. *Let there be given an admissible f and a set of six constants* $(x_0^1, x_0^2, x_0^3, X_0^1, X_0^2, X_0^3) = (x_0, X_0)$, *such that*

(a) $f(x_0, X_0) > 0$,

(b) $E(x_0, X_0, \overline{X}) > 0$ *whenever* $\overline{X} \neq 0$, ${}_u\overline{X} \neq {}_uX_0$ (5).

Then there exist two positive constants δ_1, δ_2 *such that the following holds. If* $\eta, \bar{x}^1, \bar{x}^2, \bar{x}^3, \overline{X}^1, \overline{X}^2, \overline{X}^3$ *satisfy the conditions*

(α) $0 < \eta < \delta_1$,

(β) $\overline{X}^\alpha f_\alpha(x_0, X_0) > 0$,

(γ) $|f(\bar{x}, \overline{X}) - f(x_0, \overline{X})| \leqq \eta\|\overline{X}\|$,

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(3) A repeated greek letter indicates summation with respect to that letter.

(4) Cf. Bliss [1].

(5) Observe that $X_0 \neq 0$ on account of condition (a).

then

$$f(\bar{x}, \overline{X}) - f(x_0, \overline{X}) + E(x_0, X_0, \overline{X}) \geqq -\eta\delta_2\overline{X}^\alpha f_\alpha(x_0, X_0). \tag{1}$$

Proof. By condition (a), we have $X_0 \neq 0$, and hence we can consider ${}_uX_0$. For $\sigma > 0$, let us denote by $\lambda(\sigma)$ the maximum of

$$\| {}_u\overline{X} - {}_uX_0 \|$$

for all vectors $\overline{X} \neq 0$ such that

$$E(x_0, X_0, \overline{X}) \leqq \sigma \|\overline{X}\|.$$

From condition (b) we infer easily that $\lambda(\sigma) \to 0$ for $\sigma \to 0$. Hence (cf. condition (a)), we have a $\delta_1 > 0$ such that

$$\lambda(\delta_1) < \frac{f(x_0, X_0)}{2\|X_0\| \cdot \|Y_0\|},$$

where Y_0 denotes the vector with components $f_i(x_0, X_0)$, $i = 1, 2, 3$. Note that condition (a) implies that $Y_0 \neq 0$. We define now

$$\delta_2 = \frac{2\|X_0\|}{f(x_0, X_0)},$$

and we assert that the constants δ_1 and δ_2 satisfy the requirements of the lemma. To show this, let there be given η and $(\bar{x}, \overline{X})$ such that conditions (α), (β), (γ) are satisfied. Let us put

$$H = f(\bar{x}, \overline{X}) - f(x_0, \overline{X}) + E(x_0, X_0, \overline{X}).$$

If $H \geqq 0$, then (1) is true on account of (β). Thus we can assume that $H < 0$. We have then, on account of (γ) and (α),

$$E(x_0, X_0, \overline{X}) < f(x_0, \overline{X}) - f(\bar{x}, \overline{X}) \leqq \eta\|\overline{X}\| \leqq \delta_1\|\overline{X}\|. \tag{2}$$

We note that $\overline{X} \neq 0$ on account of (β). Hence we can consider ${}_u\overline{X}$. By the definition of δ_1, (2) implies that

$$\|{}_u\overline{X} - {}_uX_0\| \leqq \lambda(\delta_1) < \frac{f(x_0, X_0)}{2\|X_0\| \cdot \|Y_0\|}. \tag{3}$$

The inequality of Schwarz yields

$$| {}_u\overline{X}^\alpha f_\alpha(x_0, X_0) - {}_uX_0^\alpha f_\alpha(x_0, X_0) | \leqq \|Y_0\| \cdot \|{}_u\overline{X} - {}_uX_0\|; \tag{4}$$

(4) and (3) yield

$$| {}_u\overline{X}^\alpha f_\alpha(x_0, X_0) - {}_uX_0^\alpha f_\alpha(x_0, X_0) | < \frac{f(x_0, X_0)}{2\|X_0\|} = \frac{1}{\delta_2}.$$

Hence

$$ {}_u\overline{X}^\alpha f_\alpha(x_0, X_0) > {}_uX_0^\alpha f_\alpha(x_0, X_0) - \frac{1}{\delta_2} = \frac{f(x_0, X_0)}{\|X_0\|} - \frac{1}{\delta_2} = \frac{1}{\delta_2}. $$

Multiplication by $\|X\|$ yields

$$ \overline{X}^\alpha f_\alpha(x_0, X_0) > \frac{1}{\delta_2}\|\overline{X}\|; \tag{5} $$

(5) and conditions (b), (γ) yield finally

$$ H > -\eta\|\overline{X}\| > -\eta\delta_2\overline{X}^\alpha f_\alpha(x_0, X_0), $$

and the lemma is proved.

1.6. We shall consider triples of functions $x^i\ (u^1, u^2)$, $i=1, 2, 3$, where the range of definition will be some simply connected Jordan region B, that is, the set of points in and on some Jordan curve. We shall use the notations

$$ T:\quad x^1(u^1, u^2),\ x^2(u^1, u^2),\ x^3(u^1, u^2), \qquad (u^1, u^2) \in B, $$

or in condensed form,

$$ T:\quad x(u), \qquad u \in B, $$

or simply (T, B) to refer to such a triple. The set of points, in (x^1, x^2, x^3)-space, with coordinates $x^1(u^1, u^2)$, $x^2(u^1, u^2)$, $x^3(u^1, u^2)$, will be denoted by $\sum(T, B)$.

1.7. The triple (T, B) will be called *quasi-linear* if the following conditions are satisfied.

(a) The boundary of B is a *polygon.*

(b) $x^1(u^1, u^2)$, $x^2(u^1, u^2)$, $x^3(u^1, u^2)$ are continuous in B.

(c) B can be subdivided into a finite number of (rectilinear) triangles in each of which the functions $x^i(u_1, u_2)$, $i=1, 2, 3$, are linear.

1.8. We shall say that the triple (T, B) is of class K_1 if the functions $x^1(u^1, u^2)$, $x^2(u^1, u^2)$, $x^3(u^1, u^2)$ are continuous in B, if their partial derivatives of the first order exist a.e. in B^0, and if the Jacobians

$$ X^1(u^1, u^2) = \frac{\partial(x^2, x^3)}{\partial(u^1, u^2)}, \quad X^2(u^1, u^2) = \frac{\partial(x^3, x^1)}{\partial(u^1, u^2)}, \quad X^3(u^1, u^2) = \frac{\partial(x^1, x^2)}{\partial(u^1, u^2)} $$

are summable in B^0(6). We shall denote by $X(u)$ the vector whose components are $X^1(u)$, $X^2(u)$, $X^3(u)$. If $(T, B)\in K_1$, then clearly $\|X(u)\|$ is summable in B^0.

1.9. If $(T, B)\in K_1$ and if f is admissible, then clearly $f(x(u), X(u))$ is measurable in B^0. Since the set $\sum(T, B)$ is bounded and closed, we have by 1.4 a constant $M>0$ such that

$$ |f(x(u), X(u))| \leqq M\|X(u)\| \quad \text{a.e. in } B^0. $$

(6) Generally, if H is a point set considered in some space, then H^0 will denote the set of its interior points relative to that space.

Hence $f(x(u), X(u))$ is summable in B^0. We shall denote its integral over B^0 by $I(T, B, f)$. For example,

$$I(T, B, \|X\|) = \iint_{B^0} \|X(u)\| du(^7).$$

1.10. Let there be given a quasi-linear triple

$$T: \quad x(u), \qquad u \in B.$$

We subject B to a topological transformation $\bar{u} = \tau(u)$ and for clarity we want to think of the image point $\bar{u}$ as being located in a different plane $(\bar{u}^1, \bar{u}^2)$. Let us denote by $\overline{B}$ the image of B, and by $u = \sigma(\bar{u})$ the inverse of τ. We have then the new triple

$$\overline{T}: \quad \bar{x}(\bar{u}) = x(\sigma(\bar{u})), \qquad \bar{u} \in \overline{B}.$$

Suppose now that τ is quasi-linear (that is, B can be subdivided into a finite number of rectilinear triangles in each of which τ is an affine transformation). Then clearly $(\overline{T}, \overline{B})$ is quasi-linear. Given then an admissible f, we find by an entirely elementary computation the formula

$$I(T, B, f) = I(\overline{T}, \overline{B}, f),$$

if τ is sense-preserving, and the formula

$$I(T, B, f) = I(\overline{T}, \overline{B}, f^*),$$

where $f^*(x, X) = f(x, -X)$, if τ is not sense-preserving.

1.11. Let there be given an admissible f and a triple

$$T_0: \quad x_0(u), \qquad u \in B_0$$

of class K_1. We shall say that (T_0, B_0) satisfies condition (c) with respect to f if

(1) there exists, in x-space, a closed bounded set A such that $\sum(T_0, B_0) \subset A^0$, and $f(x, X) \geqq 0$ for $x \in A$ and for every vector X, and

(2) for a.e. point $u \in B_0^0$ such that $X_0(u)$ exists and is not equal to 0, we have

$$E(x_0(u), X_0(u), \overline{X}) \geqq 0$$

for every vector $\overline{X} \neq 0$(6).

1.12. We shall say that (T_0, B_0) satisfies condition (+c) with respect to f if condition (2) in 1.11 is satisfied, while condition (1) there is satisfied in the following stronger form:

There exists, in x-space, a closed bounded set A such that $\sum(T_0, B_0) \subset A^0$, and $f(x, X) > 0$ for every $x \in A$ and for every vector $X \neq 0$.

1.13. We shall say that (T_0, B_0) satisfies condition (c+) with respect to f if condition (1) in 1.11 is satisfied, while condition (2) there is satisfied in the following stronger form:

(7) For brevity we write du for $du^1 du^2$.

For a.e. point $u \in B_0^0$ such that $X_0(u)$ exists and is not equal to 0, we have

$$E(x_0(u), X_0(u), \overline{X}) > 0$$

for every vector $\overline{X} \neq 0$ such that ${}_u\overline{X} \neq {}_uX_0(u)$.

1.14. For the particular integrand $\|X\|$ we find

$$E(X, \overline{X}) = \tfrac{1}{2}\|\overline{X}\| \cdot \|{}_u\overline{X} - {}_uX\|^2.$$

This identity leads immediately to the following lemmas.

1.15. Lemma. *If f is admissible and if $(T_0, B_0) \in K_1$, then for a sufficiently large value of the constant H the triple (T_0, B_0) will satisfy condition* (c+) *with respect to the integrand $H\|X\| + f$.*

1.16. Lemma. *If f is admissible, if $(T_0, B_0) \in K_1$, and if (T_0, B_0) satisfies condition* (c) *with respect to f, then for every $\epsilon > 0$ the triple (T_0, B_0) satisfies condition* (c+) *with respect to the integrand $\epsilon\|X\| + f$.*

1.17. Let there be given two continuous triples

$$T:\ x(u), \quad u \in B; \qquad \overline{T}:\ \bar{x}(\bar{u}), \quad \bar{u} \in \overline{B},$$

where, for clarity, we want to think of $\overline{B}$ as being located in a different plane $\bar{u}$. Let $\bar{u} = \tau(u)$ be a topological transformation from B to $\overline{B}$. The *Fréchet distance* of (T, B) and $(\overline{T}, \overline{B})$ is then defined as the greatest lower bound, for all possible topological transformations τ, of

$$\max_{u \in B} \|x(u) - \bar{x}(\tau(u))\| \text{ (8)}.$$

We shall denote this distance by $d[(T, B), (\overline{T}, \overline{B})]$.

1.18. If in the preceding definition we restrict τ to be *sense-preserving*, then we obtain what may be called *the oriented distance* of (T, B) and $(\overline{T}, \overline{B})$. We shall denote the oriented distance by ${}_od[(T, B), (\overline{T}, \overline{B})]$ (9).

1.19. We shall say that a triple (T_0, B_0) is of class K_2 if the following conditions are satisfied (10).

(a) $(T_0, B_0) \in K_1$.

(b) There exists a sequence of quasi-linear triples (T_n, B_n) such that

$${}_od[(T_0, B_0), (T_n, B_n)] \xrightarrow[n \to \infty]{} 0,$$

and

$$I(T_n, B_n, \|X\|) \xrightarrow[n \to \infty]{} I(T_0, B_0, \|X\|).$$

1.20. Given a continuous triple (T_0, B_0), let us consider the class $\mathfrak{K}$ of all

(8) If p, $\bar{p}$ are points in a metric space, then $\|p - \bar{p}\|$ will denote their distance.

(9) This modification of the Fréchet distance was introduced by McShane, loc. cit. (1).

(10) As we shall prove later on, *these conditions are necessary and sufficient that the Lebesgue area of the surface determined by (T_0, B_0) be given by the usual integral formula.*

continuous triples (T, B) such that

$$d[(T_0, B_0), (T, B)] = 0.$$

Clearly, $\mathfrak{K}$ is univocally determined by any one of its elements, and possesses the following properties:

(a) If $(T_1, B_1) \in \mathfrak{K}$, $(T_2, B_2) \in \mathfrak{K}$, then $d[(T_1, B_1), (T_2, B_2)] = 0$.

(b) If $(T_1, B_1) \in \mathfrak{K}$ and $d[(T_1, B_1), (T_2, B_2)] = 0$, then $(T_2, B_2) \in \mathfrak{K}$.

Conversely, every class of triples, with the properties (a) and (b), can be generated by any one of its elements in the manner described above.

1.21. A *continuous parametrized surface* S, *of the type of the circular disc*, is merely a class $\mathfrak{K}$ of continuous triples as described in 1.20. Every triple $(T, B) \in \mathfrak{K}$ will be called a representation of S. For brevity, we shall call S simply a *continuous surface.*

1.22. Given two continuous surfaces S_1, S_2, let (T_1, B_1) be a representation of S_1 and (T_2, B_2) a representation of S_2. Then $d[(T_1, B_1), (T_2, B_2)]$ is easily seen to be independent of the particular choice of these representations and can be denoted therefore by $d(S_1, S_2)$. This quantity is *the Fréchet distance of the surfaces* S_1 *and* S_2.

If (T_1, B_1), (T_2, B_2) are any two representations of the same continuous surface S, then the point sets $\sum(T_1, B_1)$, $\sum(T_2, B_2)$ are easily seen to be identical. That is, the set $\sum(T, B)$ is the same for all representations of (T, B). We shall denote this set by $\sum(S)$.

1.23. If in the statements made in 1.20, 1.21, 1.22 we replace the Fréchet distance d by the *oriented* distance ${}_od$, then we arrive at the conception of an *oriented continuous parametrized surface, of the type of the circular disc.* We shall denote such a surface by ${}_oS$ and we shall call it simply an *oriented continuous surface.* The definition of the oriented distance ${}_od({}_oS_1, {}_oS_2)$ of two oriented continuous surfaces ${}_oS_1$, ${}_oS_2$ as well as the definition of the point set $\sum({}_oS)$ associated with an oriented continuous surface ${}_oS$ is then worded in an obvious manner (cf. 1.22).

1.24. If S_0, S_n are continuous surfaces, then $S_n \to S_0$ will mean that $d(S_n, S_0) \to 0$. If $S_n \to S_0$, and if A is a point set in (x^1, x^2, x^3)-space such that $\sum(S_0) \subset A^0$ (11), then clearly $\sum(S_n) \subset A^0$ for large values of n.

1.25. If ${}_oS_0$, ${}_oS_n$ are oriented continuous surfaces, then ${}_oS_n \to {}_oS_0$ means that ${}_od({}_oS_0, {}_oS_n) \to 0$. That is, for *oriented* continuous surfaces the notion of convergence is based on the *oriented* distance. Again, if ${}_oS_n \to {}_oS_0$, and if A is a point set in (x^1, x^2, x^3)-space such that $\sum({}_oS_0) \subset A^0$, then $\sum({}_oS_n) \subset A^0$ for large values of n.

1.26. We say that a continuous surface, oriented or not, is of class K_i if it admits of a representation (T, B) that is of class K_i, $i = 1, 2$ (cf. 1.8, 1.19).

1.27. In the u-plane, let there be given a simply connected bounded Jordan region $\overline{B}$. In a $\bar{u}$-plane let there be given a region $\overline{B}$ of the same type.

(11) Cf. (6).

We assume that $\bar{B}$ is bounded by a polygon. Let there be given also a topological transformation $u=\tau(\bar{u})$ from $\bar{B}$ to B. Given then $\epsilon>0$, there exists a quasi-linear topological transformation $u=\tau^*(\bar{u})$ from $\bar{B}$ to some Jordan region B^* in the u-plane, such that

$$\|\tau(\bar{u}) - \tau^*(\bar{u})\| \leqq \epsilon \qquad \text{for } \bar{u} \in \bar{B}(^{12}).$$

If τ was sense-preserving, then τ^* can be chosen as sense-preserving (in fact, for ϵ small, τ^* will be then automatically sense-preserving)(13).

1.28. Generalization of a lemma of McShane. *In a closed square Q, in the u-plane, let there be given triples*

$$T_0:\ x_0(u), \quad u \in Q, \qquad T_n:\ x_n(u), \quad u \in Q,$$

such that the following conditions are satisfied.

(a) $(T_0, Q) \in K_1$.

(b) (T_n, Q) *is quasi-linear.*

(c) $x_n(u) \to x_0(u)$ *uniformly in Q.*

Then for every choice of the constants a^1, a^2, a^3 there exists a sequence of measurable sets V_n in Q, such that

$$\iint_{V_n} a^{\alpha} X_n^{\alpha}(u) du \to \iint_Q a^{\alpha} X_0^{\alpha}(u) du.$$

Corollary. *If*

$$\iint_Q a^{\alpha} X_0^{\alpha}(u) du > 0,$$

then the sets V_n can be chosen in such a way that $a^{\alpha} X_n^{\alpha}(u) > 0$ on V_n.

1.29. McShane(14) proved this lemma under the following additional assumptions concerning the limit triple (T_0, Q).

(d) $x_0(u)$ is absolutely continuous on the perimeter p of Q.

(e) The following equations hold

$$\iint_Q X_0^1(u) du = \frac{1}{2}\int_p (x_0^2(u) dx_0^3(u) - x_0^3(u) dx_0^2(u)),$$

$$\iint_Q X_0^2(u) du = \frac{1}{2}\int_p (x_0^3(u) dx_0^1(u) - x_0^1(u) dx_0^3(u)),$$

$$\iint_Q X_0^3(u) du = \frac{1}{2}\int_p (x_0^1(u) dx_0^2(u) - x_0^2(u) dx_0^1(u)).$$

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(12) Cf. (8).

(13) The existence of such quasi-linear approximations follows from Franklin-Wiener [1].

(14) Loc. cit. (1).

It is fundamental for our purposes that these additional assumptions are unnecessary(15). It is interesting to note that *we require solely summability of the Jacobians of the limit triple, a requirement which is necessary if the lemma is to have a meaning.*

1.30. Since (loc. cit.(15)) we did not state the corollary to the lemma, let us suggest briefly how the corollary may be derived from the lemma itself. Let us denote by Q_r the square with the same center as Q, with sides parallel to those of Q, and with side-length equal to r times the side-length of Q, $0<r\leqq 1$. Denote by V_n^+ the subset of V_n where

$$\overset{\alpha}{a}\overset{\alpha}{X}_n(u) > 0.$$

We define now a measurable subset W_n of V_n as follows. If

$$\iint_{V_n^+} \overset{\alpha}{a}\overset{\alpha}{X}_n(u)du \leqq \iint_Q \overset{\alpha}{a}\overset{\alpha}{X}_0(u)du,$$

then $W = V_n^+$. If

$$\iint_{V_n^+} \overset{\alpha}{a}\overset{\alpha}{X}_n(u)du > \iint_Q \overset{\alpha}{a}\overset{\alpha}{X}_0(u)du, \tag{6}$$

then let us denote by $E_{n,r}$ the set $Q_r \cdot V_n^+$. Then the quantity

$$\mu_n(r) = \iint_{E_{n,r}} \overset{\alpha}{a}\overset{\alpha}{X}_n(u)du \tag{7}$$

is a continuous function of r for $0<r\leqq 1$. Clearly $\mu_n(r)\rightarrow_{r\rightarrow 0}0$ and, by (6), (7),

$$\mu_n(1) > \iint_Q \overset{\alpha}{a}\overset{\alpha}{X}_0(u)du.$$

Hence there exists a value r_n between zero and one such that

$$\mu_n(r_n) = \iint_{E_{n,r_n}} \overset{\alpha}{a}\overset{\alpha}{X}_n(u)du = \iint_Q \overset{\alpha}{a}\overset{\alpha}{X}_0(u)du.$$

We put then $W_n = E_{n,r_n}$. For the sequence of sets W_n defined in this manner we have obviously

$$\iint_{W_n} \overset{\alpha}{a}\overset{\alpha}{X}_n(u)du \rightarrow \iint_Q \overset{\alpha}{a}\overset{\alpha}{X}_0(u)du,$$

and

$$\overset{\alpha}{a}\overset{\alpha}{X}_n(u) > 0 \quad \text{on} \quad W_n.$$

(15) See Radó [4].

1.31. **On functions of squares**(16). In the (u^1, u^2)-plane, we consider a fixed square Q_0 and all squares $q \subset Q_0$, all these squares having their sides parallel to the axes u^1, u^2. Given then a function $\psi(q)$ for all such squares, including Q_0 itself(17), its upper derivative $\overline{D}\psi(u)$ at an interior point u of Q_0 is defined as the least upper bound of

$$\limsup_{n\to\infty} \frac{\psi(q_n)}{|q_n|}$$

for all possible sequences of squares q_n, with sides parallel to the axes, containing u, and such that $|q_n| \to 0$. The lower derivative $\underline{D}\psi(u)$ is defined in a similar fashion. Both of these derivatives are measurable functions. Generally, they will take on the values $\pm\infty$. If $\overline{D}\psi(u)$ and $\underline{D}\psi(u)$ are finite and equal at a point u, then their common value is the derivative $D\psi(u)$. The set on which $D\psi(u)$ exists is measurable and $D\psi(u)$ is measurable on this set. In particular if $D\psi(u)$ exists a.e. in Q_0, then it is measurable in Q_0.

1.32. Let $q, q_1, q_2, \cdots, q_m$ be a system of squares in Q_0 such that $q_i \subset q$ for $i=1, 2, \cdots, m$ and such that $q_i^0 \cdot q_j^0 = 0$ for $i \neq j$. If for every such system we have

$$\sum_1^m \psi(q_i) \leqq \psi(q),$$

if $\psi(q) \geqq 0$ for every $q \subset Q_0$, and if $\psi(Q_0) < +\infty$, then we shall say that $\psi(q)$ is of type A in Q_0.

1.33. Lemma. *If $\psi(q)$ is of type A in Q_0, then its derivative $D\psi(u)$ exists a.e. in Q_0, is summable there, and satisfies the inequality*

$$\iint_q D\psi(u)du \leqq \psi(q)$$

for every $q \subset Q_0$(18).

1.34. The fundamental pattern of our semi-continuity proofs may be now described by the following

Lemma. *In the u-plane, let there be given bounded and simply connected Jordan regions $B_0, B_1, B_2, \cdots, B_n, \cdots$ such that the following condition holds: If q is any closed square in B_0^0*(19)*, then $q \subset B_n^0$ for sufficiently large values of n. In B_n^0 let there be given a non-negative summable function $f_n(u)$, $n=0, 1, 2, \cdots$. No assumption is made concerning the convergence of the sequence $f_n(u)$. For*

(16) The facts stated in 1.31, 1.32, 1.33 are immediate consequences of results in Banach [1]. Cf. also Radó [3].

(17) It is convenient to permit $\psi(q)$ to assume infinite values.

(18) While this lemma is implied by the results in Banach [1], it seems that it was first stated and used explicitly in Radó [2].

(19) Cf. (6).

every closed square $q \subset B_0^0$ let us define

(8) $$\psi(q) = \liminf_{n\to\infty} \iint_q f_n(u)du.$$

If we have, for a.e. point u in B_0^0, the inequality

(9) $$\underline{D}\psi(u) \geqq f_0(u),$$

then

(10) $$\liminf_{n\to\infty} \iint_{B_n^0} f_n(u)du \geqq \iint_{B_0^0} f_0(u)du.$$

Proof(^20). Let Q be a fixed closed square in B_0^0. Since $f_n(u) \geqq 0$, it is obvious that $\psi(q)$ is of type A in Q. Hence, by 1.33,

(11) $$\psi(Q) \geqq \iint_Q D\psi(u)du.$$

Relations (8), (9), (11) yield

(12) $$\liminf \iint_Q f_n(u)du \geqq \iint_Q f_0(u)du$$

for every $Q \subset B_0^0$. Given now any $\epsilon > 0$, we can select in B_0^0 a finite number of closed squares $Q_1, Q_2, \cdots, Q_m$ without common interior points, such that

$$\sum_1^m \iint_{Q_i} f_0(u)du > \iint_{B_0^0} f_0(u)du - \epsilon.$$

We have an N such that $Q_1, Q_2, \cdots, Q_m \subset B_n^0$ for $n > N$. Since $f_n \geqq 0$, we have then for $n > N$

$$\iint_{B_n^0} f_n(u)du \geqq \sum_1^m \iint_{Q_i} f_n(u)du,$$

and hence, by (12),

$$\liminf_{n\to\infty} \iint_{B_n^0} f_n(u)du \geqq \sum_1^m \liminf_{n\to\infty} \iint_{Q_i} f_n(u)du$$
$$\geqq \sum_1^m \iint_{Q_i} f_0(u)du > \iint_{B_0^0} f_0(u)du - \epsilon.$$

Since $\epsilon > 0$ was arbitrary, (10) is proved.

(^20) We assume that $\psi(q) < +\infty$ for every $q \in B_0^0$. This assumption is justified by the remark that if $\psi(q) = +\infty$ for some $q \subset B_0^0$, then clearly the left-hand member in (10) is $+\infty$ and then (10) is obvious.

Chapter 2. Lemmas on triples

2.1. Fundamental lemma. *Let there be given an admissible integrand f and triples*

$$T_0:\quad x_0(u),\qquad u\in B_0,\qquad\qquad T_n:\quad x_n(u),\qquad u\in B_n,$$

such that the following conditions are satisfied.

(1) $(T_0, B_0)\in K_1$.

(2) (T_n, B_n) *is quasi-linear.*

(3) (T_0, B_0) *satisfies condition* (c+) *relative to* f.

(4) *For every closed square* $q\subset B_0^0$ *there exists an* $N=N(q)$ *such that* $q\subset B_n^0$ *for* $n>N$.

(5) *On every closed square* $q\subset B_0^0$, $x_n(u)\rightarrow x_0(u)$ *uniformly.*

(6) ${}_0d[(T_0, B_0), (T_n, B_n)]\rightarrow 0$.

Then

$$\liminf_{n\rightarrow\infty} I(T_n, B_n, f) \geqq I(T_0, B_0, f).$$

Remark. By conditions (6) and (3) we have, for large $n, f(x_n(u), X_n(u))\geqq 0$ a.e. in B_n^0, and hence the application of 1.34 is justified.

2.2. **Proof.** According to the scheme outlined in 1.34, we define, for every closed square $q\subset B_0^0$,

$$\psi(q) = \liminf I(T_n, q, f).$$

Let u_0 be a point in B_0^0 and q_0 a small square in B_0^0 with center at u_0. Since we wish to obtain information about $\underline{D}\psi(u_0)$ for *almost every* point in B_0^0, we can assume that the following statements hold true for u_0 and q_0. First, $X_0(u_0)$ exists and

$$\frac{\iint_{q_0}\overset{i}{X}_0(u)du}{|q_0|}\rightarrow \overset{i}{X}_0(u_0),\qquad i=1, 2, 3, \tag{13}$$

if $|q_0|\rightarrow 0$ [21]. Next, we can assume that (cf. 1.3)

$$f(x_0(u_0), X_0(u_0)) = \overset{\alpha}{X}_0(u_0)f_\alpha(x_0(u_0), X_0(u_0)) > 0. \tag{14}$$

Indeed, we wish to prove that

$$\underline{D}\psi(u_0) \geqq f(x_0(u_0), X_0(u_0)). \tag{15}$$

But $\underline{D}\psi\geqq 0$ and $f(x_0(u_0), X_0(u_0))\geqq 0$. Thus (15) is obvious if $f(x_0(u_0), X_0(u_0)) =0$. Now it follows from (13) that

$$\frac{\iint_{q_0}\overset{\alpha}{X}_0(u)f_\alpha(x_0(u_0), X_0(u_0))du}{|q_0|}\rightarrow \overset{\alpha}{X}_0(u_0)f_\alpha(x_0(u_0), X_0(u_0)) \tag{16}$$

[21] If H is a measurable point set, then $|H|$ denotes its measure.

if $|q_0| \to 0$. By restricting the size of q_0, we shall have therefore (cf. (14))

$$\iint_{q_0} X_0^{\alpha}(u) f_\alpha(x_0(u_0), X_0(u_0)) du > 0. \tag{17}$$

From (14) it follows that

$$X_0(u_0) \neq 0.$$

Finally, we can assume that $E(x_0(u_0), X_0(u_0), \overline{X}) > 0$ whenever $\overline{X} \neq 0$ and ${}_u\overline{X} \neq {}_uX_0(u_0)$.

2.3. We now apply the lemma of 1.5 to the set of constants $(x_0(u_0), X_0(u_0))$. By that lemma, we have two positive constants δ_1, δ_2 such that whenever $0 < \eta < \delta_1$, we have

$$f(\bar{x}, \overline{X}) - f(x_0(u_0), \overline{X}) + E(x_0(u_0), X_0(u_0), \overline{X}) \geqq -\eta\delta_2 \overline{X}^{\alpha} f_\alpha(x_0(u_0), X_0(u_0))$$

for every set $(\bar{x}, \overline{X})$ for which

$$\overline{X}^{\alpha} f_\alpha(x_0(u_0), X_0(u_0)) > 0, \qquad |f(\bar{x}, \overline{X}) - f(x_0(u_0), \overline{X})| \leqq \eta \|\overline{X}\|.$$

2.4. We take an η satisfying $0 < \eta < \delta_1$. Let Q_0 be a fixed closed square in B_0^0, with center at u_0. On Q_0, by assumption, $x_n(u) \to x_0(u)$ uniformly. Hence we have an n_0 such that

$$|f(x_n(u), X_n(u)) - f(x_0(u), X_n(u))| \leqq (\eta/2)\|X_n(u)\| \text{ for } n > n_0, \text{ a.e. } u \in Q_0. \tag{18}$$

If we make the square q_0 of 2.2 sufficiently small, we shall have

$$q_0 \subset Q_0;$$

and, since $x_0(u)$ is continuous,

$$|f(x_0(u), X_n(u)) - f(x_0(u_0), X_n(u))| \leqq (\eta/2)\|X_n(u)\| \quad \text{for a.e. } u \in q_0. \tag{19}$$

Combining (18), (19) we see that

$$|f(x_n(u), X_n(u)) - f(x_0(u_0), X_n(u))| \leqq \eta\|X_n(u)\| \quad \text{for a.e. } u \in q_0,\ n > n_0. \tag{20}$$

By condition (c-!-) we have an $m_0 = m_0(q_0)$ such that

$$f(x_n(u), X_n(u)) \geqq 0 \qquad \text{for a.e. } u \in q_0,\ n > m_0. \tag{21}$$

We take now first a q_0 so small and then an n so large that all the relations (17), (20), (21) hold.

2.5. By 1.28 we have in q_0 a sequence of measurable sets V_n such that

$$\iint_{V_n} X_n^{\alpha}(u) f_\alpha(x_0(u_0), X_0(u_0)) du \xrightarrow[n\to\infty]{} \iint_{q_0} X_0^{\alpha}(u) f_\alpha(x_0(u_0), X_0(u_0)) du, \tag{22}$$

and (cf. (17) and 1.28, corollary)

$$X_n^{\alpha}(u) f_\alpha(x_0(u_0), X_0(u_0)) > 0 \qquad \text{for } u \in V_n. \tag{23}$$

We define

$$\overline{X}_n(u) = \begin{cases} X_n(u) & \text{on } V_n, \\ 0 & \text{on } q_0 - V_n. \end{cases} \tag{24}$$

By (21), (24), 1.3, 2.3, (23), (24) we can write now, for a.e. $u \in q_0$,

$$\begin{aligned} f(x_n(u), X_n(u)) &\geqq f(x_n(u), \overline{X}_n(u)) \\ &= f(x_n(u), \overline{X}_n(u)) - f(x_0(u_0), \overline{X}_n(u)) + E(x_0(u_0), X_0(u_0), \overline{X}_n(u)) \\ &\quad + \overline{X}_n^\alpha(u) f_\alpha(x_0(u_0), X_0(u_0)) \\ &\geqq (1 - \eta\delta_2)\overline{X}_n^\alpha(u) f_\alpha(x_0(u_0), X_0(u_0))du. \end{aligned}$$

Integrate over q_0 and let $n \to \infty$. By (22), (24) it follows that

$$\psi(q_0) \geqq (1 - \eta\delta_2) \iint_{q_0} X_0^\alpha(u) f_\alpha(x_0(u_0), X_0(u_0))du.$$

Divide by $|q_0|$ and let $|q_0| \to 0$. By (16) it follows that

$$\underline{D}\psi(u_0) \geqq (1 - \eta\delta_2) X_0^\alpha(u_0) f_\alpha(x_0(u_0), X_0(u_0)) = (1 - \eta\delta_2) f(x_0(u_0), X_0(u_0)).$$

Since η was arbitrary, this shows that

$$\underline{D}\psi(u_0) \geqq f(x_0(u_0), X_0(u_0)).$$

On account of 1.34, this proves the lemma of 2.1.

2.6. Lemma. *Let there be given an admissible integrand f and triples* (T_0, B_0), (T_n, B_n), *such that the following conditions are satisfied.*

(1) $(T_0, B_0) \in K_1$.
(2) (T_n, B_n) *is quasi-linear.*
(3) ${}_0d[(T_0, B_0), (T_n, B_n)] \to 0$.
(4) (T_0, B_0) *satisfies condition* (c+) *relative to f.*
Then

$$\liminf I(T_n, B_n, f) \geqq I(T_0, B_0, f). \tag{25}$$

2.7. **Proof.** For clarity, we want to think of B_n as being located in a different plane $\bar{u}$. We write therefore explicitly

$$T_0: \ x_0(u), \quad u \in B_0, \qquad T_n: \ x_n(\bar{u}), \quad \bar{u} \in B_n.$$

Let us put

$$\delta_n = {}_0d[(T_0, B_0), (T_n, B_n)].$$

We have then, for every n, a sense-preserving topological transformation

$$\bar{u} = \tau_n(u), \quad u \in B_0,$$

which carries B_0 into B_n and for which

(26) $$\|x_0(u) - x_n(\tau_n(u))\| < \delta_n + 1/n.$$

Let

$$u = \sigma_n(\bar{u}), \qquad \bar{u} \in B_n,$$

be the inverse of τ_n. Since B_n is a polygon, we have (cf. 1.27) a sense-preserving, topological, quasi-linear transformation

$$u = \sigma_n^*(\bar{u}), \qquad \bar{u} \in B_n,$$

which carries B_n into some polygonal region B_n^* in the u-plane (where B_0 itself is located) and which satisfies the condition

(27) $$\|\sigma_n(\bar{u}) - \sigma_n^*(\bar{u})\| < 1/n, \qquad \bar{u} \in B_n\text{[22]}.$$

Let us denote by

$$\bar{u} = \tau_n^*(u), \ u \in B_n^*,$$

the inverse of σ_n^* and let us introduce the triple

$$T_n^*: \ x_n(\tau_n^*(u)), \ u \in B_n^*.$$

Since (T_n, B_n) and τ_n^* are quasi-linear and since τ_n^* is sense-preserving, it follows by 1.10 that

(28) $$I(T_n, B_n, f) = I(T_n^*, B_n^*, f),$$

and further, on account of (26) and (27), that the triples (T_0, B_0), (T_n^*, B_n^*) and the integrand f satisfy the assumptions of the lemma in 2.1. Hence, by that lemma,

(29) $$\liminf I(T_n^*, B_n^*, f) \geqq I(T_0, B_0, f);$$

(28) and (29) imply (25).

2.8. Lemma. *If* $(T_0, B_0)\in K_2$, $(\overline{T}_0, \overline{B}_0)\in K_2$, *and if*

(30) $${}_0d[(T_0, B_0), (\overline{T}_0, \overline{B}_0)] = 0,$$

then

(31) $$I(T_0, B_0, \|X\|) = I(\overline{T}_0, \overline{B}_0, \|X\|).$$

Proof. Since $(T_0, B_0)\in K_2$, we have a sequence of quasi-linear triples (T_n, B_n) such that

(32) $${}_0d[(T_0, B_0), (T_n, B_n)] \rightarrow 0,$$

and

(22) Cf. (8).

(33) $$I(T_n, B_n, \|X\|) \to I(T_0, B_0, \|X\|);$$

(30) and (32) imply that

$$_od[(\overline{T}_0, \overline{B}_0), (T_n, B_n)] \to 0.$$

Applying the lemma of 2.6 to the triples $(\overline{T}_0, \overline{B}_0)$, (T_n, B_n) with $f = \|X\|$, we obtain

(34) $$\liminf I(T_n, B_n, \|X\|) \geqq I(\overline{T}_0, \overline{B}_0, \|X\|),$$

and (33) and (34) yield

(35) $$I(T_0, B_0, \|X\|) \geqq I(\overline{T}_0, \overline{B}_0, \|X\|).$$

The complementary inequality

(36) $$I(\overline{T}_0, \overline{B}_0, \|X\|) \geqq I(T_0, B_0, \|X\|)$$

is obtained in the same manner, (35) and (36) imply (31).

2.9. Lemma. *Let there be given an admissible integrand f and triples* (T_0, B_0), (T_n, B_n) *such that the following conditions are satisfied.*

(1) $(T_0, B_0) \in K_2$.
(2) (T_n, B_n) *is quasi-linear.*
(3) $_od[(T_0, B_0), (T_n, B_n)] \to 0$.
(4) $I(T_n, B_n, \|X\|) \to I(T_0, B_0, \|X\|)$.
Then

(37) $$I(T_n, B_n, f) \to I(T_0, B_0, f).$$

Proof. From 1.15 it follows easily that if the positive constant H is sufficiently large, then (T_0, B_0) satisfies condition (c+) relative to the integrand

$$H\|X\| + f.$$

Hence we have, by the lemma of 2.6,

$$\liminf \{HI(T_n, B_n, \|X\|) + I(T_n, B_n, f)\} \geqq HI(T_0, B_0, \|X\|) + I(T_0, B_0, f).$$

In view of condition (4) of the lemma it follows that

(38) $$\liminf I(T_n, B_n, f) \geqq I(T_0, B_0, f).$$

The same reasoning, applied to the integrand $-f$, yields

(39) $$\liminf (-I(T_n, B_n, f)) \geqq -I(T_0, B_0, f).$$

(38) and (39) imply (37).

2.10. Lemma. *Let there be given an admissible integrand f and triples* (T_0, B_0), $(\overline{T}_0, \overline{B}_0)$ *such that the following conditions are satisfied.*

(1) $(T_0, B_0) \in K_2$.

(2) $(\overline{T}_0, \overline{B}_0) \in K_2$.

(3) ${}_0d[(T_0, B_0), (\overline{T}_0, \overline{B}_0)] = 0$.

Then

$$I(T_0, B_0, f) = I(\overline{T}_0, \overline{B}_0, f). \tag{40}$$

Proof. Since $(\overline{T}_0, \overline{B}_0) \in K_2$, we have a sequence of quasi-linear triples (T_n, B_n) such that

$${}_0d[(\overline{T}_0, \overline{B}_0), (T_n, B_n)] \to 0, \tag{41}$$

and

$$I(T_n, B_n, \|X\|) \to I(\overline{T}_0, \overline{B}_0, \|X\|); \tag{42}$$

(41) and the condition (3) above imply that

$${}_0d[(T_0, B_0), (T_n, B_n)] \to 0. \tag{43}$$

By 2.8 we have

$$I(\overline{T}_0, \overline{B}_0, \|X\|) = I(T_0, B_0, \|X\|).$$

Hence, by (42),

$$I(T_n, B_n, \|X\|) \to I(T_0, B_0, \|X\|). \tag{44}$$

By 2.9, the relations (41), (42) and (43), (44), respectively, imply that

$$I(T_n, B_n, f) \to I(\overline{T}_0, \overline{B}_0, f) \tag{45}$$

and

$$I(T_n, B_n, f) \to I(T_0, B_0, f). \tag{45*}$$

(45) and (45*) imply (40).

CHAPTER 3. THE PRINCIPAL THEOREMS[23]

3.1. DEFINITION. *A continuous surface of class K_2, oriented or not, satisfies condition* (c), (+c), *or* (c+), *respectively, relative to an admissible integrand f if it possesses a representation of class K_2 that satisfies condition* (c), (+c), *or* (c+), *respectively, relative to f* (cf. 1.11, 1.12, 1.13).

3.2. THEOREM. *Let f be an admissible integrand and ${}_0S$ an oriented continuous surface of class K_2. If (T_1, B_1), (T_2, B_2) are any two representations of class K_2 of ${}_0S$, then*

$$I(T_1, B_1, f) = I(T_2, B_2, f).$$

This is a direct consequence of 2.10.

[23] Disregarding a few minor items, the semi-continuity theorems of this chapter are generalizations of theorems of McShane. The theorems on the Lebesgue area of surfaces are generalizations of previous results of McShane, Morrey and the author.

3.3. On account of 3.2, if f is admissible and if ${}_oS$ is of class K_2, then $I(T, B, f)$ has the same value for all representations of class K_2 of ${}_oS$. This common value may be denoted therefore by $I({}_oS, f)$.

3.4. We shall denote by ${}_o\mathfrak{S}$ the class of all oriented continuous surfaces of class K_2. By 3.3 the functional $I({}_oS, f)$ is then defined for every admissible f and for every ${}_oS \in {}_o\mathfrak{S}$.

3.5. THEOREM. *For fixed admissible f the functional $I({}_oS, f)$ is lower semi-continuous in ${}_o\mathfrak{S}$ at every oriented continuous surface ${}_oS \in {}_o\mathfrak{S}$ which satisfies condition* (c+) *relative to f.*

Proof. Take any sequence ${}_oS_n$ such that ${}_oS_n \rightarrow {}_oS_0$ and ${}_oS_n \in {}_o\mathfrak{S}$. Let (T_0, B_0), (T_n, B_n) be representations of class K_2 of ${}_oS_0$, ${}_oS_n$, respectively, where (T_0, B_0) satisfies condition (c+) relative to f. We have then

$$ {}_od[(T_n, B_n), (T_0, B_0)] \rightarrow 0, \tag{46} $$

and by 3.3

$$ I({}_oS_0, f) = I(T_0, B_0, f), \tag{47} $$

$$ I({}_oS_n, f) = I(T_n, B_n, f). \tag{48} $$

Since the triple (T_n, B_n) is of class K_2, we have by definition a sequence of quasi-linear triples (T_n^j, B_n^j) such that

$$ {}_od[(T_n^j, B_n^j), (T_n, B_n)] \xrightarrow[j\to\infty]{} 0, \tag{49} $$

and

$$ I(T_n^j, B_n^j, \|X\|) \xrightarrow[j\to\infty]{} I(T_n, B_n, \|X\|). \tag{50} $$

By 2.9 it follows from (49) and (50) that

$$ I(T_n^j, B_n^j, f) \xrightarrow[j\to\infty]{} I(T_n, B_n, f). $$

We have therefore, for every n, a j_n such that

$$ {}_od[(T_n^{j_n}, B_n^{j_n}), (T_n, B_n)] < 1/n, \tag{51} $$

and

$$ I(T_n^{j_n}, B_n^{j_n}, f) < I(T_n, B_n, f) + 1/n. \tag{52} $$

(51) and (46) imply that

$$ {}_od[(T_n^{j_n}, B_n^{j_n}), (T_0, B_0)] \rightarrow 0. \tag{53} $$

(53) implies, by 2.6, that

(54) $$\liminf_{n\to\infty} I(T_n^{i_n}, B_n^{i_n}, f) \geqq I(T_0, B_0, f).$$

(47), (48), (52), (54) imply that

$$\liminf I({}_oS_n, f) \geqq I({}_oS_0, f).$$

3.6. THEOREM. *Let f, ${}_oS_0$, ${}_oS_n$ satisfy the following conditions.*

(a) ${}_oS_0 \in {}_o\mathfrak{C}, {}_oS_n \in {}_o\mathfrak{C}$.

(b) ${}_oS_n \to {}_oS_0$.

(c) ${}_oS_0$ *satisfies condition* (c) *with respect to the admissible integrand f.*

(d) *There exists a finite constant N such that for every n*

$$I({}_oS_n, \|X\|) < N.$$

Then

(55) $$\liminf I({}_oS_n, f) \geqq I({}_oS_0, f).$$

Proof. Clearly, for every $\epsilon > 0$, ${}_oS_0$ satisfies condition (c+) with respect to the integrand $f + \epsilon\|X\|$ (cf. 1.16). Hence by 3.5

$$\liminf_{n\to\infty} \{I({}_oS_n, f) + \epsilon I({}_oS_n, \|X\|)\} \geqq I({}_oS_0, f) + \epsilon I({}_oS_0, \|X\|).$$

Since $\epsilon > 0$ was arbitrary, (55) follows on account of condition (d).

3.7. THEOREM. *For fixed admissible f, the functional $I({}_oS, f)$ is lower semi-continuous in ${}_o\mathfrak{C}$ at every surface ${}_oS \in {}_o\mathfrak{C}$ that satisfies condition* (+c) *with respect to f.*

Proof. Let us deny the assertion. Then we assert the existence of surfaces ${}_oS_0$, ${}_oS_n$ in ${}_o\mathfrak{C}$ such that ${}_oS_n \to {}_oS_0$ and

(56) $$\lim I({}_oS_n, f) < I({}_oS_0, f),$$

while ${}_oS_0$ satisfies condition (+c) relative to f. This last fact implies the existence, in x-space, of a closed bounded set A such that $\sum({}_oS_0) \subset A^0$ and $f(x, X) > 0$ for $x \in A$ and for every vector $X \neq 0$. It follows that there exists a constant $m > 0$ such that

(57) $$f(x, X) \geqq m\|X\| \qquad \text{for } x \in A$$

and for every vector X. Since ${}_oS_n \to {}_oS_0$, we have an n_0 such that

(58) $$\sum({}_oS_n) \subset A^0 \qquad \text{for } n > n_0.$$

Let

$$T_n: \quad x_n(u), \qquad u \in B_n,$$

be a representation of class K_2 of ${}_oS_n$. We have then, by 3.3, (57), (58) for $n > n_0$

(59) $I({}_oS_n, \|X\|) = I(T_n, B_n, \|X\|) \leqq (1/m)I(T_n, B_n, f) = (1/m)I({}_oS_n, f)$.

(56) and (59) imply the existence of a finite constant N such that $I({}_0S_n, \|X\|) < N$ for every n. By 3.6 this implies that

$$\liminf I({}_oS_n, f) \geqq I({}_oS_0, f),$$

in contradiction with (56).

3.8. THEOREM. *If the admissible integrand f is independent of x, that is, if* $f=f(X)$, *then the functional* $I({}_oS, f)$ *is lower semi-continuous in* ${}_o\mathfrak{C}$ *at every surface* ${}_oS_0 \in {}_o\mathfrak{C}$ *that satisfies condition* (c) *with respect to f.*

Proof. Checking through the proof of the theorem in 3.5, where the stronger condition (c+) was assumed, we find (cf. 2.5) that the positivity of the E-function was used solely to obtain an estimate for the difference

$$f(x_n(u), \overline{X}_n(u)) - f(x_0(u_0), \overline{X}_n(u)).$$

Since this difference vanishes in the present case, it is clear that condition (c+) can be replaced by the weaker condition (c).

3.9. In what precedes we worked with *oriented* continuous surfaces. Checking back through our discussion, we find that this restriction served the sole purpose of securing the invariance of the integral $I(T, B, f)$ in the quasi-linear case. From the remarks made in 1.10 it is then clear that *the restriction to oriented surfaces becomes unnecessary if the integrand f satisfies the condition*

$$f(x, X) = f(x, -X). \tag{60}$$

That is, if (60) holds, then in all that precedes we can use the *Fréchet distance* and *continuous surfaces* instead of the *oriented distance* and *oriented continuous surfaces*. By way of illustration, we state presently the theorems corresponding to those in 3.2, 3.5, 3.7, 3.8. We shall use $\mathfrak{C}$ to denote the class of all continuous surfaces of class K_2.

THEOREM. *If f is admissible, then the quantity* $I(T, B, f)$ *has the same value for all representations of class* K_2 *of the surface* $S \in \mathfrak{C}$ *if the admissible integrand f satisfies* (60). *This quantity may be therefore denoted by* $I(S, f)$.

THEOREM. *If f is an admissible integrand satisfying* (60), *then the functional* $I(S, f)$ *is lower semi-continuous on* $\mathfrak{C}$ *at every surface* $S \in \mathfrak{C}$ *that satisfies either condition* (+c) *or* (c+) *with respect to f. If in addition f is independent of x, that is, if* $f=f(X)$, *then we have lower semi-continuity at every surface* $S \in \mathfrak{C}$ *that satisfies condition* (c) *with respect to f.*

3.10. DEFINITION. *If an oriented continuous surface* ${}_oS$ *admits of a representation* (T, B) *which is quasi-linear, then* ${}_oS$ *will be called an oriented polyhedron and will be denoted by* ${}_o\mathfrak{P}$. *Clearly* ${}_o\mathfrak{P} \in {}_o\mathfrak{C}$, *since obviously every quasi-linear triple is of class* K_2.

3.11. Definition. *If a (non-oriented) continuous surface S admits of a representation which is quasi-linear, then S will be called a polyhedron and will be denoted by $\mathfrak{P}$. Clearly $\mathfrak{P} \in \mathfrak{C}$, since every quasi-linear triple is of class K_2.*

3.12. Definition. *Let ${}_oS$ be an oriented continuous surface. Its Lebesgue area $L({}_oS)$ is defined as the greatest lower bound of*

$$\liminf I({}_o\mathfrak{P}_n, \|X\|)$$

for all sequences ${}_o\mathfrak{P}_n$ such that ${}_o\mathfrak{P}_n \to {}_oS$.

3.13. Definition. *Let S be a (non-oriented) continuous surface. Its Lebesgue area $L(S)$ is defined as the greatest lower bound of*

$$\liminf I(\mathfrak{P}_n, \|X\|),$$

for all sequences $\mathfrak{P}_n$ such that $\mathfrak{P}_n \to S$.

3.14. Theorem. *Let ${}_oS$ be of class K_1. Then for every representation (T, B) of class K_1 of ${}_oS$ we have*

$$L({}_oS) \geqq I(T, B, \|X\|). \tag{61}$$

The sign of equality holds if and only if (T, B) is of class K_2[24].

Proof. Let ${}_o\mathfrak{P}_n$ be any sequence such that ${}_o\mathfrak{P}_n \to {}_oS$, and let (T_n, B_n) be a quasi-linear representation of ${}_o\mathfrak{P}_n$. Then

$${}_od[(T_n, B_n), (T, B)] \to 0.$$

Obviously *every* triple of class K_1 satisfies condition (c+) with respect to the integrand $\|X\|$. Hence, by 2.6,

$$\liminf I(T_n, B_n, \|X\|) \geqq I(T, B, \|X\|).$$

Since this holds for every sequence ${}_o\mathfrak{P}_n$ such that ${}_o\mathfrak{P}_n \to {}_oS$, and since

$$I(T_n, B_n, \|X\|) = I({}_o\mathfrak{P}_n, \|X\|)$$

by (3.3), the inequality (61) follows. *Suppose now that*

$$L({}_oS) = I(T, B, \|X\|). \tag{62}$$

It follows from the definition of $L({}_oS)$ that we have a sequence ${}_o\mathfrak{P}_n^*$ such that

$${}_o\mathfrak{P}_n^* \to {}_oS \tag{63}$$

and

$$I({}_o\mathfrak{P}_n^*, \|X\|) \to L({}_oS). \tag{64}$$

Let (T_n^*, B_n^*) be a quasi-linear representation of ${}_o\mathfrak{P}_n^*$. We have then, by (63),

[24] This theorem could also be inferred from Radó [2].

(65) $$_od[(T_n^*, B_n^*), (T, B)] \to 0,$$

while (64) and (62) imply that (cf. 3.3)

(66) $$I(T_n^*, B_n^*, \|X\|) \to I(T, B, \|X\|).$$

By (65), (66) the triple (T, B) is of class K_2 (cf. 1.19).

Suppose conversely that (T, B) *is of class* K_2. We have then, by definition, a sequence of quasi-linear triples $(\overline{T}_n, \overline{B}_n)$ such that

(67) $$_od[(\overline{T}_n, \overline{B}_n), (T, B)] \to 0,$$

and

(68) $$I(\overline{T}_n, \overline{B}_n, \|X\|) \to I(T, B, \|X\|).$$

The quasi-linear triple $(\overline{T}_n, \overline{B}_n)$ is a representation of an oriented polyhedron $_o\overline{\mathfrak{P}}_n$. Using (3.3), (1.22), the relations (67), (68) can be rewritten in the form

(69) $$_o\overline{\mathfrak{P}}_n \to {_oS},$$

(70) $$I(_o\overline{\mathfrak{P}}_n, \|X\|) \to I(T, B, \|X\|).$$

By the definition of $L(_oS)$, (69) and (70) imply that

(71) $$I(T, B, \|X\|) \geqq L(_oS).$$

(71) and (61) (which we have already proved) imply that

$$L(_oS) = I(T, B, \|X\|).$$

3.15. THEOREM. *The oriented continuous surface* $_oS$ *is of class* K_2 *if and only if it admits of a representation*

$$T:\quad x(u), \qquad u \in B,$$

such that the following conditions hold.

(a) *The Jacobians* $X^1(u)$, $X^2(u)$, $X^3(u)$ *exist a.e. in* B^0 *and are summable there.*

(b) $L(_oS) = \iint_{B_0} \|X(u)\| du.$

Briefly: $_oS$ *is of class* K_2 *if and only if it admits of a representation where the Lebesgue area is given by the usual integral formula.*

This theorem is merely a rewording of the second half of the theorem in 3.14.

3.16. THEOREM. *If* (T, B) *is a quasi-linear representation of the oriented polyhedron* $_o\mathfrak{P}$, *then*

$$L(_o\mathfrak{P}) = I(T, B, \|X\|).$$

Briefly: *The Lebesgue area of an oriented polyhedron is equal to its area in the elementary sense.*

This theorem is merely a very special case of 3.14, since every quasi-linear triple is of class K_2.

3.17. Similar theorems hold for non-oriented surfaces. Since the proofs are entirely analogous to those in 3.14, 3.15, 3.16, we only state the results in the following theorems.

3.18. THEOREM. *Let S be of class K_1. Then for every representation (T, B) of class K_1 of S we have*

$$L(S) \geqq I(T, B, \|X\|).$$

The sign of equality holds if and only if (T, B) is of class K_2.

3.19. THEOREM. *A continuous surface S is of class K_2 if and only if it admits of a representation*

$$T\colon x(u), \quad u \in B$$

such that the following conditions hold.

(a) *The Jacobians $X^1(u)$, $X^2(u)$, $X^3(u)$ exist a.e. in B^0 and are summable there.*

(b) $L(S) = \iint_{B^0} \|X(u)\| du$.

Briefly: S is of class K_2 if and only if it admits of a representation where the Lebesgue area of S is given by the usual integral formula.

3.20. THEOREM. *If (T, B) is a quasi-linear representation of a (non-oriented) polyhedron $\mathfrak{P}$, then*

$$L(\mathfrak{P}) = I(T, B, \|X\|).$$

Briefly: The Lebesgue area of a polyhedron is equal to its area in the elementary sense.

3.21. *In conclusion, we want to compare our surfaces of class K_2 with the surfaces used by McShane and Morrey in their researches*(25). For brevity, we shall restrict our remarks to *non-oriented* continuous surfaces, and we shall follow the presentation of Morrey.

3.22. According to Morrey, a continuous surface S is of class L if it admits of a representation

$$T_0\colon \quad x_0(u), \qquad u \in R_0, \tag{72}$$

where R_0 is the square given by

$$R_0\colon \quad 0 \leqq \overset{1}{u} \leqq 1, \qquad 0 \leqq \overset{2}{u} \leqq 1,$$

such that the following conditions are satisfied(26).

(25) See McShane [1, 2] and Morrey [1].

(26) For convenience, we have split the two conditions (i), (ii) of Morrey [1, p. 701], into three conditions (a), (b), (c).

(a) $\overset{i}{x}_0(u)$, $i=1, 2, 3$, is absolutely continuous in the sense of Tonelli.

(b) The Jacobians

$$\frac{\partial(\overset{2}{x}_0, \overset{3}{x}_0)}{\partial(u^1, u^2)}, \qquad \frac{\partial(\overset{3}{x}_0, \overset{1}{x}_0)}{\partial(u^1, u^2)}, \qquad \frac{\partial(\overset{1}{x}_0, \overset{2}{x}_0)}{\partial(u^1, u^2)}$$

are summable in R_0.

(c) For every rectangle R: $a^1 \leqq u^1 \leqq b^1$, $a^2 \leqq u^2 \leqq b^2$, completely interior to R_0, we have

$$\iint_R \left| \overset{i}{X}_0(u) - \overset{i}{X}_h(u) \right| du \underset{h\to 0}{\longrightarrow} 0, \qquad i = 1, 2, 3,$$

where $\overset{1}{X}_0(u)$, $\overset{2}{X}_0(u)$, $\overset{3}{X}_0(u)$ are the Jacobians corresponding to the given triple T_0, while $\overset{1}{X}_h(u)$, $\overset{2}{X}_h(u)$, $\overset{3}{X}_h(u)$ are the Jacobians corresponding to the triple

$$T_h: \quad x_h(u), \qquad u \in R_h,$$

where

$$\overset{i}{x}_h(u) = \frac{1}{4h^2}\int_{-h}^{h}\int_{-h}^{h} \overset{i}{x}_0(\overset{1}{u} + \overset{1}{v}, \overset{2}{u} + \overset{2}{v}) d\overset{1}{v} d\overset{2}{v}, \qquad i = 1, 2, 3,$$

and R_h is the rectangle

$$R_h: \quad h \leqq u^1 \leqq 1 - h, \qquad h \leqq u^2 \leqq 1 - h.$$

If these conditions are satisfied, then the representation (72) and the triple T_0 will be also said to be of class L.

3.23. *Suppose the representation* (72) *is of class L.* Let us define, for every positive integer n, a rectangle

$$(73) \qquad R^{(n)}: \quad \frac{1}{n} \leqq u^1 \leqq 1 - \frac{1}{n}, \qquad \frac{1}{n} \leqq u^2 \leqq 1 - \frac{1}{n} \cdot$$

Clearly then

$$(74) \qquad {}_0d[(T_0, R^{(n)}), (T_0, R_0)] \underset{n\to\infty}{\longrightarrow} 0,$$

and

$$(75) \qquad \iint_{R^{(n)}} \| X_0(u) \| du \underset{n\to\infty}{\longrightarrow} \iint_{R_0} \| X_0(u) \| du.$$

For fixed n, the rectangle $R^{(n)}$ will be completely interior to the rectangle R_h if h is sufficiently small. By a well known property of integral means, we have $\overset{i}{x}_h(u) \to_{h\to 0} \overset{i}{x}_0(u)$, $i=1, 2, 3$, uniformly in $R^{(n)}$ for fixed n. Combining this remark with condition (c) in 3.22, we find that for every n we have an $h_n > 0$ such that

(76) $$ {}_0d[(T_{h_n}, R^{(n)}), (T_0, R^{(n)})] < \frac{1}{n}, $$

(77) $$ \iint_{R^{(n)}} \|X_0(u) - X_{h_n}(u)\| du < \frac{1}{n} \cdot $$

Since for fixed n and small h the functions $x_h^i(u)$ are continuous in $R^{(n)}$ together with their partial derivatives of the first order, we have by a familiar reasoning a quasi-linear triple $(\overline{T}^{(n)}, R^{(n)})$ such that

(78) $$ {}_0d[(\overline{T}^{(n)}, R^{(n)}), (T_{h_n}, R^{(n)})] < \frac{1}{n}, $$

(79) $$ \iint_{R^{(n)}} \|X_{h_n}(u) - \overline{X}^{(n)}(u)\| du < \frac{1}{n}, $$

where $\overline{X}^{(n)}(u)$ is the vector whose components are the Jacobians corresponding to $(\overline{T}^{(n)}, R^{(n)})$. Combining (74)–(79), we now obtain the relations

$$ {}_0d[(\overline{T}^{(n)}, R^{(n)}), (T_0, R_0)] \xrightarrow[n\to\infty]{} 0, $$

$$ \iint_{R^{(n)}} \|\overline{X}^{(n)}(u)\| du \xrightarrow[n\to\infty]{} \iint_{R_0} \|X_0(u)\| du. $$

Since the triples $(\overline{T}^{(n)}, R^{(n)})$ are quasi-linear, the last two relations imply that the triple (T_0, R_0) is of class K_2 (cf. 1.19).

3.24. In other words: *Every triple* (T_0, R_0) *which is of class L in the sense of Morrey is of class* K_2 *in our sense. In fact, in establishing this result we did not use condition* (a) *in* 3.22 *at all,* as a glance through 3.23 shows. *Hence, every triple* (T_0, R_0) *which satisfies conditions* (b) *and* (c) *in* 3.22, *is of class* K_2 *in our sense.*

3.25. *Morrey proved that if conditions* (a), (b), (c) *in* 3.22 *are satisfied, then the Lebesgue area of the surface S determined by the triple* (T_0, R_0) *is given by the usual integral formula.* Condition (a), requiring that the coordinate functions $x_0^i(u)$ be *absolutely continuous in the Tonelli sense,* played an essential part in his proof, and consideration of the case of surfaces given in non-parametric form seemed to suggest that condition (a) could not be dispensed with. However the present author found some years later that *condition* (a) *could be replaced by the weaker condition requiring only bounded variation in the sense of Tonelli* on the part of $x_0^i(u)$(27). *At present we see that even this weaker condition is unnecessary.* Indeed, by 3.24, conditions (b) and (c) in 3.22 are sufficient to insure that (T_0, R_0) is of class K_2. Hence, by 3.14, the Lebesgue area of the surface represented by (T_0, R_0) is given by the usual integral formula as soon as (T_0, R_0) satisfies only conditions (b) and (c) in 3.22.

(27) See Radó [1].

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