ON n-ARC CONNECTEDNESS

BY

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1. **Introduction.** In this paper a simple elementary proof will be given for the well known theorem that any two points a and b of a locally connected continuum M which are not separated in M by any set of less than n points can be joined in M by a set of n independent simple arcs, that is, arcs intersecting by pairs in just $a+b$. The theorem in this form was first proven by Rutt [1](1) for plane continua M and for general M by Nöbeling [2] and Menger [3] and has been extended by Zippin [4]. Our proof represents an extension to arbitrary n of the inductive type of reasoning used by the author [5] in simplifying the proof of the cyclic connectedness theorem (which is the case $n=2$ of the above theorem). It holds for locally compact M, as in Zippin's generalized form

Our terminology in the main will be that of Menger-Nöbeling and of the author's book [5]. We recall that a *continuum* is a connected compact metric space and a *generalized continuum* is the same except that compactness is replaced by local compactness. A metric set W is *regular* at a point $x \in W$ if x has arbitrarily small neighborhoods in W whose boundaries are finite sets. A subset X of W *separates* two disjoint sets P and Q in W provided $W-X$ is the sum of two separated sets one containing P and the other Q. Such a set X *separates P and Q in the broad sense* in W provided $W-X$ is the sum of two separated sets (possibly empty), one containing $P-X\cdot P$ and the other $Q-X\cdot Q$. The set W is *n-point connected* between P and Q provided no set of less than n points separates P and Q in W, and W is *n-point strongly connected* between P and Q provided no set of less than n points separates P and Q in the broad sense in W. An arc *joins* two sets if it has an end point in each of them.

2. **Theorem**(2). *If the locally connected separable complete metric space M is n-point strongly connected between two disjoint closed sets P and Q, then M contains n disjoint arcs joining P and Q.*

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(1) Numbers in brackets refer to the bibliography at the end of the paper.

(2) This is called the *second n-arc theorem* by Menger [3]. It seems to have been proved heretofore only for locally compact spaces M, except in the cases $n=1$, where it becomes the arcwise connectedness theorem established for complete spaces by Moore [6] and Menger [7], and $n=2$, where it was proved for complete spaces by the author [8]. It should be noted that the main theorem, §3, *does not hold* for complete spaces even in case $n=2$, as was shown by the author in [8].

The proof will be by induction on n. The case $n=1$ is an immediate consequence of the arcwise connectedness theorem, since then some component of M must intersect both P and Q. Suppose the theorem holds for all positive integers less than n.

Let S denote the set of all $x \in M$ such that there exists a sum S_x of n disjoint arcs, $n-1$ of which join P and Q and the remaining one px joins P and x. Then S is open in M and nonempty. For if $x \in S$ and R is any region in M with $x \in R \subset M - S_x + px$, the arc px can be extended in R to any $y \in R$ thus giving a system S_y for y so that $S \supset R$. Also if $y \in M$, $M-y$ is $(n-1)$-point strongly connected between $(P+y)-y$ and $(Q+y)-y$, because if a set X of less than $n-1$ points separated these two sets in $M-y$ in the broad sense, $X+y$ would similarly separate P and Q in M. Hence $P \subset S$ since for any y, $M-y$ contains a set of $n-1$ disjoint arcs joining P and Q.

The set S is also closed in M. For let y be any limit point of S. Then as just shown, $M-y$ contains a sum A of $n-1$ disjoint arcs $\alpha_1, \alpha_2, \cdots, \alpha_{n-1}$ each joining P and Q. Let R be a region about y with $\overline{R} \subset M-(P+A)$. Let $x \in R \cdot S$ and let S_x be a sum of n arcs determined by x as above. Then $S_x - R \cdot S_x$ contains a sum B of n disjoint arcs $\beta_1, \beta_2, \cdots, \beta_n$ each joining P and $Q+\overline{R}$. Let $\alpha_i = s_i q_i$, $\beta_i = p_i r_i$ where $p_i, s_i \in P$, $q_i \in Q$, $r_i \in (Q+\overline{R})$. On α_i in the order q_i, s_i let q_i^1 be the first point of $B+P$, let $Q_1 = \sum_1^{n-1} q_i^1$ and $A_1 = \sum_1^{n-1} q_i q_i^1$. On β_i in the order p_i, r_i let p_i^1 be the first point of $Q+\overline{R}+Q_1$ and let $B_1 = \sum_1^n p_i p_i^1$. Repeating: on α_i in the order q_i, s_i let q_i^2 be the first point of B_1+P, let $Q_2 = \sum_1^{n-1} q_i^2$ and $A_2 = \sum_1^{n-1} q_i q_i^2$; on β_i in the order p_i, r_i let p_i^2 be the first point of $Q+\overline{R}+Q_2$ and $B_2 = \sum_1^n p_i p_i^2$; and so on. Continue this process. For some integer m we must have $A_j = A_m$ and $B_j = B_m$ for all $j > m$. For otherwise $B_j \neq B_{j+1}$ for all j; and hence for some $i \leqq n$ and infinitely many j's: $j_1, j_2, \cdots$, we have $p_i^j \neq p_i^{j+1}$. But this is impossible since the open subarcs $[p_i^{j_k} p_i^{j_k+1}]$ of β_i are disjoint and $p_i^{j_k}$ and $p_i^{j_k+1}$ lie on *different* arcs α_i.

Now A_m and B_m are sums of $n-1$ and n disjoint arcs respectively; and $A_m \cdot B_m$ is contained in the set Q_m of end points q_i^m of the arcs in A_m. Thus if we keep intact each arc in A_m having its end point q_i^m in P and to each remaining arc of A_m, say $q_k q_k^m$, we add the unique arc $p_t p_t^m$ which it intersects so that $p_t^m = q_k^m$, we obtain a set of $n-1$ disjoint arcs $e_1, e_2, \cdots, e_{n-1}$ each joining P and Q.

Now since Q_m contains exactly $n-1$ points, there is at least one arc β_i, say β_k, with $Q_m \cdot \beta_k = 0$. But this gives $Q_i \cdot \beta_k = 0$ for all $i \leqq m$, since by construction $\beta_k \cdot Q_i \neq 0$ gives $\beta_k \cdot Q_j \neq 0$ for all $j > i$ and hence $\beta_k \cdot Q_m \neq 0$. Accordingly, $p_k r_k = \beta_k = p_k p_k^m \subset B_m$ so that $\beta_k \cdot E = 0$ where $E = \sum_1^{n-1} e_i$.

Now if $r_k \in \overline{R}$, $\beta_k + R + W$ (W a region about r_k not intersecting E) contains an arc $p_k y$ which, together with the arcs in E, forms a system S_y so that $y \in S$. On the other hand, if $r_k \in Q$, $E + \beta_k$ consists of n disjoint arcs joining P and Q; and even in this case, $R+$the arc of S_x containing $x+$one of the arcs of $E+\beta_k$ contains an arc py which together with the remaining $n-1$ arcs in $E+\beta_k$ forms a system S_y so that $y\in S$.

Thus S is both open and closed in M; and since M contains arcs joining P and Q and $P\subset S$, we have $S\cdot Q\neq 0$. Hence if $x\in S\cdot Q$, the system S_x determined by x satisfies our theorem.

3. **Theorem.** *If the locally connected generalized continuum M is n-point connected between two of its points a and b, it contains a set of n independent arcs joining a and b.*

Again the proof is by induction on n. For $n=1$ the conclusion follows by arcwise connectedness. We suppose it holds for all $k<n$. If $M-a-b$ contains more than one component Q with $\overline{Q}\supset a+b$, $M_1=\overline{Q}$ and $M_2=M-Q$ are locally connected generalized continua; and since if a set K_i separates a and b in M_i $(i=1, 2)$, K_1+K_2 separates a and b in M, it follows that M_i is n_i-point connected between a and b $(i=1, 2)$, where $0<n_i\leqq n-1$ and $n_1+n_2=n$; and our induction hypothesis gives a set of n_i independent arcs in M_i joining a and b whose sum is a set of n such arcs as required by the theorem. Thus we may suppose there is only one such component Q and we set $\overline{Q}=E$. Since any set separating a and b in E also separates them in M, E is n-point connected between a and b.

Lemma. *E contains a locally connected generalized continuum N which is n-point connected between a and b and has a as a regular point.*

Proof of the lemma. Let U be a neighborhood of a such that $\overline{U}$ is compact and does not contain b. Since each point of E is interior to an arbitrarily small locally connected continuum in E and the boundary $F(U)$ of U is compact, $F(U)$ is contained in the interior of the sum L of a finite number of locally connected continua in $E-a-b$; and if we add to L an arc in $E-a-b$ joining each pair of components of L, we obtain a locally connected continuum X_1 in $E-a-b$ containing $F(U)$ in its interior. Thus if we set $H_1=G+X_1$, where G is the component of $E-F(U)$ containing b, H_1 is a locally connected generalized continuum because every $x\in H_1$ is interior either to G or to X_1 in H_1.

Now let (V_i) be a sequence of neighborhoods of a such that $(U-U\cdot X_1)\supset\overline{V}_1\supset V_1\supset\overline{V}_2\supset V_2\supset\cdots$ and $\delta(V_i)<1/i$. Let us set $Y_1=0$, and for each $i>1$ let X_i be the sum of a finite number of locally connected continua such that $F(V_i)\subset X_i\subset V_{i-1}-V_{i-1}\cdot X_{i-1}-\overline{V}_{i+1}$, and let Y_i be the sum of a finite number of arcs in $E-a$ each joining X_i and H_{i-1} and having just one point in H_{i-1} and such that Y_i intersects every component of X_i. Let

$$H_i = H_{i-1} + Y_i + X_i \qquad \text{for } i > 1, \text{ and } H = a + \sum_1^\infty H_i.$$

Then since H_1 is a locally connected generalized continuum and, for each

$i>1$, $X_i+Y_i\subset V_{i-1}$, it follows that H is a locally connected generalized continuum.

Since for any $x\in(E-a-b)$, $E-x$ is $(n-1)$-point connected between a and b, it contains a sum S_x of $n-1$ independent arcs joining a and b. Since the open sets $E-S_x$ cover the compact sets X_i+Y_i, for each i there exists a finite collection S_1^i, S_2^i, $\cdots$, $S_{m_i}^i$ of the sets S_x such that $X_i+Y_i\subset\sum_{j=1}^{m_i}(E-S_j^i)$. Set $S_j^1=T_j^1$, $j\leqq m_1$, and for each $i>1$, and $j\leqq m_i$, let T_j^i be the closure of the component of $S_j^i-S_j^i\cdot H_{i-1}$ containing a. Finally, set

$$N = H + \sum_{i=1}^{\infty}\sum_{j=1}^{m_i} T_j^i.$$

Then since for each $i>1$ and all $j\leqq m_i$, $T_j^i\subset V_i$, it follows that N is a locally connected generalized continuum. Further, since for any $i>1$, $W_i=N\cdot(V_{i-1}-\overline{V}_i)\subset X_{i-1}+X_i$ plus a finite number of the arcs of Y_s and of T_j^s for $s\leqq i$, since W_i contains a compact set not intersecting $X_{i-1}+X_i$ and separating a and $F(V_{i-1})$ in N, and since any finite arc sum is everywhere regular as it can contain no continuum of condensation, it follows that N is regular at a.

It remains to show that N is n-point connected between a and b. Suppose, on the contrary, that some set K of less than n points separates a and b irreducibly in N. We shall prove this impossible by showing that $K\cdot H=0$. In the first place, $K\cdot X_1=0$ because for any $x\in X_1$ some T_j^1 does not contain x whereas every T_j^1 must contain K, as otherwise one of the arcs of T_j^1 from a to b fails to intersect K. Also, $K\cdot G=0$. For if not, then if y is the last point of X_1 on an arc ab in $E-K$, $yb\subset G+X_1\subset N$. Then since $K-K\cdot G$ would not separate a and b in N, $N-(K-K\cdot G)$ would contain an arc aub; and if u is the first point of X_1 on this arc, $au\cdot K=0$ and we have a connected subset $au+X_1+yb$ of $N-K$ containing $a+b$. Thus $K\cdot H_1=0$. Hence if $K\cdot H\neq 0$ and k is the least integer such that $K\cdot H_k\neq 0$, we have $k>1$. But $H_k=H_{k-1}+Y_k+X_k$; and if Y_k+X_k contains a point x of K, some set T_j^k fails to contain x and hence contains an arc az with $az\cdot K=0$, $z\in H_{k-1}$, since T_j^k consists of $n-1$ arcs joining a and H_{k-1} and intersecting by pairs in just a. This is impossible since then $H_{k-1}+az$ is connected and contains $a+b$ but does not intersect K. Thus $K\cdot H=0$, which also is absurd because $N\supset H\supset a+b$. Thus the lemma is proven.

Returning to the proof of the theorem, we note that since the lemma can be applied in N at b, we may suppose N regular at both a and b. Thus there exist infinite sequences (P_i) and (Q_i) of regions in N such that $\overline{P}_1\cdot\overline{Q}_1=0$ and, for each $i>1$,

$$a\subset P_i\subset\overline{P}_i\subset P_{i-1}\subset V_{1/i}(a),\qquad b\subset Q_i\subset\overline{Q}_i\subset Q_{i-1}\subset V_{1/i}(b),$$

the boundaries $F(P_i)$ and $F(Q_i)$ contain finite numbers p_i and q_i of points, and the boundary of any open set lying in P_i (or Q_i) and containing a (b) contains not less than p_i (q_i) distinct points. The set $N_0=N-P_1-Q_1$ is

n-point strongly connected between $F(P_1)$ and $F(Q_1)$, because any set separating N_0 between these two sets in the broad sense also separates N between a and b. Hence, by §2, N_0 contains a sum T_0 of n disjoint arcs joining $F(P_1)$ and $F(Q_1)$ and we suppose these arcs reduced so that each has only its end points in the sum of these two sets.

Let $P_1' = T_0 \cdot F(P_1)$, $Q_1' = T_0 \cdot F(Q_1)$. Then each of these sets contains exactly n points, namely, one end point from each of the n arcs in T_0. Now if $N_1 = \bar{P}_1 - P_2$, N_1 must be n-point strongly connected between P_1' and $F(P_2)$. For if some set H of less than n points separated these two sets in N_1 in the broad sense, the set $K = H + F(P_1) - P_1'$ contains less than p_1 points; and the component of $N - K$ containing a is in P_1 and has less than p_1 boundary points, contrary to the choice of P_1. Hence N_1 contains a sum T_1 of n disjoint arcs joining P_1' and $F(P_2)$. Similarly $M_1 = \bar{Q}_1 - Q_2$ contains a sum S_1 of n disjoint arcs joining Q_1' and $F(Q_2)$. We suppose the arcs in T_1 and S_1 reduced so that the set $P_2' = T_1 \cdot F(P_2)$ and $Q_2' = S_1 \cdot F(Q_2)$ consist of exactly n end points as before. Continuing, $N_2 = \bar{P}_2 - P_3$ and $M_2 = \bar{Q}_2 - Q_3$ contain sums T_2 and S_2 respectively of n disjoint arcs joining P_2' to $F(P_3)$ and Q_2' to $F(Q_3)$, and so on indefinitely. Clearly the set

$$S = a + b + T_0 + \sum_{1}^{\infty} (T_i + S_i)$$

consists of exactly n independent arcs in N (and thus in M) from a to b as required by the theorem.

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