ULTRAFILTERS AND COMPACTIFICATION OF UNIFORM SPACES(1)

BY

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Introduction

In point set topology—or, more precisely, in the part of it once called "theory of abstract spaces"—there are two principal methods of investigation. The first one refers to the topological space alone(2); we shall call it the internal method: for example the separation axioms T_0, T_1, T_2 [A-H], the notions of regularity, normality, compactness, are expressed in terms of the topological space only. The second method uses the real numbers as a tool for investigating the topological space S, and will be called the external method. Here the real numbers appear via the channel of real valued continuous functions defined on S. One may consider either a restricted class of these functions, like distances, pseudo distances, separating functions, or the whole ring of all continuous real valued functions defined on S. Examples of this method are the notions of complete regular space and of metric space, the Tietze extension theorem, the reconstruction of the space from its ring of real valued continuous functions [G](3).

In both methods one recognizes rapidly that, in order for the topological space to have many interesting properties, one must impose more restrictive conditions. About these conditions, mathematicians now seem in better agreement than some years ago [W] [Tu]. From the internal standpoint the interesting spaces are the uniform spaces(4) and the compact spaces(5). From the external standpoint, if one wants the topological space described accurately

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Presented to the Society, September 5, 1947; received by the editors May 8, 1947.

(1) A dissertation presented to the faculty of Princeton University in candidacy for the degree of doctor of philosophy.

(2) We mean here, by topological space, the composite object $(S, \mathfrak{T})$ of a set S and a family $\mathfrak{T}$ of subsets of S satisfying the axioms for closed sets. It seems convenient logically and harmless in practice to call the family of closed subsets of S the "topology" of S. We shall often use the condensed notation S to denote a topological space. But, when several different topologies of the same set S are under consideration, we shall come back to the rigorous and convenient notation $(S, \mathfrak{T})$. S is called the underlying set of the topological space $(S, \mathfrak{T})$.

(3) Roman letters in brackets refer to the bibliography at the end of the paper.

(4) The notion of uniform space was introduced in 1937 by A. Weil [W]. They are generalizations of metric spaces, and they have essentially all their properties, except those arising from countability.

(5) By compact space we mean here "bicompact Hausdorff space"; that is, a space S which satisfies the separation axiom T_2 of Hausdorff (any two distinct points of S can be separated by two disjoint open sets), and the Lebesgue covering axiom (from every open covering of S one can extract a finite subcovering), or any one of the numerous properties equivalent with it.

by real valued continuous functions, one must have a sufficient number of them, and that is expressed by the complete regularity of the space.

The relations between these three interesting kinds of spaces are very close: immediately after having defined the uniform spaces, Weil recognized, by using only internal methods [W, p. 24], that every compact space, and therefore every subspace of a compact space, can be considered as a uniform space. In order to prove the converse, he had to use real numbers: he proves first [W, p. 13], generalizing a method due to Pontrjagin, that any uniform space is completely regular, and, as proved by Tychonoff(6), every completely regular space is a subspace of a compact space.

The aim of this paper is to give a proof of this fact using only internal methods. In other words, given a uniform space S, to construct, without using real numbers, a compact space having a subspace homeomorphic with S. This operation is called "compactification." Some partial solutions of this problem exist already: if, in the Hausdorff space S, the open-closed sets form a base for the open sets, the method of Stone [S] gives a compactification of S. On the other hand, Wallman [Wa], given any T_1 space S, constructs a T_1 space Ω_W containing an everywhere dense subspace homeomorphic with S and satisfying the Lebesgue covering axiom; but the fact that S is a Hausdorff space, or even a completely regular space, does not insure that Ω_W is compact(7); in fact Wallman proved that a necessary and sufficient condition for Ω_W to be compact is that S be normal.

The method used by both Stone and Wallman can be interpreted as making a topological space out of the set of all "ultrafilters"(8) of a suitable directed set. The method used here is based on the same principle.

The first part of this paper contains the definition and the study of the main tool used here: the notion of ultrafilter. Filters and ultrafilters were invented in 1937 by H. Cartan [Ca] as a generalization of sequences and of the diagonal process, and are used systematically by N. Bourbaki and his collaborators. We define here filters and ultrafilters in a more general situation, that is, in any directed set, we study their general properties and give, in terms of filters, a characterization of boolean rings considered as directed sets. The next section is a review of the uses of filters in general topology(9).

The first part ends with a series of counter examples.

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(6) See [T]. Tychonoff imbeds the completely regular space in some, finite or transfinite, cube. A more elegant method, due to Gelfand [G] uses the normed ring of continuous real valued functions defined on the given space.

(7) We recall that we do not call "compact" a space which is not Hausdorff. A non Hausdorff space which satisfies the covering axiom has none of the nice properties of compact spaces.

(8) The notions of filter and ultrafilter will be defined later. For a lattice, what is meant here by "filter" is called "ideal" by some authors.

(9) This is only, with slight modifications of terminology, a restatement of the main points of the Chapters I and II of the Bourbaki treatise [B]. We shall make constant use, in this paper, of the notation and terminology of Bourbaki.

The second part is the study of the natural topologies on the set of all ultrafilters in a directed set; it contains also the parallel theory of the set of all maximal ideals in a commutative ring with unit element [GS] [J_2].

The third part gives the method of compactifying any uniform space by the use of ultrafilters. The main result is the following: for any uniformizable space S, with the uniform structure $\mathfrak{S}$, is constructed a compactification $\mathfrak{S}^{\smile}$ which contains as topological subspace the completion $S^{\frown}$ of S with respect to $\mathfrak{S}$. Since the underlying topological space of S may be considered as imbedded in $S^{\smile}$, the unique [W] [B] uniform structure of the compact space $S^{\smile}$ induces on S a precompact uniform structure [B] $\mathfrak{S}^*$ compatible with the topology of S; $\mathfrak{S}\supset\mathfrak{S}^*$, and the correspondence $\mathfrak{S}\rightarrow\mathfrak{S}^*$ is monotone. This correspondence gives us a tool for studying the diverse uniform structures compatible with the topology of a uniformizable space; we give proofs, by our methods, of the main properties of the Čech [C] and Wallman compactifications. We give also a purely topological characterization of uniformizable spaces. The paper ends by uniform structural characterizations of locally compact spaces and normal sequentially compact(10) spaces.

In the preparation of this paper I received much valuable advice from Professors C. Chevalley and J. Dieudonné, and great encouragement from Professor S. Lefschetz.

Contents

Chapter I

1. **Definitions.** A partially ordered set E will be called a *directed set*, if, given any two elements a and b of E, there exists an element $c\in E$ such that

(1) $c \leqq a$ and $c \leqq b$,

(2) "$x \leqq a$ and $x \leqq b$" implies $x \leqq c$(11).

(10) A space S is called sequentially compact if every sequence has a cluster point.

(11) In general this second condition is not required for a directed set. It would not have been difficult to get all the results of this paper without imposing this condition. But the exposition would then become long and unsatisfactory. And, since all the directed sets which are used in mathematics satisfy this second condition, any greater generality would have been pointless.

The axioms for a partially ordered set imply immediately that such an element c is unique. It is called the *greatest lower bound* (g.l.b.) (or the *meet*, or the *intersection*) of a and b—and will be denoted by ab.

For example any simply ordered set, lattice, or boolean ring is a directed set.

The mapping $(a, b) \to ab$ of $E \times E$ onto E is an associative, commutative and idempotent law of composition in E. The relations "$ab = b$" and "$b \leqq a$" are equivalent. The converse is obviously true.

Notion of filter. We suppose from now on that E has a minimal element ω; then ω is the smallest element of E since, for all $x \in E$, $\omega x \leqq \omega$, hence $\omega x = \omega$, which means $\omega \leqq x$.

We can define inductively the g.l.b. of any finite family of elements of E. The g.l.b., being associative and commutative, is independent of the order in which the elements of the finite family are written; being idempotent this g.l.b. depends only on the set of elements belonging to the family[12]. A subset C of E is called a *compatible subset* if every finite subset of C has a g.l.b. different from ω.

An elements a and a subset D are called compatible when $D \cup \{a\}$ is compatible. If D is already known to be compatible, one says that a is compatible with D.

Two elements a and b are called compatible if the subset $\{a, b\}$ is compatible. In other words: $ab \neq \omega$.

When the directed set E is the set $\mathfrak{P}(S)$ of all subsets of a set S, ordered by inclusion, a compatible subset $\mathfrak{F}$ of $\mathfrak{P}(S)$ is called a family of subsets with the finite intersection property: the subsets of any finite subfamily of $\mathfrak{F}$ have a nonempty intersection.

Instead of considering any compatible subsets, it is more convenient to consider compatible subsets of a certain type called *filters*.

A *filter* F is a subset of the directed set E which fulfills the three following conditions:

F_I: If $a \in F$ and $x \geqq a$, then $x \in F$.

F_{II}: If $a \in F$ and $b \in F$, then $ab \in F$.

F_{III}: $\omega \notin F$.

One deduces from F_{II} and F_{III} by ordinary induction that a filter is a compatible subset.

The whole directed set, which satisfies F_I, F_{II} but not F_{III} is sometimes called the improper filter.

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(12) We recall that a family of elements of E is not a subset of E, but a mapping of some indexing set into E. For example, I being the set of integers between 1 and n, a family of n elements of E is a mapping: $j \to a(j)$, $j \in I$, $a(j) \in E$. Of course the same element of E may occur several times in $a(I)$, and since in most algebraic laws of composition $aa \neq a$, the product of the elements of the set and the product $\prod_{j \in I} a(j)$, which is called the product of the family of elements, are in general different. Any set may be considered as a family, since it may be indexed by itself.

One sees immediately that one obtains a filter F from a compatible subset C by adjoining to C:

(1) All the g.l.b. of finite subsets of C;

(2) All the elements of E greater than any one of these g.l.b.

The compatible set C is then called a *subbase* of the filter F; one says that F is the filter generated by C.

The subset B obtained by the first adjunctions only is called a *base* of the filter F. In general a subset of E is a base for a filter if it satisfies:

B_{I}: If $a\in B$ and $b\in B$, there exists $C\in B$ such that $c\leqq a$, $c\leqq b$.

B_{II}: $\omega\notin B$.

Two bases B and B' are called *equivalent* if they generate the same filter F; it is equivalent to say: given any $b\in B$ there exists $b'\in B'$ smaller than b, and given any $b_1'\in B'$ there exists $b_1\in B$ smaller than b_1'.

It is readily seen that the intersection $F\cap G$ of two filters F and G in E is a filter; it is called the *filter intersection* of F and G.

Two filters F and G are called *compatible* if the subset $F\cup G$ is compatible. This condition is, by F_{II} and ordinary induction, equivalent with the following: "every element of F is compatible with every element of G." In this case the filter H generated by $F\cup G$ is called *the filter generated by F and G*. H has as base the set of elements fg, where $f\in F$ and $g\in G$; using the usual algebraic notation one can denote this base by FG. Then $FG\neq E$ means that the filters F and G are compatible.

Remark. If E is a distributive lattice, the l.u.b. of a and b being denoted by $a\vee b$, the set of all elements greater than a is $a\vee E$. Then any element of the filter H may be written $fg\vee x=(f\vee x)(g\vee x)=f'g'$ $(f'\in F,\ g'\in G)$. Hence H is the set FG.

If $F\supset G$ the filter F is said to be *finer* than the filter G, and the filter G *coarser* than the filter F.

The set of all elements $x\in E$ greater than a given element a is clearly a filter, called the *principal filter* generated by a and denoted by F_a.

Classification of the elements of E with respect to a filter $F\subset E$. Let a be any element of E. Three mutually exclusive cases may happen.

(1) $a\in F$—or equivalently $F_a\subset F$.

(2) $a\notin F$ and there exists $x\in F$ such that $ax=\omega$; a is incompatible with F. Equivalently: F_a and F are incompatible filters.

(3) $a\notin F$ and, for every $x\in F$, $ax\neq\omega$; a is compatible with the filter F; equivalently F_a and F are compatible filters. In this case the filter generated by the compatible subset $F\cup\{a\}$ (or $F\cup F_a$) is strictly finer than F.

2. **Ultrafilters.** The family of all filters in E is ordered by inclusion. One calls *ultrafilter* a maximal element of this family, that is, a filter F such that there exists no filter strictly finer than F.

The existence of ultrafilter is insured by:

THEOREM I. *Given any filter F in E, there exists an ultrafilter U finer than F.*

This is an immediate consequence of Zorn's lemma, since the family of all filters finer than F, ordered by inclusion, is inductively ordered [B, Tu], that is, the union $\bigcup_\alpha F_\alpha$ of an increasing family of filters is a filter.

Remark. Let us operate in the family of all compatible subsets of E containing a given compatible subset C; the fact that, in the family of all subsets of E containing C, the property of being compatible is of finite character([13]) shows that there exists a maximal compatible family U containing C; it is easy to prove that U is actually an ultrafilter [W_a, L].

Characterization of ultrafilters. Two distinct ultrafilters are incompatible. Every filter is either contained in an ultrafilter U, or incompatible with U, in particular the principal filter F_a. An element $a \in E$ is either an element of U, or incompatible with U. Conversely let F be a filter such that, for all $a \notin F$, there exists $x \in F$ such that $ax = \omega$. An ultrafilter U containing F can not contain a, since the family $\{a\} \cup F$ is incompatible. Hence $U = F$. Therefore we have the following theorem.

THEOREM II. *A necessary and sufficient condition for a filter F to be an ultrafilter is that every element of $E - F$ be incompatible with F.*

Intersections of ultrafilters. Every filter F is contained in the intersection $\bigcap_\alpha U_\alpha$ of the ultrafilters U_α finer than F. Suppose that $F = \bigcap_\alpha U_\alpha$ for every F. Then, to two distinct filters correspond two distinct families of finer ultrafilters. In particular given the principal filters F_a and F_b, there exists, for example, an ultrafilter U finer than F_a but not finer than F_b; that means that there exists $x \in U$ such that:

$$ax \neq \omega, \qquad bx = \omega.$$

If this condition, or the condition obtained by exchanging a and b, holds for every pair (a, b) of distinct elements of E, the directed set E is called *disjunctive*.

We suppose conversely that E is disjunctive. Let F_a be any principal filter in E, $\{U_\alpha\}$ the family of all ultrafilters finer than F_a. If $G = \bigcap_\alpha U_\alpha$ is a filter different from F_a, there exists $b \in G$, $b \notin F_a$. To the two principal filters F_a and F_b corresponds the same family of finer ultrafilters. We now use the fact that E is disjunctive: if there exists x such that $ax \neq \omega$, $bx = \omega$, let U be an ultrafilter finer than F_{ax}; it contains ax, hence a, hence is one of the U_α; hence it contains b, but it contains x; since $bx = \omega$, we get a contradiction. If there exists y such that $ay = \omega$, $by \neq \omega$, let V be an ultrafilter finer than F_{by}; it contains by, hence b, hence is one of the U_α, hence contains a; but it contains y, hence $ay = \omega$; a contradiction. We can therefore state the following theorem.

THEOREM III (WALLMAN). *A necessary and sufficient condition for every*

([13]) A property of a set D is said to be of finite character if the set D has this property whenever all finite subsets of D have it. Most algebraic properties, like being linearly free, are of finite character.

principal filter F to be the intersection of the ultrafilters finer than F is that the directed set E be disjunctive.

Ultrafilters finer than the filter generated by a family of filters. Let (F_λ) be a family of filters in E, and Φ_λ the family of all ultrafilters finer than F_λ. Let F be the (proper or improper) filter generated by the F_λ. Every ultrafilter U finer than F is finer than all the F_λ, hence belongs to $\bigcap_\lambda \Phi_\lambda$. Conversely every ultrafilter in $\bigcap_\lambda \Phi_\lambda$ contains all the F_λ, hence F. We can therefore state:

PROPOSITION 1. *The family Φ of all ultrafilters finer than the filter F generated by the filters F_λ is the intersection $\bigcap_\lambda \Phi_\lambda$ of the families Φ_λ of ultrafilters finer than F.*

Remark. If G is the intersection of the filters F_λ, the family Ψ of all ultrafilters finer than G obviously contains $\bigcap_\lambda \Phi_\lambda$. But in general it is strictly greater than this union—an example will be given later.

3. **Ultrafilters in complemented directed sets.** The notion of ultrafilter takes its full significance, and all its interesting properties, only when to every element $a \in E$ corresponds an element $a' \in E$ which has the main features of the complement of a subset in a directed set $\mathfrak{P}(S)$. Here we must, of course, express these properties in terms of the operation $(a, b) \rightarrow ab$ only.

DEFINITION. A directed set E is called *complemented* if, to every $a \in E$, corresponds $a' \in E$ such that:

C_{I}. $aa' = \omega$,

C_{II}. $ax = \omega$ implies $x \leqq a'$.

C_{II} implies the uniqueness of such an element a'.

We can also state these conditions in the following way: the set of all elements incompatible with a has a greatest element a'.

The element a' is called the *complement* of a.

The conditions PC_I and PC_{II} imply that E has a greatest element $\epsilon = \omega'$. Hence $\epsilon' = \omega$.

The complement $(a')'$ of a' is denoted a''. In general, $a'' \neq a$, and we shall see that $a'' = a$ characterizes the boolean rings with a unit element; also, it is easy to see that the set of all filters in a boolean ring ordered by inclusion is a complemented directed set where $F'' \neq F$ in general.

Since $aa' = \omega$, applying PC_{II} to a', we get $a \leqq a''$.

On the other hand $a'a'' = \omega$, and, if $xa'' = \omega$, we have a fortiori $xa = \omega$; hence $x \leqq a'$. Therefore $a''' = a'$.

Notice also that $c \leqq d$ implies $c' \geqq d'$ (since $d'c \leqq d'd = \omega$).

Set C of the complements of elements of E. (1) If $x \leqq a'$, $xa'' = \omega$ hence $a'' \leqq x'$, hence $x'' \leqq a''' = a'$. From $a'b' \leqq a'$ and $a'b' \leqq b'$, we deduce therefore $(a'b')'' \leqq a'$, $(a'b')'' \leqq b'$; hence $(a'b')'' \leqq a'b'$. But, since $x'' \geqq x$, $(a'b')'' = a'b'$, and $a'b'$ is a complement. *The set C is closed under the law of composition $(a, b) \rightarrow ab$.*

(2) We shall write: $a' \vee b' = (a''b'')'$.

(a) $a''b'' \leqq a''$; hence $a' \vee b' = (a''b'')' \geqq a''' = a'$. Similarly $a' \vee b' \geqq b'$.

(b) If $x' \geqq a'$ and $x' \geqq b'$, then $x'' \leqq a''$ and $x'' \leqq b''$; hence $x'' \leqq a''b''$, and $x''' = x' \geqq (a''b'')' = a' \vee b'$. *$a' \vee b'$ is therefore the* l.u.b. *of a' and b' in the set C* (but not in general in the set E).

We could now prove that each of the laws of composition $a'b'$ and $a' \vee b'$ in C is distributive with respect to the other, and therefore C is a boolean ring with unit. But it is more elegant to use the theory of ultrafilters to prove this theorem.

In spite of the fact that it is not a l.u.b. in E we shall use the notation $a_1 \vee a_2 \vee \cdots \vee a_n = (a_1' \cdots a_n')'$; notice that this law of composition is not idempotent: $a \vee a = a''$.

The element $a_1 \vee a_2 \vee \cdots \vee a_n$ is greater than each a_i: $a_i a_i' \cdots a_n \leqq a_i a_i' = \omega$, hence $a_i \leqq (a_1' \cdots a_n')'$.

Remarks about the situation of the elements a and a' with respect to a filter F. If $a \in F$, a' is incompatible with F.

If $a' \in F$, a is incompatible with F.

If a is incompatible with F, there exists $x \in F$ such that $ax = \omega$; hence $x \leqq a'$ and $a' \in F$.

If a' is compatible with F and $a' \notin F$, a is compatible with F and $a \notin F$. (Otherwise there would exist $x \in F$, such that $ax = \omega$, hence $a' \geqq x$ and $a' \in F$.) These considerations lead immediately to the following theorems:

THEOREM IV (CARTAN). *A necessary and sufficient condition for a filter F to be an ultrafilter is that for every $a \in E$, either $a \in F$ or $a' \in F$.*

(1) If F is an ultrafilter and if $a \notin F$, a is incompatible with F, hence $a' \in F$.

(2) If, conversely, for every $a \in E$, F contains either a or a', give any $x \notin F$, $x' \in F$; since $xx' = \omega$, x is incompatible with F. Every element of $E - F$ being incompatible with F, F is an ultrafilter by Theorem II.

THEOREM V (ULTRAFILTER THEOREM). *If $a_1 \vee a_2 \vee \cdots \vee a_n$ belongs to an ultrafilter U, then at least one of the a_i belongs to U.*

In fact, if U does not contain any of the a_i, it contains all the a_i' by Theorem IV, and hence also $a_i' \cdots a_n'$. Then $(a_1' \cdots a_n')' = a_1 \vee a_2 \vee \cdots \vee a_n$ does not belong to U; a contradiction.

Corollaries. (1) *If a'' belongs to an ultrafilter U, then $a \in U$.* In fact $a \vee a = (a'a')' = a''$.

(2) *If a'' is compatible with a filter F, a is compatible with F.* In fact it means that a'' belongs to an ultrafilter U finer than F.

(3) *Every ultrafilter U finer than the intersection $G = F_1 \cap \cdots \cap F_n$ of a finite number of filters contains at least one of them.* Otherwise there would exist for each i an element $a_i \in F_i$ such that $a_i \notin U$. Then $a_i' \in U$ for every i, and therefore $a_i' \cdots a_n' \in U$. But $a = a_1 \vee \cdots \vee a_n = (a_i' \cdots a_n')'$; it is

greater than every a_i. Hence $a \in F_1 \cap \cdots \cap F_n = G$—a contradiction.

Intersections of ultrafilters. Let us see under what condition every principal filter is the intersection of all ultrafilters finer than itself, in other words what can be said about a complemented disjunctive directed set.

Since $a'' \geqq a$, $a''x = \omega$ implies $ax = \omega$. The disjunctivity would then imply, if $a'' \neq a$, the existence of y such that $a''y \neq \omega$, $ay = \omega$. But then $y \leqq a'$ by PC_{II}, and, since $a'a'' = \omega$, $a''y = \omega$; a contradiction.

Hence, for every $a \in E$, $a'' = a$ and $C = E$.

In this case we have a result stronger than Wallman's theorem.

THEOREM VI (STONE). *In a complemented directed set E in which $a'' = a$ for every $a \in E$, every filter is the intersection of the ultrafilters finer than itself.*

Let (U_α) be the family of all ultrafilters finer than F, and $G = \bigcap_\alpha U_\alpha$; clearly $F \subset G$. If $F \neq G$, there exists an element $b \in G$, $b \notin F$. We can write $b = (b')' = a'$; a' being compatible with F and not contained in F, a is compatible with F. Let U be an ultrafilter containing $F \cup \{a\}$; it cannot contain $b = a'$; this contradicts the definition of b. Hence $F = G$, and the theorem is proved.

THEOREM VII. *A complemented directed set E in which $a'' = a$ for every $a \in E$ is a boolean ring with a unit element.*

Let Φ be the set of all, proper and improper, filters in E. Φ has two laws of composition: $(F_1G) \rightarrow F \cap G$ and $(F,G) \rightarrow E(F,G)$, the filter generated by F and G. By Theorem VI every filter F corresponds in a one-to-one way to a subset Ω_F of the set Ω of ultrafilters in E: Ω_F is the family of ultrafilters containing F. Proposition 1 (Part II) and Corollary 3 to Theorem 5 show that to the filter $E(F_\lambda)$ generated by any family (F_λ) of filters corresponds the subset $\bigcap_\lambda \Omega_{F_\lambda}$ of Ω, and to the filter intersection $\bigcap_i F_i$ of a finite family (F_i) of filters corresponds the subset $\bigcup_i \Omega_{F_i}$ of Ω.

Since we are now reduced to a calculus of subsets of the set Ω, we get the following distributivity relations:

(1) $F \cap E(F_\lambda) = E_\lambda(F \cap F_\lambda)$ (the family (F_λ) being any family of filters).

(2) $E(F, \bigcap_i F_i) = \bigcap_i (E(F, F_i)$ (the family (F_i) being finite).

In particular, if we are dealing with principal filters, $E(F_a, F_b) = F_{ab}$, $F_a \cap F_b = F_{a \smile b}$ (since $a \vee b$ is the l.u.b. of a and b). Therefore:

(1)′ $a \vee bc = (a \vee b)(a \vee c)$.

(2)′ $a(b \vee c) = ab \vee ac$.

That makes out of E a distributive lattice.

If we defined $x + y = xy' \vee yx'$, it follows in the usual way that E is a boolean ring with respect to the operations $(x, y) \rightarrow x + y$ and $(x, y) \rightarrow xy$. Observe that $x \vee y = x + y + xy$.

Remarks. (1) Let $F \subset E$ be a filter, and F' the set of complements of elements of F; F' is sometimes called an *antifilter. The antifilters coincide with the proper ideals of the boolean ring.* (The proof is straightforward.)

Hence, for a boolean ring, the notions of filter and of ideal are closely related. To an ultrafilter corresponds a maximal ideal; Theorem VI is equivalent to the following well known result: "In a boolean ring every ideal is the intersection of the maximal ideals containing it."

Theorem IV corresponds to the fact that the factor ring of a boolean ring by a maximal ideal is a field with two elements.

(2) In the propositional calculus of the first kind (that is, without quantifiers "Whatever $x \cdots$," and "There exists $x \cdots$") the propositions form a boolean ring E[14] (α' corresponds to "not α," $\alpha\beta$ to "α and β," $\alpha \vee \beta$ to "α or β"). If we attach to some propositions a truth value, "true" or "false," the true propositions form a filter F, and the false ones the ideal (or antifilter) $F' = \mathfrak{A}$. Equivalence between two propositions "$\alpha\beta \vee \alpha'\beta'$ is true" corresponds to congruence modulo $\mathfrak{A}$. More generally one can say that the assignment of truth values (in a many-valued logic) corresponds to the passage to a factor ring. The fact that the number of elements of a finite boolean ring is of the form 2^n, excludes 3, 5, 6, $\cdots$-valued logics (if one wants to keep the usual rules of the propositional calculus).

A theory with complete determination (that is, every proposition is either true or false) corresponds to the case where the filter of true propositions is an ultrafilter. The existence theorem of ultrafilters (Theorem I) means that, given a theory with dubious propositions, one can extend it to a theory with complete determination.

Notice also that the axiom $F_{III}(\omega \notin F)$ corresponds to the non contradiction principle.

(3) A consequence of the distributivity conditions (1)′ and (2)′ (and of them only) is that one can denote the filter intersection $F \cap G$ and the filter generated by F and G by $F \vee G$ and FG respectively, these notations having their usual algebraic meaning.

4. Use of filters and ultrafilters in general topology. We shall review here the principal applications of filters and ultrafilters in general topology. For more details the reader is referred to the Bourbaki treatise [B, III, chaps. I and II].

DEFINITIONS. *Filter over a set.* Given a set S we denote by $\mathfrak{P}(S)$ the set of all subsets of S. $\mathfrak{P}(S)$, ordered by inclusion, is a complemented directed set, and even a boolean ring.

A filter $\mathfrak{F}$ in the directed set $\mathfrak{P}(S)$ is called a *filter over* S.

If f is a mapping of the set S into the set T, the images of the subsets which are elements of a filter $\mathfrak{F}$ over S form a family of subsets of T, denoted by $f(\mathfrak{F})$, which is obviously the base of a filter over T. The filter over T generated by $f(\mathfrak{F})$ is called the *direct image of the filter* $\mathfrak{F}$. Notice that the direct image of an ultrafilter is an ultrafilter.

[14] We realize that it is not at all logical to speak about "sets of propositions." Therefore these considerations about logic have only an interpretative value.

If we are given a filter $\mathfrak{G}$ over T, and if none of the inverse images under f of the elements of $\mathfrak{G}$ is empty, they form the base of a filter over S, denoted by $f^{-1}(\mathfrak{G})$; the filter generated by $f^{-1}(\mathfrak{G})$ is called the *inverse image of the filter* $\mathfrak{G}$.

Filters over a topological space. Let S be the underlying set of a topological space $(S, \mathfrak{T})$. The family of all neighborhoods of a point $x \in S$(15) forms a filter over S, called the *filter of neighborhoods* of x and denoted by $\mathfrak{V}(x)$.

A filter $\mathfrak{F}$ over a topological space $(S, \mathfrak{T})$(16) is said to *converge* to a point $x \in S$, if $\mathfrak{F}$ is finer than $\mathfrak{V}(x)$. A filter which converges to some point of S is called *convergent.*

The Hausdorff separation axiom can now be stated in the following form: "a filter cannot converge to more than one point." This point, if it exists, is called the *limit* of the filter $\mathfrak{F}$.

A point x of S is called a *cluster point of the filter* if the filters $\mathfrak{F}$ and $\mathfrak{V}(x)$ are compatible. The set A of all cluster points of a filter $\mathfrak{F}$ is a closed set, the intersection of the closures of all elements of $\mathfrak{F}$, and is called the *closure* of the filter $\mathfrak{F}$.

Let f be a mapping of a set S into the topological space T and $\mathfrak{F}$ be a filter over S. If the direct image $f(\mathfrak{F})$ has a limit $x \in T$, x is called the limit value of f with respect to the filter $\mathfrak{F}$; one writes $x = \lim_{\mathfrak{F}}(f(t))$.

Similarly one defines the cluster values of a function with respect to a filter. If S is the set N of positive integers, a mapping f of S into T is called a *sequence* of points of T: $n \rightarrow a_n$. The complements of finite subsets of N form a filter over N (this is a general property of the infinite sets), called the *Fréchet-filter*. A limit value of the sequence with respect to the Fréchet-filter is called a limit of the sequence, a cluster value a cluster point of the sequence. The connection with the usual notions, in a T_1 space, is readily established.

Comparison of topologies. Given a set S and two topologies $\mathfrak{T}_1$ and $\mathfrak{T}_2$ over this set, it may happen that every closed set for $\mathfrak{T}_1$ is a closed set for $\mathfrak{T}_2$, in other words $\mathfrak{T}_1 \subset \mathfrak{T}_2$. We say in this case that $\mathfrak{T}_2$ is *finer* than $\mathfrak{T}_1$, or that $\mathfrak{T}_1$ is *coarser* than $\mathfrak{T}_2$. The discrete topology is the finest one. Some authors say "stronger" and "weaker" instead of "finer" and "coarser."

If $\mathfrak{T}_1$ is finer than $\mathfrak{T}_2$, the filter of neighborhoods $\mathfrak{V}_1(x)$ (in $\mathfrak{T}_1$) is finer than $\mathfrak{V}_2(x)$ (in $\mathfrak{T}_2$).

The family $\mathfrak{T}_1 \cap \mathfrak{T}_2$ satisfies the axioms for closed sets; it is therefore a topology called the *intersection topology* of $\mathfrak{T}_1$ and $\mathfrak{T}_2$.

By adjoining to the family $\mathfrak{T}_1 \cup \mathfrak{T}_2$ all the finite unions of sets of it, and all the intersections of these finite unions, one gets a topology $\mathfrak{T}$, which is the l.u.b. of $\mathfrak{T}_1$ and $\mathfrak{T}_2$ in the partially ordered set of the topologies over S; it is called the l.u.b. *topology* of $\mathfrak{T}_1$ and $\mathfrak{T}_2$, and denoted by $\mathfrak{T}_1 \vee \mathfrak{T}_2$.

(15) We mean by neighborhood of x, any subset of S containing an open set U containing x.

(16) We say "filter over a topological space" as short cut for the rigorous "filter over the underlying set of a topological space."

The notions of intersection topology and of l.u.b. topology can obviously be defined for any set $(\mathfrak{T}_\lambda)$ of topologies over S.

The l.u.b. topology may be obtained in the following way: consider, in the product space $\prod_\lambda(S, \mathfrak{T}_\lambda)$ the diagonal Δ, the set of elements with coordinates all equal. As a set of points Δ can be identified with S; and the topology induced on Δ by the product space topology is the l.u.b. of the topologies $\mathfrak{T}_\lambda$. That proves immediately that if S has an algebraic structure (group, ring, or field), and if each topology $\mathfrak{T}_\lambda$ is compatible with the algebraic structure (that is, makes out of S a topological group, ring, or field), the l.u.b. topology is also compatible with the algebraic structure of S.

It is easy to see that the filter of neighborhoods of x in the l.u.b. topology is the filter generated by the filters of neighborhoods $\mathfrak{B}_\lambda(x)$ in the topologies $\mathfrak{T}_\lambda$.

The intersection topology has none of these properties: the filter of neighborhoods of x in it is only coarser than the filter $\bigcap_\lambda \mathfrak{B}_\lambda(x)$. The intersection of two group topologies may not be a group topology (see §5, "counter examples").

This difference of behavior is due to the fact that the l.u.b. topology and the filter generated are obtained by an essentially finite (or at least algebraic) construction from the data. But the intersection topology (and filter) are obtained by a set-theoretical intersection effected at the highest level we are operating in. We shall notice later a similar situation with uniform structures.

The property of being a filter is essentially settled in the finite case; in fact there is essentially nothing more in a filter than in a family with the finite intersection property (F.I.P.), and many authors, like Lefschetz and Wallman, work successfully with families with F.I.P. If we prefer filters it is only for aesthetic reasons; instead of equivalence between families with F.I.P., we have, the filters being very big collections of objects, identity between filters. More generally the relations of inclusion and compatibility between filters are easier to express than the corresponding relations between families with F.I.P.

After this digression we come back to our point, which is the significance of filters in topology. Topology deals essentially with infinite sets, while it is much easier to operate with finite sets. It is therefore necessary to have a tool permitting the passage from the finite to the infinite (or conversely by using dual methods). The necessary tool has to have finite features in its definition, but to be infinite in its essence; and the filters fulfill both requirements. Of course, in order to be significant, filters should have some relations with topological spaces furnished by the notions of convergence and cluster points. In other words we want to get, from the "finite" compatibility (or cohesion) expressed by the main filter axioms, some sort of infinite cohesion (expressed by the limit or the cluster point). The most interesting spaces, the ones where we can expect to be able to transform infinite considerations into finite ones, will be those where all filters are related with the topology; these spaces are the compact spaces.

Compact spaces. A Hausdorff space S is called *compact* if it satisfies the following equivalent conditions:

C_I: Every ultrafilter over S is convergent.

C_{II}: Every filter over S has a cluster point.

C_{III}: Every family of closed sets with the F.I.P. has a nonempty intersection.

C_{IV}: From every open covering of S, one can extract a finite covering.

It is not the place to review here the classical properties of compact spaces.

Identification spaces. Let $(S, \mathfrak{T})$ be a topological space and R an equivalence relation in S. We denote by S/R the set of equivalence classes, by ϕ the *canonical* mapping of S onto S/R: ϕ maps every element of S upon its equivalence class. A subset $A \subset S$ is called *saturated* if it is a union of equivalence classes; in other words: $\phi^{-1}(\phi(A)) = A$. (In general one has only $\phi^{-1}(\phi(B)) \supset B$. B being an arbitrary subset of S, $\phi^{-1}(\phi(B))$ *is saturated and is called the saturated set of* B.)

We introduce a topology in S/R by taking as closed (open) sets in S/R the images under ϕ of the saturated closed (open) sets of S. The topology one obtains is called the identification topology of S/R. Notice that it is the finest topology making the canonical mapping continuous.

The main difficulty with identification spaces is the question of separation. We shall only study it when S is a compact space: it is clear that S/R, as continuous image of S, will satisfy the covering axiom C_{IV} (for example). If we want S/R to be a Hausdorff space (and therefore a compact space), the image $\phi(A)$ of a closed set $A \subset S$ will have to be compact, hence closed in S/R. Therefore the saturated set $\phi^{-1}(\phi(A))$ of a closed set A must be closed.

We suppose that conversely the saturated set of any closed set $A \subset S$ is closed. Let C and D be two distinct equivalence classes; they are closed in S by assumption; S being normal there exist two open sets U and V such that $C \subset U$, $D \subset V$, $U \cap V = \phi$. Let $A = S - U$, $B = S - V$. The saturated sets A' and B' of A and B are closed by assumption. Consider the saturated open sets $U' = S - A'$, $V' = S - B'$. Clearly $U' \subset U$, $V' \subset V$; since U' is saturated, $C \subset U'$ and similarly $D \subset V'$. Therefore, in the identification space S/R, the points $\phi(C)$ and $\phi(D)$ are separated by the disjoint open sets $\phi(U')$ and $\phi(V')$. Therefore we have the following theorem.

THEOREM. *A necessary and sufficient condition for the identification space of a compact space to be compact is that the saturated set of any closed set be closed.*

Uniform spaces. Consider a set S and its Cartesian square $S \times S$. The set of pairs (a, a) $(a \in S)$ is called the *diagonal* of $S \times S$ and is denoted by Δ. If V is a subset of $S \times S$ we denote by V^{-1} the set of pairs (b, a) where $(a, b) \in V$; if $V = V^{-1}$ the set V is called *symmetric*; by V^2 is meant the set of all pairs (a, b) such that there exists $x \in S$ such that $(a, x) \in V$, $(x, b) \in V$; V^n is defined inductively.

If $A \subset S$, and $V \subset S \times S$, we denote by $V(A)$ the image of the set $(S \times A) \cap V$ under the projection pr_1 of $S \times S$ onto its first factor. If $a \in S$ we write $V(a)$ instead of $V(\{a\})$. If $\Delta \subset V$, $V(A) \supset A$.

By *a uniform space* is meant a pair $(S, \mathfrak{S})$ composed of a set S and of a filter $\mathfrak{S}$ over $S \times S$, having the following properties:

$\mathrm{U_I}$: If $V \in \mathfrak{S}$, $V^{-1} \in \mathfrak{S}$.

$\mathrm{U_{II}}$: If $V \in \mathfrak{S}$, there exists $W \in \mathfrak{S}$ such that $W^2 \subset V$.

$\mathrm{U_{III}}$: $\bigcap_{V \in \mathfrak{S}} : V \supset \Delta$.

The uniform space $(S, \mathfrak{S})$ is called *separated* if it also satisfies:

$\mathrm{U_{IV}}$: $\bigcap_{V \in \mathfrak{S}} : V = \Delta$.

The filter $\mathfrak{S}$ is called the *uniform structure* of $(S, \mathfrak{S})$. Its elements V are called the *surroundings*, and sometimes $\mathfrak{S}$ is called the filter of surroundings.

Given two uniform spaces $(S_1, \mathfrak{S}_1)$, $(S_1, \mathfrak{S}_2)$ with the same underlying set S, the uniform structure $\mathfrak{S}_1$ is said to be *finer* than $\mathfrak{S}_2$ if $\mathfrak{S}_1 \supset \mathfrak{S}_2$; $\mathfrak{S}_2$ is also said to be *coarser* than $\mathfrak{S}_1$. It is clear that a uniform structure finer than a separated one is separated.

Given a family $(\mathfrak{S}_\lambda)$ of uniform structures on the set S, one verifies that the filter $\mathfrak{S}$ generated by the filters $\mathfrak{S}_\lambda$ satisfies the axioms $\mathrm{U_I}$, $\mathrm{U_{II}}$, $\mathrm{U_{III}}$ (proof is in [B, II, Chap. II, p. 88]); $\mathfrak{S}$ is called the l.u.b. *uniform structure* of the family. Notice that when one has defined the product of a family of uniform spaces, the l.u.b. uniform structure, like the topology, is the uniform structure of the diagonal.

A mapping f of a uniform space $(S_1, \mathfrak{S}_1)$ into a uniform space $(S_2, \mathfrak{S}_2)$ is called *uniformly continuous* if $f^{-1}(\mathfrak{S}_2) \subset \mathfrak{S}_1$ (the mapping f being extended to the products). It is clear, in general, that the inverse image of a filter of surroundings is the base of a filter of surroundings.

Given a uniform space $(S, \mathfrak{S})$ and a surrounding V, a subset $A \subset S$ is called *small of order* V if $A \times A \subset V$. A uniform structure $\mathfrak{S}$ is called *totally bounded* if, for every $V \in \mathfrak{S}$, there exists a finite covering of S by sets small of order V. A totally bounded and separated uniform space is called *precompact*. A filter $\mathfrak{F}$ over S is called a *Cauchy filter* if, for every $V \in \mathfrak{S}$, $\mathfrak{F}$ contains a set small of order V.

To a uniform space $(S, \mathfrak{S})$ is attached a topology $\mathfrak{T}(\mathfrak{S})$ in the following way: we take as filter of neighborhoods of a point $a \in S$ the filter $\mathfrak{B}(a)$ of all sets $V(a)$ where $V \in \mathfrak{S}$. $\mathrm{U_{IV}}$ means that the topology $\mathfrak{T}(\mathfrak{S})$ satisfies the Hausdorff separation axiom.

If $\mathfrak{S}_1$ is finer than $\mathfrak{S}_2$, $\mathfrak{T}(\mathfrak{S}_1)$ is finer than $\mathfrak{T}(\mathfrak{S}_2)$. But two distinct uniform structures may give the same topology. If a topology $\mathfrak{T}$ is already given on

S, and if there exists a uniform structure $\mathfrak{S}$ such that $\mathfrak{T}=\mathfrak{T}(\mathfrak{S})$, $\mathfrak{S}$ is called *compatible* with the topology $\mathfrak{T}$. Such a topological space $(S, \mathfrak{T})$ is called *uniformizable*. We shall study in this paper the conditions for a topological space $(S, \mathfrak{T})$ to be uniformizable, and the set of uniform structures compatible with its topology. Uniformizability is clearly a hereditary property.

Notice the following properties:

(1) Every uniform structure coarser than a totally bounded one is totally bounded.

(2) The l.u.b. of a family of totally bounded uniform structures is totally bounded.

(3) The inverse image of a totally bounded uniform structure is totally bounded.

(4) The topology deduced from the l.u.b. uniform structure is the l.u.b. topology. As consequences:

(a) The l.u.b. of a family of uniform structures compatible with a given topology $\mathfrak{T}$ is compatible with $\mathfrak{T}$.

(b) The l.u.b. of a family of uniformizable topologies is uniformizable.

A uniform space S is called *complete* if every Cauchy filter is convergent. By a generalization of the Cantor method, one can consider every uniform space S is an everywhere dense subset of a complete space $S^\frown$, the uniform structure of S being identical with the one induced on S by that of $S^\frown$.

One proves that every compact space S is uniformizable in a unique way: its uniform structure $\mathfrak{S}$ is the filter of neighborhoods of Δ in $S\times S$. A base of the filter $\mathfrak{S}$ is formed by the graphs $\bigcup_i(V_i\times V_i)$ of the finite open coverings (V_i) of S.

One deduces immediately that every subspace of a compact space is uniformizable. We shall prove later the converse.

A necessary and sufficient condition for the completion $S^\frown$ of S to be compact is that S be precompact.

5. **Counter examples.**

Ultrafilters finer than the intersection of an infinite family of filters. In opposition with the set of ultrafilters finer than the filter generated by a family of filters which has the property expressed in Proposition I (Chap. II), the set of all ultrafilters finer than the filter intersection of an infinite family of filters contains the union of the sets of ultrafilters finer than a given filter of the family, but is not equal to it. This is the reason why, for the laws of composition between filters, infinite distributivity holds only in one direction.

For example the principal filter $\mathfrak{F}_{\{a\}}$ over the infinite set S $(a\in S)$ is an ultrafilter. $\bigcap_{a\in S}\mathfrak{F}_{\{a\}}$ is the minimal filter which contains only S. But there are other ultrafilters than the trivial ones $\mathfrak{F}_{\{a\}}$, since there exist filters with empty intersection.

On the other hand it could be predicted, for filters over an infinite set S, that distinct families of ultrafilters may have the same intersection: in fact the set of all filters and the set of all ultrafilters have the same cardinal $2^{2^{\text{card } S}}$. Therefore the correspondence between filters and sets of ultrafilters cannot be one-to-one.

Now we give two counter examples involving infinite distributivity (of formula (2) Chap. III).

(1) Consider the filters over the real line R. Let $\mathfrak{F}$ be the filter of neighborhoods of 0, $\mathfrak{G}_n$ the principal filter of all subsets containing $[1/n, 1]$. $E(\mathfrak{F}, \mathfrak{G}_n)$ is the improper filter, hence $\bigcap_n E(\mathfrak{F}, \mathfrak{G}_n)$ also.

But $\bigcap_n \mathfrak{G}_n$ is the principal filter of all subsets containing $]0, 1]$ and $E(\mathfrak{F}, \bigcap_n \mathfrak{G}_n)$ is a proper filter.

(2) If one wants to avoid the improper filter, one can demand that $\mathfrak{F}$ and $\mathfrak{G}_n$ contain a given set, for example $\{-1\}$.

We may also consider in R the equivalence relation "$a-b$ is rational." Each equivalence class, translated from the rational line $\mathfrak{R}$, is everywhere dense in R, and countable. The set of all classes has the power of the continuum, and we may established a one-to-one correspondence: $x \rightarrow A_x$ between strictly positive numbers x and classes A_x.

Let $\mathfrak{F}$ be the filter of neighborhoods of 0, $\mathfrak{G}_x$ the principal filter generated by A_x. $\bigcap_x \mathfrak{G}_x$ is the minimal filter $\{R\}$; hence $E(\mathfrak{F}, \bigcap_x \mathfrak{G}_x)^x = \mathfrak{F}$.

$E(\mathfrak{F}, \mathfrak{G}_x)$ is a proper filter since A_x is dense in R; it contains the set $B_x = A_x \cap [-x, +x]$.

The union $B = \bigcup_x B_x$ belongs therefore to $\bigcap_x E(\mathfrak{F}, \mathfrak{G}_x)$. But it is not a neighborhood of 0 since no point of $A_x \cap (R-B_x)$ lies in $[-a, +a]$ $(a<0, x<a)$, and no other A_y can have introduced them in B since the (A_z) form a partition of R.

We see therefore that only the following inclusion holds:

$$E(F, \bigcap_\lambda \mathfrak{G}_\lambda) \subset \bigcap_\lambda E(F, G_\lambda).$$

Filters of neighborhoods and intersection topology. Consider in the plane R^2 the two following topologies:

$\mathfrak{T}_H$: a set $A \subset R^2$ is called open if, with any point a, it contains a horizontal segment with center a.

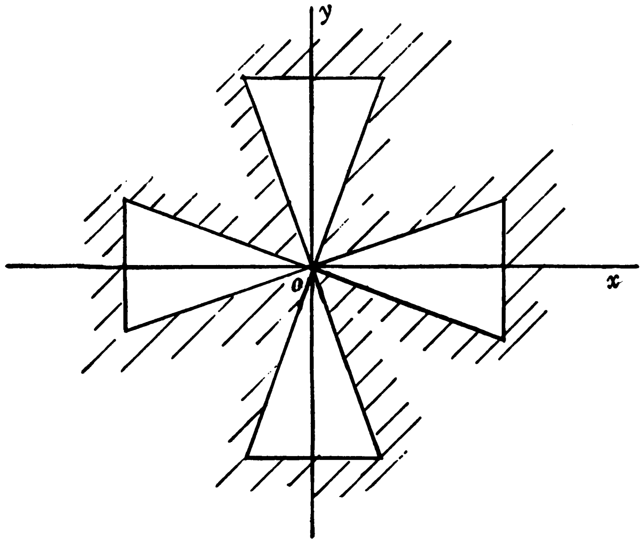
$\mathfrak{T}_V$: similarly with vertical segments.

$\mathfrak{T}_H$ and $\mathfrak{T}_V$ are the order topologies deduced from the two lexicographic orderings of the plane.

In the intersection toplogy a set A is open if, with any point a, it contains a horizontal segment and a vertical segment of center a.

The union of a vertical segment and of a horizontal segment of center 0 is in the filter intersection of the filters of neighborhoods of 0 in $\mathfrak{T}_H$ and $\mathfrak{T}_V$; but it is not a neighborhood of 0 in the intersection topology.

Intersection of group topologies which is not a group topology. We use the same example: $\mathfrak{T}_H$ and $\mathfrak{T}_V$ are compatible with the additive group structure of the plane.



The following set V, in the form of a Maltese cross around O, is open in $\mathfrak{T}_H \cap \mathfrak{T}_V$. (The boundaries are not in V, except the point O.) If $\mathfrak{T}_H \cap \mathfrak{T}_V$ were a group topology, this neighborhood V of O would contain a neighborhood U of O, such that $U+U\subset V$. But U must contain a horizontal segment and a vertical segment with centers O, hence $U+U$ contains a rectangle with center O and cannot be contained in V.

Filter intersection of two filters of surroundings. We use again the same example. The topologies $\mathfrak{T}_H$ and $\mathfrak{T}_V$, being group topologies, generate additive uniform structures: to every neighborhood V_1 (V_2) of O in $\mathfrak{T}_H$ ($\mathfrak{T}_V$) corresponds the surrounding $\tilde{V}_1$ ($\tilde{V}_2$) defined by: "$(x, y)\in\tilde{V}_1$" means "$x-y\in V_1$" ("$(x, y)\in\tilde{V}_2$" means "$x-y\in V_2$"). Let $\mathfrak{S}_1$ and $\mathfrak{S}_2$ be these uniform structures. The filter $\mathfrak{S}_1\cap\mathfrak{S}_2$ is composed of the sets $\tilde{V}_1\cup\tilde{V}_2$; $(x, y)\in\tilde{V}_1\cup\tilde{V}_2$ means $x-y\in V_1\cup V_2$. If $\mathfrak{S}_1\cap\mathfrak{S}_2$ is a uniform structure, there exists $\tilde{W}=\tilde{W}_1\cup\tilde{W}_2\in\mathfrak{S}_1\cap\mathfrak{S}_2$ such that $\tilde{W}^2\subset\tilde{V}_1\cap V_2$; translated in terms of neighborhoods of O it means that $W+W\subset V_1\cup V_2$. We take for V_1 and V_2 a horizontal and a vertical segment with center O; $W=W_1\cup W_2$ must contain a horizontal and a vertical segment with center O, and $W+W$ a rectangle with center O; hence no $W+W$ can be contained in $V_1\cup V_2$, and $\mathfrak{S}_1\cap\mathfrak{S}_2$ is not a uniform structure.

Chapter II

6. **Spaces of ultrafilters.** Let E be a directed set with a minimal element

ω; we denote by Ω the set of all ultrafilters in E, by Ω_a the set of all ultrafilters containing the element $a \in E$, by Ω_a^* the complement of Ω_a in Ω. We define on the set Ω the two following topologies:

$\mathfrak{T}_0$: the subsets Ω_a are taken as a subbase for open sets.

$\mathfrak{T}_F$: the subsets Ω_a are taken as a subbase for closed sets; in other words the subsets Ω_a are taken as a subbase for open sets.

Description of the open sets and closed sets in $\mathfrak{T}_0$ *and* $\mathfrak{T}_F$. (1) Every finite intersection of sets Ω_a is a set Ω_a: $\Omega_{a_1} \cap \cdots \cap \Omega_{a_n} = \Omega_{a_1 \ldots a_n}$. Hence the subsets Ω_a^* form a base for open sets in $\mathfrak{T}_0$. Any open set in $\mathfrak{T}_0$ is a union of them.

(2) Using Proposition I, §2, and noticing that every filter is generated by the principal filters determined by its elements, we see that the set Ω_F of all ultrafilters finer then a given filter $F \subset E$ is closed in $\mathfrak{T}_F$. Every closed set (in $\mathfrak{T}_F$) is a finite union of sets Ω_F.

If, furthermore, E is complemented, Corollary 3 of §3 shows that any finite union of sets Ω_F is a set Ω_F. Hence the sets Ω_F are the only closed sets in $\mathfrak{T}_F$.

Comparison of $\mathfrak{T}_0$ *and* $\mathfrak{T}_F$. (1) Consider an open set Ω_a^* of the subbase for $\mathfrak{T}_F$, and the family (b_α) of all elements of E such that $ab_\alpha = \omega$. Every ultrafilter U containing some b_α cannot contain a, and conversely every ultrafilter not containing a contains some b_α: $\Omega_a^* = \bigcup_\alpha \Omega_{b_\alpha}$ and Ω_a^* is open in $\mathfrak{T}_0$; therefore:

PROPOSITION 1: $\mathfrak{T}_0$ *is finer than* $\mathfrak{T}_F$.

(2) Consider now a closed set Ω_a^* of the base for $\mathfrak{T}_0$, and let us see under what conditions it is a closed set for $\mathfrak{T}_F$. Then there must exist a finite number of filters $F_1, \cdots, F_n$ such that the ultrafilters not containing a are the same as the ultrafilters containing at least one F_i. Hence the family $F_i \cup \{a\}$ is incompatible (otherwise there would exist an ultrafilter containing both F and a), and, in each F_i, there exists b_i such that $ab_i = \omega$. The ultrafilters containing at least one b_i cannot contain a, and because the condition of containing b_i is weaker than the condition of containing F_i, the ultrafilters not containing a must be the same as the ultrafilters containing at least one b_i. Therefore:

PROPOSITION 2. *A necessary and sufficient condition for* $\mathfrak{T}_F$ *to be finer than* $\mathfrak{T}_0$ *(and therefore identical with it) is that, for every* $a \in E$, *there exists a finite number of elements* $b_1, \cdots, b_n$ *of* E *such that* $ab_i = \omega$ *for every* i, *and that every ultrafilter in* E *contains either* a, *or one of the* b_i.

If E is complemented the family $\{a'\}$ fulfills the requirement of Proposition 2 (Theorem IV, §3). Hence:

PROPOSITION 3. *For a complemented directed set, the topologies* $\mathfrak{T}_0$ *and* $\mathfrak{T}_F$ *of its set of ultrafilters are the same.*

Separation axioms. (1) Since two distinct ultrafilters U and V are incompatible, there exist $a \in U$ and $b \in V$ such that $ab = \omega$. Hence Ω_a and Ω_b are two disjoint open sets (in $\mathfrak{T}_0$) containing U and V:

PROPOSITION 4. *$\mathfrak{T}_0$ satisfies the Hausdorff separation axiom* (T_2).

(2) U being an ultrafilter, Ω_u is closed in $\mathfrak{T}_F$:

PROPOSITION 5. *$\mathfrak{T}_F$ satisfies the Frechet separation axiom* (T_1).

Remarks. The fact that $\mathfrak{T}_F$ satisfies the Hausdorff axiom does not imply that E is complemented or disjunctive. (Take for E a simply ordered set.) Nor does it imply that $\mathfrak{T}_0 = \mathfrak{T}_F$: one sees easily [Wa] that if S is a compact space the space $(\Omega_W, \mathfrak{T}_F)$, Ω_W being the set of ultrafilters of the directed set $\mathfrak{F}(S)$ of closed sets in S, is homeomorphic with S, hence a Hausdorff space; but the closed sets of S have, as images in Ω_W, the basic open sets for $\mathfrak{T}_0$, and $\mathfrak{T}_0$ is therefore discrete.

Compactness.

THEOREM VIII. *$\mathfrak{T}_F$ satisfies the Lebesgue covering axiom.*

We shall consider this axiom in its dual form: every family Φ of closed sets $\mathfrak{F}_\alpha$ of $(\Omega, \mathfrak{T}_F)$ with the finite intersection property has a nonempty intersection. Let Ψ be an ultrafilter over Ω containing the compatible family Φ. Each $\mathfrak{F}_\alpha$ is a finite union of basic closed sets Ω_F:

$$\mathfrak{F}_\alpha = \Omega_{F_{\alpha 1}} \cup \cdots \cup \Omega_{F_{\alpha_n}}.$$

Hence, since $\mathfrak{F}_\alpha \in \Psi$, Ψ contains some $\Omega_{F_{\alpha_{i(\alpha)}}} = \Omega_{F_\alpha}$ for each α (Theorem V, §3); and these Ω_{F_α} have the finite intersection property.

If $\bigcap_\alpha \Omega_{F_\alpha} = \emptyset$, no ultrafilter contains all the filters F_α; the family $\bigcup_\alpha F_\alpha$ is incompatible, and there exists a finite number of elements $a_1 \in F_{\alpha 1}, \cdots, a_n \in F_{\alpha_n}$ such that $a_1 a_2 \cdots a_n = \omega$. Then $F_{\alpha_1} \cup F_{\alpha_2} \cup \cdots \cup F_{\alpha_n}$ is incompatible, in contradiction to the assumption that $\bigcap_{i=1}^n \Omega_{F\alpha_i}$ is nonempty and contains at least one ultrafilter.

Remarks. If $\mathfrak{T}_0 = \mathfrak{T}_F$ the space of ultrafilters is compact.

In this case every open set contains a basic Ω_a which is open and closed, and the space Ω is totally disconnected. This is, in particular, the case when E is complemented.

7. **Spaces of maximal ideals.** We consider a commutative ring A, with a unit element. By Krull's theorem every proper ideal $\mathfrak{a}$ is contained in a maximal ideal. Let Ω be the set of all maximal ideals of A; we denote by Ω_a the set of all maximal ideals containing a, by $\Omega_{\mathfrak{a}}$ the set of all maximal ideals containing the ideal $\mathfrak{a}$. We define on Ω the two following topologies:

$\mathfrak{T}_0$: The sets Ω_a are taken as a subbase for open sets.

$\mathfrak{T}_F$: The sets $\Omega_{\mathfrak{a}}$ are taken as a subbase for closed sets.

Description of the open sets and of the closed sets in $\mathfrak{T}_0$ and $\mathfrak{T}_F$.

(1) For $\mathfrak{T}_0$ the sets Ω_a form a subbase for open sets.

(2) If the family (a_α) of elements of A generates the ideal $\mathfrak{a}$, it is clear that $\bigcap_\alpha \Omega_{\mathfrak{a}_2} = \Omega_{\mathfrak{a}}$. Hence the closed sets for $\mathfrak{T}_F$ are finite unions $\Omega_{\mathfrak{a}_1} \cup \cdots \cup \Omega_{\mathfrak{a}_n}$. Every maximal ideal belonging to this union contains $\mathfrak{a} = \mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_n$.

Suppose conversely that a maximal ideal M contains $\mathfrak{a}$ but does not contain any $\mathfrak{a}_i$; then $A = m + \mathfrak{a}_i$, $1 = m_i + a_i$, $m_i \in M$, $a_i \in \mathfrak{a}_i$.

By multiplication $1 = a_1 \cdots a_n + m$, $m \in \mathfrak{m}$ and $\mathfrak{m} + \mathfrak{a} = A$, which contradicts the assumption $\mathfrak{a} \subset \mathfrak{m}$.

Hence $\Omega_{\mathfrak{a}_1} \cup \cdots \cup \Omega_{\mathfrak{a}_n} = \Omega_{\mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_n}$, and the only closed sets in $\mathfrak{T}_F$ are the sets $\Omega_{\mathfrak{a}}$.

Comparison between $\mathfrak{T}_0$ *and* $\mathfrak{T}_F$. (1) Given any $a \in A$, consideration of the family b_α of all elements of A such that the ideal (a, b_α) is the whole ring proves, as above, that: $\mathfrak{T}_0$ *is finer than* $\mathfrak{T}_F$.

(2) Consider now a closed set $\Omega_a^* = \Omega - \Omega_a$ of the subbase for $\mathfrak{T}_0$, and let us see under what conditions it is a closed set for $\mathfrak{T}_F$. There must exist an ideal $\mathfrak{b}$ such that $\Omega_a^* = \Omega_{\mathfrak{b}}$ This implies that $(a) + b = A$, hence there exist $b \in \mathfrak{b}$ and $x \in A$ such that $xa + b = 1$. The maximal ideals containing b cannot contain a. Since $\Omega_b \supset \Omega_{\mathfrak{b}}$, we can write the condition $\Omega_a^* = \Omega_b$, or:

A necessary and sufficient condition that $\mathfrak{T}_0 = \mathfrak{T}_F$ *is that, for every* $a \in A$, *there exists* $b \in A$ *such that the maximal ideals not containing* a *are the same as the maximal ideals containing* b.

In some respect b is a "complement of a." This condition may be written also:

$$(a) + (b) = A,$$

$(a) \cap (b)$ is contained in the intersection of all maximal ideals, the radical [J1].

In other words:

A necessary and sufficient condition that $\mathfrak{T}_0 = \mathfrak{T}_F$ *is that the factor ring of* A *by its radical be a regular ring* [VN].

Separation axioms. (1) $\mathfrak{T}_F$ *satisfies the axiom* T_1.

This is obvious. The example of the ring of integers (where the only nontrivial closed sets are the finite ones) shows that $(\Omega, \mathfrak{T}_F)$ is not always a Hausdorff space.

(2) Given two distinct maximal ideals $\mathfrak{a}$ and $\mathfrak{b}$, $\mathfrak{a} + \mathfrak{b} = A$; hence there exist $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$ such that $a + b = 1$. Ω_a and Ω_b are two disjoint open set in $\mathfrak{T}_0$ containing $\mathfrak{a}$ and $\mathfrak{b}$: Hence:

$(\Omega, \mathfrak{T}_0)$ *is a Hausdorff space.*

Compactness. $\mathfrak{T}_F$ *satisfies the Lebesgue covering axiom.*

Let $\{\Omega_{\mathfrak{a}_\alpha}\}$ be a family of closed sets for $\mathfrak{T}_F$ with the finite intersection property. This means that no finite family of ideals $(\mathfrak{a}_{\alpha_i})$ generates the whole ring. Hence the ideal generated by the whole family $(\mathfrak{a}_\alpha)$ cannot contain 1, and is contained in some maximal ideal. Hence:

$$\bigcap_\alpha \Omega_{\mathfrak{a}_\alpha} \neq \emptyset. \qquad Q.E.D.$$

Remark. If $\mathfrak{T}_0=\mathfrak{T}_F$ (for example if A is a commutative regular ring or, in particular a boolean ring), the space of maximal ideals is compact. For a boolean ring there is a natural isomorphism between the space of ultrafilters and the space of maxmal ideals. The boolean rings form the link between §§6 and 7.

Chapter III

8. **Compactification of uniform spaces.** Given a topological space $(S, \mathfrak{T})$ we can consider several directed sets associated with it: the set of all subsets, the set of all closed sets, or open sets, or open-closed sets. The filters and ultrafilters in these directed sets will be called filters, closed filters, open filters, open-closed filters. Let us review the associated spaces of ultrafilters.

(1) Space of open-closed ultrafilters Ω_S, or Stone's space [S] $-\Omega_S$, is compact since its directed set is complemented. If every point of S has a base of neighborhoods composed of open-closed sets, S is homeomorphic with an everywhere dense subset of Ω_S. This proves, without real numbers, that S is a uniformizable space.

(2) Space of closed ultrafilters Ω_W, or Wallman's space [W_a]. Its directed set is not complemented, and, in general, $\mathfrak{T}_0\neq\mathfrak{T}_F$.

$\mathfrak{T}_0$, totally disconnected, is not very interesting: if a closed ultrafilter converges to a point $x\in S$, x must belong to every set of the ultrafilter, which must be the principal ultrafilter $\mathfrak{U}_{\{a\}}$; this gives a natural correspondence ϕ between S (if S is a T_1 space, of course) and a subset of Ω_W; but, with $(\Omega_W, \mathfrak{T}_0)$ the image of a closed set of S is an open set in $\phi(S)$, and $\phi(S)$ is a discrete space.

With the topology $\mathfrak{T}_F$, S and $\phi(S)$ are homeomorphic. The main question is to see under what condition $(\Omega_W, \mathfrak{T}_F)$ is a Hausdorff space, and therefore a compact space. It was settled by Wallman who proved the following elegant result: a necessary and sufficient condition for $(\Omega_W, \mathfrak{T}_F)$ to be compact is that $(S, \mathfrak{T})$ be a normal space. This shows, again without real numbers, that every normal space is uniformizable.

(3) Space of ultrafilters Ω. Since its directed set is complemented, $\mathfrak{T}_0=\mathfrak{T}_F$, and Ω is compact. It is obtained by completing the set S, considered as a discrete space, with respect to the uniform structure of finite partitions [B]. Since there is nothing in Ω which reflects the given topology $\mathfrak{T}$ of S, Ω cannot be used for topological purposes without preparation.

If S is not discrete there is no natural one-to-one correspondence between S and a subset of Ω: in fact there is in general more than one ultrafilter converging to a point $x\in S$. We shall have to identify these ultrafilters converging to x, and put S in one-to-one correspondence with a subset of an identification space of Ω; of course we shall try to find an identification space which is Hausdorff. In this case S will be homeomorphic with a dense subset of a compact space, hence a Hausdorff uniformizable space. We must therefore suppose that the topology $\mathfrak{T}$ of S is compatible with a uniform structure $\mathfrak{S}$. We shall operate only with symmetric surroundings ($V=V^{-1}$).

Envelope of a filter—equivalent filters. Let $\mathfrak{F}$ be a filter over S. We call *envelope* of $\mathfrak{F}$ and denote by $\mathfrak{F}^*$ the filter having as base the family of all sets $V(A)$, $V\in\mathfrak{S}$, $A\in\mathfrak{F}$. It is clear that:

$$\mathfrak{F}\supset\mathfrak{F}^*,$$
$$\mathfrak{F}^{**}=\mathfrak{F}^* \qquad (\text{if } W^2\subset V,\ W(W(A))\subset V(A)).$$

If $\mathfrak{F}\supset\mathfrak{G}$, $\mathfrak{F}^*\supset\mathfrak{G}^*$.

If $\mathfrak{F}$ converges to x, its envelope is the filter $\mathfrak{V}(x)$ of neighborhoods of x.

Two filters $\mathfrak{F}$ and $\mathfrak{G}$ are called *equivalent* if they have the same envelope. Notice that, if $\mathfrak{F}$ and $\mathfrak{G}$ are Cauchy filters, this equivalence coincides with the equivalence used in the completion theorem [B].

If $\mathfrak{F}\supset\mathfrak{G}^*$ and $\mathfrak{G}\supset\mathfrak{F}^*$, then $\mathfrak{F}^*\supset\mathfrak{G}^{**}=\mathfrak{G}^*$ and $\mathfrak{G}^*\supset\mathfrak{F}^{**}=\mathfrak{F}^*$, hence $\mathfrak{F}^*=\mathfrak{G}^*$ and $\mathfrak{F}$ and $\mathfrak{G}$ are equivalent.

We suppose now that $\mathfrak{G}$ is an ultrafilter, and that $\mathfrak{F}\supset\mathfrak{G}^*$. If $\mathfrak{G}\not\supset\mathfrak{F}^*$, $\mathfrak{G}$ and $\mathfrak{F}^*$ are incompatible ($\mathfrak{G}$ being an ultrafilter). Hence there exists $A\in\mathfrak{F}$, $B\in\mathfrak{G}$ and $V\in\mathfrak{S}$ such that $B\cap V(A)=\emptyset$. But this implies $V(B)\cap A=\emptyset$ (since V is supposed to be symmetric), in contradiction with $\mathfrak{F}\supset\mathfrak{G}^*$. Hence $\mathfrak{G}\supset\mathfrak{F}^*$, and $\mathfrak{F}$ and $\mathfrak{G}$ are equivalent.

The identification space S. If, in the space of ultrafilters Ω, we identify equivalent ultrafilters, we obtain an identification space $S^\smile$. We denote by ϕ the natural mapping of Ω onto $S^\smile$.

$S^\smile$ is a compact space. In Ω every closed set is a set $\Omega_{\mathfrak{F}}$ of all ultrafilters finer than a given filter $\mathfrak{F}$. We shall prove that the saturated set of $\Omega_{\mathfrak{F}}$ is the set $\Omega_{\mathfrak{F}^*}$, hence a closed set, which proves that S is compact (§4).

(1) If $\mathfrak{U}\supset\mathfrak{F}$ and if $\mathfrak{V}$ is equivalent to $\mathfrak{U}$, $\mathfrak{V}\supset\mathfrak{V}^*=\mathfrak{U}^*\supset\mathfrak{F}^*$.

(2) Let $\mathfrak{V}\supset\mathfrak{F}^*$. We seek an ultrafilter $\mathfrak{U}\supset\mathfrak{F}$ equivalent to $\mathfrak{V}$. For every $A\in\mathfrak{F}$, every symmetric $V\in\mathfrak{S}$, and every $B\in\mathfrak{V}$, $B\cap V(A)\neq\emptyset$, since the filters $\mathfrak{V}$ and $\mathfrak{F}^*$ are compatible. Hence $V(B)\cap A\neq\emptyset$, which expresses the fact that $\mathfrak{V}^*$ and $\mathfrak{F}$ are compatible. Let $\mathfrak{U}$ be an ultrafilter finer than $\mathfrak{V}^*$ and $\mathfrak{F}$. Since $\mathfrak{U}\supset\mathfrak{V}^*$ and since $\mathfrak{V}$ is an ultrafilter, $\mathfrak{U}$ and $\mathfrak{V}$ are equivalent. Q.E.D.

Imbedding of S in $S^\smile$. We have noticed before that the set of all ultrafilters converging to a given point $x\in S$ is an equivalence class $x^\cdot\in S^\smile$. This gives us a natural correspondence between S and a subset $S^\cdot$ of $S^\smile$.

Every open set of Ω, and, in particular, every saturated open set, contains an Ω_A, which contains a convergent ultrafilter. Hence $S^\cdot$ is dense in $S^\smile$.

The trace on $S^\cdot$ of the closed set $\phi(\Omega_{\mathfrak{F}^*})$ of S is obviously the image of the closure of the filter $\mathfrak{F}^*$ over S (that is, the set of all limit points of ultrafilters finer than $\mathfrak{F}^*$). Conversely every closed set $A\subset S$, being the closure of the filter $\mathfrak{F}^*$ generated by the $V(A)$ ($V\in\mathfrak{S}$) which is an envelope, has as image in $S^{\cdot}$ the closed set $S^{\cdot}\cap\phi(\Omega_{\mathfrak{F}^*})$. That shows that the correspondence between S and $S^{\cdot}$ is a homeomorphism. Therefore:

THEOREM IX. *Every Hausdorff uniformizable space can be imbedded in a compact space.*

Remark. Let S be any Hausdorff space, Ω_C the set of all convergent ultrafilters over S with the topology induced by that of Ω. Identifying the ultrafilters converging to the same point we obtain a one-to-one mapping ψ of S onto an identification space T of Ω_C.

A closed set of ultrafilters in Ω_C is the set of all convergent ultrafilters containing a given filter $\mathfrak{F}$; hence the inverse image by ψ of any closed set A of T (A corresponds to a saturated closed set of Ω_C) is the closure of a certain filter $\mathfrak{F}$, hence a closed set.

Therefore ψ *is continuous.*

If ψ is bicontinuous, the image by ψ of a closed set A in S is closed in T. In other words, the set of all convergent ultrafilters with limit in A is a set $\Omega_C\cap\Omega_{\mathfrak{F}}$. But the intersection of all these ultrafilters is the filter $\mathfrak{V}(A)$ of neighborhoods of A; hence "ψ is bicontinuous" means that, if A is closed, the closure of $\mathfrak{V}(A)$ is A itself; in other words S is a *regular* space.

Comparison between the uniform structures of S and $S^{\cdot}$. The unique uniform structure of the compact space $S^{\smile}$, defined by the graphs of the finite open coverings, induces on $S^{\cdot}$ (and therefore on S) a new uniform structure $\mathfrak{S}^*$: $\mathfrak{S}^*$ is precompact and compatible with the topology of S. We shall compare $\mathfrak{S}$ and $\mathfrak{S}^*$.

PROPOSITION 6. $\mathfrak{S}^*$ *is coarser than* $\mathfrak{S}$.

By taking complements, there corresponds to every finite open covering of $S^{\smile}$ a finite family (B_i) of closed sets with empty intersection. Each B_i is the set of all envelopes of ultrafilters containing a given envelope $\mathfrak{F}_i^*$. The fact that $\bigcap_i B_i=\emptyset$ means that the family $\mathfrak{F}_1^*\cup\cdots\cup\mathfrak{F}_n^*$ is incompatible. Denoting by A_i the closure of the filter $\mathfrak{F}_i$ in S, that means that there exists a surrounding $V\in\mathfrak{S}$ such that

$$V(A_1)\cap\cdots\cap V(A_n)=\emptyset. \tag{1}$$

If we identify S and $S^{\cdot}$ we notice that $A_i=B_i\cap S$. Hence a system of generators of the filter $\mathfrak{S}^*$ is composed of the graphs

$$U=\bigcup_{i=1}^{n}(U_i\times U_i)$$

where the $A_i=S-U_i$ are closed sets satisfying (1).

Suppose that the V of the condition (1) is a symmetric surrounding. Let $(x, y)\in V$; since the $\{U_i\}$ form a covering of S, we may suppose that $x\in U_i$.

Let $U_{i_1}\cdots U_{i_q}$ be all the sets U_j to which x belongs; if y belongs to none of them, y belongs to all the sets $A_{i_1},\cdots,A_{i_q}$; hence $x\in V(A_{i_1})\cap\cdots\cap V(A_{i_n})$.

By condition (1) there exists another index j such that $x \notin V(A_j)$; a fortiori $x \notin A_j$, $x \in U_j$; a contradiction. Hence y belongs to some U_{i_k}, and $(x, y) \in U$; therefore $V \subset U$ and the proposition is proved.

COROLLARY. *If* $\mathfrak{S}_1$ *is finer than* $\mathfrak{S}_2$ *and compatible with the same topology,* $\mathfrak{S}_1^*$ *is finer than* $\mathfrak{S}_2^*$.

In fact condition (1) is easier to fulfill with $\mathfrak{S}_1$ then with $\mathfrak{S}_2$.

Relation with the classical completion of $(S, \mathfrak{S})$. If $\mathfrak{S} = \mathfrak{S}^*$, $\mathfrak{S}$ is a precompact uniform structure. In order to prove the converse we show first that the space $S^\frown$, the completion of S with respect to the uniform structure $\mathfrak{S}$, is homeomorphic with a subspace of the compact space $S^\smile$ (of course this is only a topological homeomorphism, and not a uniform structural isomorphism).

PROPOSITION 7. $S^\frown$ *is homeomorphic with a subspace of* $S^\smile$.

If the ultrafilter $\mathfrak{U}$ is a Cauchy filter, its envelope $\mathfrak{U}^*$ is also, and therefore all the ultrafilters equivalent to $\mathfrak{U}$. If we notice that, for Cauchy filters, equivalence by envelopes is the same as the classical equivalence (used in the completion theorem), we get a natural one-to-one correspondence between $S^\frown$ and the subset T of $S^\smile$ which is the image of the set of Cauchy ultrafilters. We identify $S^\frown$ and T considered as point sets. We recall that the uniform structure of $S^\frown$ is defined by the surroundings $\tilde{V}$: the relation "$(\mathfrak{X}, \mathfrak{Y}) \in \tilde{V}$" for Cauchy filters $\mathfrak{X}$ and $\mathfrak{Y}$ meaning "$\mathfrak{X}$ and $\mathfrak{Y}$ have in common a set $D \subset S$ small of order V."

Let $\Phi \subset T$ be closed in T; Φ is the set of all envelopes of Cauchy ultrafilters $\mathfrak{U}^*$ containing a given envelope $\mathfrak{F}^*$. Let $\mathfrak{X}^*$ be an envelope of a Cauchy ultrafilter adherent to Φ in the topology of $S^\frown$. Let B be any element of $\mathfrak{F}$, $W \in \mathfrak{S}$ any surrounding, and $V \in \mathfrak{S}$ a surrounding such that, $V^2 \subset W$, $-\mathfrak{X}^*$ being in the closure of Φ, there exists $\mathfrak{U}^* \in \Phi$ such that $(\mathfrak{X}^*, \mathfrak{U}^*) \in \tilde{V}$: in other words there exists D small of order V, $D \in \mathfrak{X}^* \cap \mathfrak{U}^*$, $-D \cap V(B) \neq \emptyset$ since $\mathfrak{U}^* \supset \mathfrak{F}^*$; hence $D \subset V(B)$, and $W(B) \in \mathfrak{X}^*$. That means that $\mathfrak{X}^*$ is finer than $\mathfrak{F}^*$, and therefore $\mathfrak{X}^* \in \Phi$. Therefore Φ is closed in $S^\frown$.

Conversely let Ψ be closed in $S^\frown$, $\mathfrak{F}$ be the intersection of the filters $\mathfrak{U}_\alpha^*$ elements of Ψ, $\mathfrak{F}^*$ the envelope of $\mathfrak{F}$. Let $W \in \mathfrak{S}$ be any surrounding, $V \in \mathfrak{S}$ symmetric and such that $V^3 \subset W$. Let $\mathfrak{X}^*$ be any envelope of a Cauchy ultrafilter finer than $\mathfrak{F}^*$. We pick in $\mathfrak{X}^*$ a set A small of order V, and in each $\mathfrak{U}_\alpha^*$ a set D_α small of order V. $D = \bigcup_\alpha D_\alpha$ belongs to $\mathfrak{F}$; hence $V(D) \in \mathfrak{F}^*$. Since $\mathfrak{X}^* \supset \mathfrak{F}^*$, $A \cap V(D) \neq \emptyset$ (V being symmetric) it follows that $V(A) \cap D \neq \emptyset$. Hence there exists an index α such that $V(A) \cap D_\alpha \neq \emptyset$; the set $V(A) \cup D_\alpha$ is therefore small of order $V^3 \subset W$, and belongs to $\mathfrak{X}^*$ and $\mathfrak{U}_\alpha^*$. This means that, whatever $W \in \mathfrak{S}$, there exist a $\mathfrak{U}_\alpha^* \in \Psi$ such that $(\mathfrak{X}^*, \mathfrak{U}_\alpha^*) \in \tilde{W}$; in other words $\mathfrak{X}^*$ is in the closure of Ψ in $S^\frown$, and $\mathfrak{X}^* \in \Psi$. Therefore Ψ is closed in T. Q.E.D.

Now if $\mathfrak{S}$ is precompact, every ultrafilter is a Cauchy filter, hence $S^\frown$ can be identified with $S^\smile$. From the uniqueness of the uniform structure of the compact space $S^\frown$, we deduce that $\mathfrak{S}=\mathfrak{S}^*$. Therefore:

PROPOSITION 8. $\mathfrak{S}=\mathfrak{S}^*$ *characterizes the precompact uniform structures.*

COROLLARY 1. $\mathfrak{S}^{**}=\mathfrak{S}^*$.

COROLLARY 2. *A necessary and sufficient condition for a uniform space to be precompact is that every ultrafilter be a Cauchy filter.*

The necessity is clear (every ultrafilter over a compact space being convergent). The sufficiency is also clear (it means that $S^\smile=S^\frown$).

Remark. It may happen that $\mathfrak{S}$ is very simple but $\mathfrak{S}^*$ very complicated; this is the case when $\mathfrak{S}$ is the additive uniform structure of the real line. Then $S^\smile$ is neither the circle, nor the closed interval, but a space almost as complicated as the Čech compactification of the real line.

9. **Relations with existing compactifications, characterization of a uniformizable spaces.** Let $(S, \mathfrak{T})$ be a topological space. We consider the *class* of all continuous mappings of $(S, \mathfrak{T})$ *onto* all uniform spaces. Since the cardinal number of any image of S is smaller than the cardinal number of S, the possible image spaces of S which are uniform form a *set*, and therefore so do all possible continuous mappings of $(S, \mathfrak{T})$ onto them. Therefore there is no logical difficulty in considering the set of all continuous mappings of $(S, \mathfrak{T})$ onto all uniform spaces. We denote this set by $\mathfrak{M}=\{\mathfrak{f}_\alpha\}$.

Let $\mathfrak{f}_\alpha$ be a continuous mapping of $(S, \mathfrak{T})$ onto the uniform space $(S'_\alpha, \mathfrak{T}'_\alpha)$ and $\mathfrak{S}'_\alpha$ the uniform structure of S'_α. We denote by $\mathfrak{S}_\alpha=\mathfrak{f}_\alpha^{-1}(\mathfrak{S}'_\alpha)$ the inverse image of $\mathfrak{S}'_\alpha$; $\mathfrak{S}_\alpha$ is a uniform structure on S. Since $\mathfrak{f}_\alpha$ is continuous the topology $\mathfrak{T}_\alpha$ deduced from $\mathfrak{S}_\alpha$ is coarser than $\mathfrak{T}$.

Let $\mathfrak{U}$ denote the l.u.b. of all the uniform structures $\mathfrak{S}_\alpha$; $\mathfrak{U}$ is called the *universal uniform structure* of the space $(S, \mathfrak{T})$; in general the topology $\mathfrak{T}_1$ deduced from $\mathfrak{U}$ is coarser than $\mathfrak{T}$ ($\mathfrak{T}_1=\text{l.u.b.}_\alpha\, \mathfrak{T}_\alpha$).

But, if $(S, \mathfrak{T})$ is uniformizable, the identity mapping of S onto itself is in the family $\{\mathfrak{f}_\alpha\}$, and $\mathfrak{U}$ is finer than any uniform structure of S compatible with $\mathfrak{T}$. Hence $\mathfrak{T}_1=\mathfrak{T}$ and $\mathfrak{U}$ itself is compatible with $\mathfrak{T}$. Consequently:

THEOREM X. *For any uniformizable space $(S, \mathfrak{T})$ the universal uniform structure $\mathfrak{U}$ is compatible with $\mathfrak{T}$ and finer than any other uniform structure compatible with $\mathfrak{T}$. Every continuous mapping of $(S, \mathfrak{T})$ onto any uniform space is uniformly continuous with respect to $\mathfrak{U}$. This last property characterizes $\mathfrak{U}$ among the uniform structures compatible with $\mathfrak{T}$.*

The last statement is clear: consider the identity mapping of $(S, \mathfrak{S})$ onto $(S, \mathfrak{U})$, $\mathfrak{S}$ being different from $\mathfrak{U}$ and compatible with $\mathfrak{T}$.

Čech uniform structure [C]. Instead of considering the whole family $\{\mathfrak{f}_\alpha\}$ of continuous mappings of $(S, \mathfrak{T})$ onto uniform spaces, we shall consider the subfamily $\{\mathfrak{f}_\beta\}$ of continuous mappings of $(S, \mathfrak{T})$ onto totally bounded uniform spaces. The corresponding uniform structures $\mathfrak{S}_\beta$ are totally bounded (as inverse images of totally bounded ones), and $\mathfrak{T}(\mathfrak{S}_\beta)$ is coarser than $\mathfrak{T}$. We denote by $\mathfrak{C}$ and call the *Čech uniform structure* of the topological space $(S, \mathfrak{T})$ the l.u.b. of the $\mathfrak{S}_\beta$. $\mathfrak{C}$ is clearly totally bounded; the topology $\mathfrak{T}(\mathfrak{C})$ is in general coarser than $\mathfrak{T}$.

But if $(S, \mathfrak{T})$ is Hausdorff and uniformizable, it has at least one precompact uniform structure (some $\mathfrak{S}^*$) compatible with $\mathfrak{T}$. By considering the identity mapping of S onto itself, we see that this $\mathfrak{S}^*$ is coarser than $\mathfrak{C}$. Hence $\mathfrak{T}=\mathfrak{T}(\mathfrak{C})$. Consequently:

THEOREM XI. *For any Hausdorff unformizable space $(S, \mathfrak{T})$ the Čech uniform structure $\mathfrak{C}$ is compatible with $\mathfrak{T}$, precompact, and finer than any other uniform structure compatible with $\mathfrak{T}$. Every continuous mapping of $(S, \mathfrak{T})$ onto any totally bounded uniform space is uniformly continuous with respect to $\mathfrak{C}$. This last property characterizes $\mathfrak{C}$ among the precompact uniform structures compatible with $\mathfrak{C}$.*

Since the correspondence $\mathfrak{S}\rightarrow\mathfrak{S}^*$ is monotonic, we see immediately that:

$$\mathfrak{C} = \mathfrak{U}^*.$$

The completion of S with respect to $\mathfrak{C}$ is a compact space called the Čech compactification of S and commonly denoted by $\beta(S)$.

COROLLARY (ČECH). *$\beta(S)$ is the greatest compact extension of S.*

This means that every compact space T in which S is everywhere dense is an identification space of $\beta(S)$.

Let $\mathfrak{f}$ be the homeomorphism between S and an everywhere dense subset S_1 of T. If we put the Čech structure on S and on S_1, the precompact structure induced by T, $\mathfrak{f}$ is uniformly continuous, and hence can be extended to the completion $\beta(S)$. $\mathfrak{f}(\beta(S))$ is compact, contains S_1; hence $\mathfrak{f}(\beta(S))=T$.

Since the continuous image of a closed set in $\beta(S)$ is compact, it is closed in T and T has the identification topology. One can say that T is obtained by identifying points in $\beta(S)-S$.

Čech structure and Wallman structure of a normal space S [C] [A]. Let S be a normal space. The Wallman compactification consists in imbedding S in the space $(\Omega_W, \mathfrak{T}_F)$ of closed ultrafilters over S. A closed set in Ω_W is the set of all closed ultrafilters finer than some closed filter of a finite family $(\mathfrak{F}_\alpha)$. Let $\mathfrak{F}=\bigcap_\alpha\mathfrak{F}_\alpha$. Every ultrafilter of the closed set is finer than $\mathfrak{F}$. Let conversely $\mathfrak{U}$ be a closed ultrafilter finer than $\mathfrak{F}$. If it does not contain any $\mathfrak{F}_\alpha$, it is incompatible with all of them. Hence, for each α, there exists $A_\alpha\in\mathfrak{F}_\alpha$, $B_\alpha\in\mathfrak{F}_\alpha$ such that $A_\alpha\cap B_\alpha=\emptyset$. $A=\bigcap_\alpha A_\alpha$ is in $\mathfrak{U}$ since the family (A_α) is finite. $B=\bigcup_\alpha B_\alpha$ is in $\mathfrak{F}$. But then $B\cap A=\emptyset$, which contradicts $\mathfrak{F}\subset U$. Notice that this fact, namely that every closed set of $(\Omega_W, \mathfrak{T}_F)$ is a set $\Omega_{\mathfrak{F}}$, depends only on the fact that the directed set of closed sets in S is a distributive lattice. Notice also that the trace of $\Omega_{\mathfrak{F}}$ on S is the closure of the filter f in S (or the intersection of all the closed sets in $\mathfrak{F}$).

The uniform structure of the Wallman space Ω_W is defined by the graphs of the finite open coverings, or equivalently by the finite families of closed sets with empty intersection. These are defined by the finite families of closed filters $(\mathfrak{F}_i)$ such that $\bigcup_i \mathfrak{F}_i$ is incompatible. Let A_i be the closure of $\mathfrak{F}_i$ in S. The trace on S of the finite open covering of Ω_W determined by the $\mathfrak{F}_i$ is clearly composed of the open sets $U_i = S - A_i$, which form a covering of S.

Conversely every finite open covering $\{U_i\}$ of S is obtained in this way. (Take $A_i = S - U_i$, for $\mathfrak{F}_i$ the principal closed filter generated by A_i; since $\bigcap_i A_i = \emptyset$, the filters $\mathfrak{F}_i$ are incompatible. Hence $\bigcap_i \Omega_{\mathfrak{F}_i} = \emptyset$, and the $\Omega_W - \Omega_{\mathfrak{F}_i}$ form a finite open covering of Ω_W.) Therefore:

Proposition 9. *The Wallman structure $\mathfrak{W}$ of a normal space S is defined by the graphs of the finite open coverings of S.*

Remark. The fact that the graphs of the finite open coverings form a base for a uniform structure characterizes normal spaces: given 2 disjoint closed sets A and B, their complements form a finite open covering. Let $V = (S-A) \times (S-A) \cup (S-B) \times (S-B)$ be its graph. Suppose there exists a symmetric surrounding W such that $W^2 \subset V$; then $V(A) = S - B \supset W^2(A)$, so that $B \cap W^2(A) = \emptyset$, and hence $W(A) \cap W(B) = \emptyset$; and this implies normality.

A proof of the converse, which does not use Wallman's results, is the following:

Let S be a normal space, V the graph of the binary open covering $\{A, B\}$. Let A_1 and B_1 be two closed sets such that $A_1 \subset A$, $B_1 \subset B$, $A_1 \cup B_1 = S$. It is clear that the three open sets $\{S - A_1, S - B_1, A \cup B\}$ form a covering of S. Let W be the graph. One checks easily that $W^2 \subset V$. Noticing now that the graph V of any finite open covering is the intersection of a finite number $\{V_1, \cdots, V_n\}$ of graphs of binary open coverings, we deduce from the existence for each V_i of a W_i with $W_i^2 \subset V_i$ that $W = W_1 \cap \cdots \cap W_n$ is such that $W^2 \subset V$. Consequently the filter generated by the graphs of the finite open coverings of a normal space S satisfies the axioms for a uniform structure; and this uniform structure is clearly compatible with the topology of S.

From Proposition 9 and the proof of Proposition 7, we deduce that every precompact uniform structure on S is defined by the graphs of the finite open coverings $\{U_i\}$ which satisfy some condition (condition (1) of the proof of Proposition 9). Hence the Wallman structure is finer than all of them. Consequently:

Theorem XII. *For a normal space the Wallman structure $\mathfrak{W}$ and the Čech structure $\mathfrak{C}$ are identical.*

Corollary. *If A is closed in the normal space S, the sets $V(A)$ ($V \in \mathfrak{W}$) form a base of neighborhoods of A.*

That is: every open set B containing A contains some $V(A)$. In fact the Wallman surrounding $V = (B \times B) \cup (S-A) \times (S-A)$ works: $V(A) = B$.

Remark. The same is true, a fortiori, with the universal uniform structure of S.

Remarks on the Čech uniform structure. (1) The Čech structure of a product of uniformizable spaces is, in general, different from the product of the Čech structures of its factors.

In fact, if they were identical, one could use the Čech structure of a topological group G in order to complete it: by Theorem XI the continuous mapping $(x, y) \rightarrow xy^{-1}$ would be uniformly continuous, hence could be extended to the completion. Thus every topological group G would be a subgroup of a compact group $G^\frown$. This is clearly false: the unique uniform structure of $G^\frown$ is invariant by translations, hence also the precompact uniform structure induced by it on G. Any infinite discrete group or the additive group of real numbers gives a counter example.

(2) The trace of the Čech uniform structure of S on a subspace A is not always the Čech structure of A: take A infinite and discrete, S the Alexandroff compactification (by one point "at infinity"). The induced structure is defined by the graphs of the finite partitions $\{A_i\}$ of A, one of the A_i being the complement of a finite subset. The Čech structure on A is defined by the graphs of all finite partitions, without restriction. One could have noticed also that the Čech compactification of A is the space of all ultrafilters over A, and not S.

The Čech structure of A and the induced structure are identical in the case where A is a *retract* of S (in particular if A is open-closed in S, or if A is a factor when S is a product space). In fact let r be the retraction. For a continuous mapping f of A in a precompact space F, the composite mapping maps continuously S into F, and is therefore uniformly continuous for the Čech structure of S. Its contraction to A, which is $\mathfrak{f}$, is therefore uniformly continuous for the induced structure, and this property, holding for all $\mathfrak{f}$, characterizes the Čech structure of A (Theorem XI).

By Theorem X the same is true for the universal uniform structures.

The Čech structure of A and the structure on A induced by the Čech structure of S are also equal when S is normal and A closed in S: this result is clear if one uses the Wallman form of the Čech structure. Čech has proved that, if this equality holds for every closed subset of S, then S is normal [C].

Topological characterization of uniformizable spaces. We consider again a topological space $(S, \mathfrak{T})$, and now the class of all continuous mappings of $(S, \mathfrak{T})$ into all compact spaces. For such a mapping $\mathfrak{f}$ the cardinal number of $\mathfrak{f}(S)$ is at most the cardinal number of S; and the cardinal number of the compact space, closure of $\mathfrak{f}(S)$, is at most $2^{2^{\text{card}\, S}}$ (the cardinal of the set of ultrafilters over $\mathfrak{f}(S)$ [P]). Consequently we can restrict ourselves to compact spaces with bounded cardinal number. The continuous mappings of $(S, \mathfrak{T})$ into all compact spaces with cardinal $\leqq 2^{2^{\text{card } S}}$ form a set $\{\mathfrak{f}_\gamma\}$. Let $(T_\gamma, \mathfrak{T}'_\gamma)$ be the compact space corresponding to the mapping $\mathfrak{f}_\gamma$; since $\mathfrak{f}_\gamma$ is continuous, the inverse image topology $\mathfrak{f}_\gamma^{-1}(\mathfrak{T}'_\gamma) = \mathfrak{T}_\gamma$ is coarser than $\mathfrak{T}$. Let $\mathfrak{T}_0 = \text{l.u.b.}_\gamma\, \mathfrak{T}_\gamma$. In general $\mathfrak{T}_0$ is coarser than $\mathfrak{T}$. Each $\mathfrak{T}'_\gamma$ is a uniformizable topology. If $\mathfrak{S}'_\gamma$ denotes the uniform structure of the compact space $(T_\gamma, \mathfrak{T}'_\gamma)$, $\mathfrak{T}'_\gamma = \mathfrak{T}(\mathfrak{S}'_\gamma)$. Passing to the inverse image we define $\mathfrak{S}_\gamma = \mathfrak{f}_\gamma^{-1}(\mathfrak{S}'_\gamma)$, and it is clear that $\mathfrak{T}_\gamma = \mathfrak{T}(\mathfrak{S}_\gamma)$. In other words $\mathfrak{T}_\gamma$ is uniformizable. Consequently the topology $\mathfrak{T}_0$ is deduced from the Čech uniform structure $\mathfrak{C}$ of $(S, \mathfrak{T})$. Note that $\mathfrak{C}$ and $\mathfrak{T}_0$ may very well be non-separated.

If $(S, \mathfrak{T})$ is uniformizable and separated, its injection into any one of its compactifications is a mapping $\mathfrak{f}_\gamma$. Hence $\mathfrak{T}$ is among the $\mathfrak{T}_\gamma$, and $\mathfrak{T}_0 = \mathfrak{T}$. Consequently:

THEOREM XIII. *A necessary and sufficient condition for a topological space* $(S, \mathfrak{T})$ *to be uniformizable Hausdorff is that the topology* $\mathfrak{T}_0$ *defined by the mappings of* $(S, \mathfrak{T})$ *into compact spaces be identical with* $\mathfrak{T}$.

Remark. Using the fact that every compact space can be imbedded in some "cube," we shall prove that the condition of the preceding theorem is *equivalent to the complete regularity of* $(S, \mathfrak{T})$. In fact, using products of compact spaces if necessary, the condition of Theorem XIII means: "given any closed set A in $(S, \mathfrak{T})$ there exists a compact space T, a closed subset B of T, and a continuous mapping $\mathfrak{f}$ of S into T such that $A = \mathfrak{f}^{-1}(B)$."

We may take for T a cube. Let $x \in S - A$, then $\mathfrak{f}(x) \notin B$. T is the product space $\prod_\alpha I_\alpha$ of a family (I_α) of intervals $I_\alpha = [0, 1]$. Since B is closed, there exists a basic neighborhood $\prod_\alpha V_\alpha$ of $\mathfrak{f}(x)$ which does not intersect B. All the V_α are equal to I_α, except a finite number of them $\{V_{\alpha_1}, \cdots, V_{\alpha_n}\}$, by definition of the product space topology. Let π denote the projection of T onto the partial product $P = V_{\alpha_1} \times \cdots \times V_{\alpha_n}$. By construction $\pi(\mathfrak{f}(x))$ does not belong to $\pi(B)$. Because $\pi(B)$ is compact, and hence closed in P, the Euclidean distance (in P) from $\pi(\mathfrak{f}(x))$ to $\pi(B)$ is not zero. We define now for every $y \in S$ the following continuous bounded real-valued function: $g(y) = \text{distance}_{\text{in } P}(\pi(\mathfrak{f}(y), \pi(B))$; g is zero on A, different from zero at x. Hence S is completely regular.

Conversely let S be completely regular. Given a closed set A in S, we consider, for every $x \in S - A$, a separating real-valued continuous function $\mathfrak{f}_x$: $\mathfrak{f}_x(A) = \{0\}$, $\mathfrak{f}_x(x) = 1$.

Consider also a family I_x of unit intervals, indexed by the set $S - A$. Let T be the product space $\prod_{x \in S-A} I_x$. The mapping $\mathfrak{f}$ of S into T defined by $\mathfrak{f}(y) = (\mathfrak{f}_x(y))$ is continuous, and A is the inverse image under $\mathfrak{f}$ of the point of T with all coordinates equal to zero. Hence the equivalence is proved.

10. **Characterizations of locally compact spaces and of normal sequentially**

compact spaces. One can also ask the question "Do there exist uniformizable spaces other than the compact ones which are capable of only one uniform structure?" The answer is yes [D]. The examples given by Dieudonné issue from the transfinite line; they are locally compact and sequentially compact([17]). We shall see that local compactness is always a necessary condition, and sequential compactness also if one deals with normal spaces([18]).

Locally compact spaces. A locally compact space S, as a subspace of its Alexandroff compactification $\overline{S}$, is uniformizable. We denote by $\mathfrak{A}$ and call the *Alexandroff structure* of S the uniform structure induced on S by the unique one of $\overline{S}$. $\mathfrak{A}$ is precompact, and can clearly be defined by the graphs of the finite open coverings $\{A_0, A_1, \cdots, A_n\}$ of S where A_0 is the complement of a compact set and the other A_i have compact closures in S.

The locally compact spaces are characterized by the following property:

THEOREM XIV. *The existence of a coarsest uniform structure $\mathfrak{A}$ in the set of all uniform structures compatible with the topology of a uniformizable (non-compact) space S characterizes locally compact spaces. $\mathfrak{A}$ is then the Alexandroff structure of S.*

(1) Let S be locally compact, and $\mathfrak{A}$ its Alexandroff structure. Let $\mathfrak{S}$ be any uniform structure compatible with the topology of S. We shall prove that $\mathfrak{S}$ is finer than $\mathfrak{A}$. Since $\mathfrak{S}\supset\mathfrak{S}^*$, it is sufficient to prove it for a precompact $\mathfrak{S}$. Let $S^\frown$ be the (compact) completion of S with respect to $\mathfrak{S}$. Every point $x\in S$ has a compact neighborhood V_x in S; hence $V_x^\frown$ is the same as V_x and, by the completion theorem, $V_x^\frown$ is a neighborhood of x in $S^\frown$. Consequently S, being a neighborhood of all its points in S, is open in $S^\frown$. We identify in $S^\frown$ all the points of $S^\frown - S$; since the saturated set of any closed set of $S^\frown$ is closed, the identification space F is compact.

Let ϕ be the canonical mapping of $S^\frown$ onto F; ϕ clearly maps S homeomorphically onto $\phi(S)$. Hence F is the Alexandroff compactification of S (since the compactification of a locally compact space by adjunction of one point is unique). The continuous mapping ϕ of the compact space $S^\frown$ onto F being uniformly continuous, its restriction to S is also. But this restriction is the identity mapping of the uniform space $(S, \mathfrak{S})$ onto the uniform space $(S, \mathfrak{A})$. That means $\mathfrak{S}\supset\mathfrak{A}$. Q.E.D.

(2) We now suppose that the uniformizable space S is capable of a coarsest uniform structure $\mathfrak{B}$. Since $\mathfrak{B}\supset\mathfrak{B}^*$, $\mathfrak{B}=\mathfrak{B}^*$ and $\mathfrak{B}$ is precompact. Let $S^\frown$ be the (compact) completion of S with respect to $\mathfrak{B}$. If $S^\frown - S$ contains at least two points a and b, we consider the identification space E of $S^\frown$ obtained in

([17]) For the definition of this term, see below.

([18]) These conditions are not sufficient: if E denotes the space of all countable ordinals with the order topology, both E and $E\times E$ are locally compact and sequentially compact. E is capable of only one uniform structure, but not $E\times E$ which can be compactified in at least two different ways (the Alexandroff process, and by $E_0\times E_0$, E_0 being the Alexandroff compactification of E).

identifying a and b. The saturated set of any closed set of $S^\frown$ being closed, E is compact. Let ϕ be the canonical mapping of $S^\frown$ onto E; ϕ maps S homeomorphically onto $\phi(S)$. Let $\mathfrak{S}$ be the uniform structure of S deduced from the one of $\phi(S)$ induced on $\phi(S)$ by the unique uniform structure of E. $\mathfrak{S}$ and $\mathfrak{B}$ are distinct since the completions $S^\frown$ and E are distinct. On the other hand ϕ is uniformly continuous, as continuous mapping from a compact space. Hence the restriction of ϕ to S is also uniformly continuous, and that means that $\mathfrak{B}$ is finer than $\mathfrak{S}$. Since $\mathfrak{B}\neq\mathfrak{S}$, we get a contradiction. Consequently $S^\frown - S$ contains only one point (it is not empty, since we have excluded the trivial case where S is compact). Now S, as an open set of the compact space $S^\frown$, is locally compact. And $\mathfrak{B}$ is clearly its Alexandroff compactification. Theorem XIII is thus proved.

Remark. Another proof of the first part is the following. Let $\mathfrak{A}$ be the Alexandroff structure of the locally compact space S, $\mathfrak{S}$ any uniform structure of S compatible with its topology. We are given a basic surrounding $U\in\mathfrak{A}$ defined by the open covering $\{A_0A_1 \cdots A_n\}$, A_i having compact closure for $i\neq 0$. Let B be the complement of A_0; $B\subset\bigcup_{i=1}^n A_i$, hence B is compact, and contained in the open set $\bigcup_{i=1}^n A_i = A$.

For every $x\in B$, there exists a surrounding $V_x\in\mathfrak{S}$ such that $V_x(x)\subset A$.

Let W_x be a symmetric surrounding of $\mathfrak{S}$ such that $W_x^2\subset V_x$. For any $y\in W_x(x)$, $W_x(y)\subset W_x^2(x)\subset V_x(x)\subset A$.

One can cover B by a finite number of neighborhoods $W_x(x)$, say $W_{x_1}(x_1), \cdots, W_{x_q}(x_q)$. Let W' be the surrounding $\bigcap_{j=1}^q W_{x_j}\in\mathfrak{S}$; clearly $W'(a)\subset A$ for every $a\in B$; in other terms $W'(B)\subset A$. Because it is a symmetric surrounding, $W'\subset(A\times A)\cup(A_0\times A_0)$.

Now let us operate in the compact set B. The trace on $B\times B$ of the surrounding $\mathfrak{U}\in\mathfrak{A}$ is a surrounding of the unique uniform structure of B, and is therefore also induced on B by a surrounding W'' of the uniform structure $\mathfrak{S}$. The surrounding $W = W''\cap W'$ of $\mathfrak{S}$ is clearly contained in U. This proves again that $\mathfrak{S}$ is finer than $\mathfrak{A}$.

Sequentially compact spaces. We call *sequentially compact* a Hausdorff space in which every sequence has a cluster point. By reasoning entirely similar to that establishing the equivalence of the diverse forms of the compactness axiom, one proves that, for a Hausdorff space, sequential compactness is equivalent to each of the following:

(1) Every filter with countable base has a cluster point.

(2) Every decreasing sequence of closed sets has a nonempty intersection.

(3) From every countable open covering one can extract a finite covering.

(4) Every closed discrete subspace is finite.

One proves readily that:

(1) Every closed subset of a sequentially compact space is sequentially compact (but not conversely).

(2) Any separable, or any metrizable, sequentially compact space is compact.

Examples of nonregular sequentially compact spaces are known[19].

THEOREM XV. *All the uniform structures compatible with the topology of a uniformizable sequentially compact space are precompact. Conversely if all the uniform structures compatible with the topology of a normal space S are precompact, S is sequentially compact.*

(1) Let S be sequentially compact, and $\mathfrak{S}$ a uniform structure of S. If $\mathfrak{S}$ were not precompact, there would exist a surrounding $V\in\mathfrak{S}$ such that no finite union of sets small of order V is the entire space S. Let $W\in\mathfrak{S}$ be symmetric and such that $W^2\subset V$. We form the following sequence of points: x_1 arbitrary, x_2 in the complement of $W(x_1), \cdots, x_{n+1}$ in the complement of $\bigcup_{j=1}^{n} W(x_j)$ (this complement is nonempty, since every $W(x_j)$ is small of order V). Since for $p\neq q$, $(x_p, x_q)\notin W$, the sequence $\{x_n\}$ cannot have any cluster point in S; a contradiction. Hence $\mathfrak{S}$ is precompact.

(2) Let S be normal but nonsequentially compact. Then there exists a closed, countable, discrete subspace $A=\{a_n\}$. On A we define a real-valued function $\mathfrak{f}$ by $\mathfrak{f}(a_n)=n$. Because A is discrete, $\mathfrak{f}$ is continuous. By Tietze's extension theorem we can extend $\mathfrak{f}$ to a continuous real-valued function $\mathfrak{g}$ defined on S. $\mathfrak{g}$ is uniformly continuous for the universal uniform structure $\mathfrak{U}$ of S. Hence so is its restriction $\mathfrak{f}$ for the structure induced on A. But the condition $|\mathfrak{f}(a_p)-\mathfrak{f}(a_q)|<1/2$ can only be fulfilled for $a_r=a_q$. Hence the diagonal of $A\times A$ is a surrounding of the induced structure. This one is consequently not precompact (A is an infinite set). Hence $\mathfrak{S}$ cannot be precompact. Theorem XIV is proved.

Remark. If one wants the uniformizable space S to be capable of only one uniform structure $\mathfrak{S}$, Theorem XIII shows that S must be locally compact. If one requires also normality, the fact that $\mathfrak{S}=\mathfrak{S}^*$ is precompact shows, by Theorem XIV, that S has to be sequentially compact.

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