CONCERNING UPPER SEMI-CONTINUOUS COLLECTIONS OF CONTINUA(1)

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Hurewicz [1](2) and Mazurkiewicz [2] showed independently that if M is any compact metric continuum, there exist a one-dimensional continuum K in three-dimensional Euclidean space and an upper semi-continuous collection [3] of mutually exclusive continua filling up K which with respect to its elements as points is topologically equivalent to M. In the case of each of these solutions the method of proof used does not lend itself readily to the solution of the principal result of this paper which is the demonstration that if M is any compact continuous curve, there exist a one-dimensional continuous curve K in three-dimensional Euclidean space and an upper semi-continuous collection of mutually exclusive continua filling up K which with respect to its elements as points is topologically equivalent to M. It is known that, in the problem of Hurewicz and Mazurkiewicz, if M is not a continuous curve then K cannot be a continuous curve.

The principal result in this paper, Theorem II, was proposed to me as a problem by Professor R. L. Moore. I wish to express my sincere appreciation to Professor Moore for his patient and stimulating teaching and for his contagious enthusiasm for mathematical research.

DEFINITION. A collection Q of continuous curves will be said to have the X property if the common part of the continua of any subcollection of Q has only a finite number of components, each a continuous curve.

THEOREM I. *If M is any compact metric continuous curve, there exists a sequence $G_1, G_2, G_3, \cdots$ such that for each i,* (1) *G_i is a finite collection of continuous curves covering M,* (2) *$G_1+G_2+\cdots+G_i$ has the X property, and* (3) *each continuum of G_i is of diameter less than $1/i$.*

In order to prove this theorem, four lemmas will be used. The following notation will be adopted: if Y is a finite collection of continua, $N(Y)$ will be the number of elements of Y; if z is a continuum, $D(z)$ will be the diameter of z.

LEMMA 1. *If the collection $y_1, y_2, \cdots, y_k, y$ has the X property and the collection $y_1, y_2, \cdots, y_k, x$ has the X property and x and y have a point in common then the collection $y_1, y_2, \cdots, y_k, y+x$ has the X property.*

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(2) Numbers in brackets refer to the bibliography at the end of the paper.

Suppose the contrary. Then the common part of the continuous curves of some subcollection $y_{n_1}, y_{n_2}, \cdots, y_{n_i}, y+x$ is not the sum of a finite number of continuous curves. But $y_{n_1} \cdot y_{n_2} \cdot \cdots \cdot y_{n_i}$ is the sum of continuous curves $m_1, m_2, \cdots, m_e$, and $m_j \cdot (y+x)$ is the sum of a finite number of continuous curves as $m_j \cdot y$ and $m_j \cdot x$ each are, and every point in $m_j \cdot (y+x)$ is in $m_j \cdot y$ or $m_j \cdot x$.

LEMMA 2. *If the finite collection Y of continuous curves has the X property, and z, a subcontinuous curve of a continuum N of Y, is of diameter less than ϵ, then z is a subset of a subcontinuous curve z' of N of diameter less than ϵ, such that the collection whose elements are z' and the elements of Y has the X property.*

Suppose the contrary. Then there exists a set Y, z, and ϵ as in the hypothesis of the lemma for which the conclusion does not hold such that if Y', z', and ϵ' is any set as in the hypothesis for which the conclusion does not hold, $N(Y) \leqq N(Y')$.

Let the continua of Y be $y_1, y_2, \cdots, y_k$ $[k=N(Y)]$ with y_k corresponding to N of the hypothesis. Therefore $z<y_k$. There exists a continuous curve z'' such that $z''>z$, z'' is of diameter less than ϵ, $z''<y_k$, and the collection $y_2, y_3, \cdots, y_k, z''$ has the X property. The contradiction is immediate if y_1 and z'' have no point in common. Suppose they do have a point in common. For each point P of $y_1 \cdot z''$ let R containing P be a subset of and a domain with respect to $y_1 \cdot y_k$ such that $\overline{R}$ is a continuum of diameter less than $\delta = [\epsilon - D(z'')]/2$. For each such R there exists a continuous curve K_R of diameter less than δ containing $\overline{R}$ and lying in $y_1 \cdot y_k$ such that the collection $K_R, y_2, y_3, \cdots, y_{k-1}, C_R$ has the X property where C_R is the component of $y_1 \cdot y_k$ containing $\overline{R}$. But some finite collection α of these continua K_R covers $y_1 \cdot z''$. The collection $y_1, y_2 \cdots, y_k, z''+\alpha^*$ has the X property and $z''+\alpha^*$ is a continuum z' yielding the desired contradiction.

LEMMA 3. *If M is a compact continuous curve, Y is a finite collection of subcontinuous curves of M having the X property, and z is a subcontinuous curve of M of diameter less than ϵ, then there exists a subcontinuous curve z' of M of diameter less than ϵ, containing z, such that the collection whose elements are z' and the elements of Y has the X property.*

This lemma follows from Lemma 2 by adding M to the collection Y making z a subset of an element of the new collection.

LEMMA 4. *If the finite collection Y of subcontinuous curves of the compact continuous curve M has the X property, and Z is a finite collection of subcontinuous curves, $z_1, z_2, \cdots, z_k$ of M each of diameter less than ϵ, then there exists a collection Z' of subcontinuous curves $z_1', z_2', \cdots, z_k'$ of M such that for each i, $z_i' > z_i$ and is of diameter less than ϵ, and the collection $Y+Z'$ has the X property.*

Lemma 4 may be easily established with the aid of Lemma 3.

Theorem I follows as a consequence of Lemma 4.

THEOREM II. *If M is a compact metric continuous curve, there exists in three-dimensional Euclidean space a one-dimensional continuous curve K which is filled up by an upper semi-continuous collection G of mutually exclusive continua which with respect to its elements as points is topologically equivalent to M.*

In order to prove this theorem, there will first be described a particular upper semi-continuous collection F of mutually exclusive closed point sets filling up a totally disconnected closed subset t' of an interval t such that with respect to its elements as points F is topologically equivalent to M. In order to define the collection F it is necessary and sufficient to define a continuous transformation τ of t' into M.

Let $G_1, G_2, G_3, \cdots$ be a sequence as in Theorem I. Let $g_1, g_2, \cdots, g_s$ be a finite ordering U_1 of continua of G_1 such that for each i $(i=1, \cdots, s-1)$, g_i has a point in common with g_{i+1}, and such that each continuum of G_1 occurs at least once in this ordering. Let t be a straight line interval whose end points are a and b. Let $t_1, t_2, \cdots, t_s$ be s mutually exclusive subintervals of t of equal length such that t_1 contains a and t_s contains b and such that on t, t_i separates a from t_j if and only if $1<i<j$. Call the end points of the arc t_i, a_i and b_i in the order on t from a to b. Let $\tau(a_1)$ be a point P_1 of g_1, let $\tau(b_s)$ be a point P_{s+1} of g_s, and let $\tau(a_i)$ and $\tau(b_{i-1})$ $(i=2, 3, \cdots, s)$ be the same point P_i of $g_i \cdot g_{i-1}$. The transformation τ will be such that $\tau(t_i \cdot t')$ will be g_i. Call the initial decomposition of M into $g_1, g_2, \cdots, g_s$ and of t into $t_1, t_2, \cdots, t_s$ a decomposition D_1. With D_1 associate a finite set γ_1, of points of M, one in each component of $g_{j_1} \cdot g_{j_1'}$ for all possible values of j_1 and j_1' with $j_1 \neq j_1'$ for which the common part exists.

For each continuum g_i there exists a finite ordering U_{1i} of continuous curves $g_{i1}, g_{i2}, \cdots, g_{is_i}$ whose sum is g_i such that, for each j $(j=1, 2, \cdots, s_i-1)$, g_{ij} has a point in common with g_{ij+1}, for each j, g_{ij} is a component of the common part of a continuum of G_2 and g_i and is therefore of diameter less than $1/2$, and such that g_{i1} contains P_i and g_{is_i} contains P_{i+1}. There exists a set $t_{i1}, t_{i2}, \cdots, t_{is_i}$ of mutually exclusive subintervals of t_i of equal length such that t_{i1} contains a_i, t_{is_i} contains b_i, and t_{ij} separates a_i from t_{ik} on t_i if and only if $1<j<k$. Call the end points of t_{ij}, a_{ij} and b_{ij} in the order on t_i from a_i to b_i. Let $\tau(a_{ij})$ and $\tau(b_{ij-1})$ be the same point of $g_{ij} \cdot g_{ij-1}$. We now have a second decomposition of M and t by means of an initial decomposition of each g_i and t_i. Call this second decomposition D_2 and associate with D_2 a finite set γ_2 of points of M, one in each component of $g_{j_1k_1} \cdot g_{j_1k_2}$ for all possible values of the subscripts with $k_1 \neq k_2$ for which the common part exists. With D_2 also associate a finite set δ_1 of points of M, one in each component of $g_{j_1k_1} \cdot g_{j_1k_2} \cdot g_{j_1'k_3}$ for all possible values of the subscripts with $k_1 \neq k_2$, $j_1 \neq j_1'$, for which the common part exists.

Similarly a decomposition D_3 of M and t may be defined by a decomposition D_1 on D_2 using G_3 as G_2 was used for D_2. With D_3 associate a finite set γ_3 of points of M, one in each component of $g_{j_1j_2k_1}\cdot g_{j_1j_2k_2}$ for all possible values of the subscripts with $k_1\neq k_2$ for which the common part exists. With D_3 also associate a finite set δ_2 of points of M, one in each component of $g_{j_1j_2k_1}\cdot g_{j_1j_2k_2}\cdot g_{j_1'j_2'k_3}$ for all possible values of the subscripts with $k_1\neq k_2$ and either $j_1\neq j_1'$ or $j_2\neq j_2'$ for which the common part exists.

A decomposition D_1 on D_3 defines a decomposition D_4 and in this manner a sequence of decompositions may be defined, $D_1, D_2, D_3, \cdots$, wherein with each D_i two finite point sets γ_i and δ_{i-1} of M are associated. The set γ_i consists of a point of each component of $g_{j_1j_2\cdots j_{i-1}k_1}\cdot g_{j_1j_2\cdots j_{i-1}k_2}$ for all possible values of the subscripts with $k_1\neq k_2$ for which the common part exists. The set δ_{i-1} consists of a point of each component of $g_{j_1j_2\cdots j_{i-1}k_1}\cdot g_{j_1j_2\cdots j_{i-1}k_2}\cdot g_{j_1'j_2'\cdots j_{i-1}'k_3}$ for all possible values of the subscripts with $k_1\neq k_2$ and at least one $j_e\neq j_e'$ for which the common part exists.

By means of the procedure outlined above, a continuous transformation τ of a closed totally disconnected subset t' of t into M is defined and there exists, defined by τ, an upper semi-continuous collection F of mutually exclusive closed point sets filling up t' such that, with respect to its elements as points, F is topologically equivalent to M. If s is any component of $t-t'$, the end points of s belong to the same element of F.

Let the interval t be thrown into the interval $0\leqq x\leqq 1$, $y=0$, $z=0$ under a reversibly continuous transformation T carrying a into $(0, 0, 0)$ and let t be thrown into the interval $x=0$, $0\leqq y\leqq 1$, $z=0$ under a reversibly continuous transformation S carrying a into $(0, 0, 0)$ such that if P is any point of t, $T(P)$ and $S(P)$ are the same distance from $(0, 0, 0)$. Let X' be the set of x coordinates of the image of t' under T and Y' be the set of y coordinates of the image of t' under S. Let K' be a continuum in three-dimensional Euclidean space consisting of the following points: the set μ of points (x, y, z) such that x belongs to X', $0\leqq y\leqq 1$, $z=0$; the set ν of points (x, y, z) such that y belongs to Y', $0\leqq x\leqq 1$, $z=1$; and the set ω of all points P lying between the planes $z=0$ and $z=1$ such that for some element Q of F the projection of P onto the x-axis is in $T(Q)$ and the projection of P onto the y-axis is in $S(Q)$.

K' is not a continuous curve. There exists an upper semi-continuous collection E' of continua filling up K' such that with respect to its elements as points E' is topologically equivalent to M. Let G' be the particular collection defined as follows: a subcontinuum of K' belongs to G' if and only if for some element Q of F it consists of (1) all points μ_x in the set μ whose projections x on the x-axis satisfy the property that $T^{-1}(x)$ belongs to Q, (2) all points ν_y in the set ν whose projections y on the y-axis satisfy the property that $S^{-1}(y)$ belongs to Q, and (3) all points in K' in some vertical line containing a point of μ_x and a point of ν_y.

To the continuum K' will be added certain horizontal straight line seg ments parallel either to the y-axis or to the x-axis so that a new continuum K will be defined. K will be a continuous curve.

In the set ϕ of all the segments to be added to K' there will not be infinitely many segments all of diameter greater than the same positive number.

Let θ_1' denote a countable set of segments in the plane $z=0$ parallel to the x-axis each having only its end points in K', such that $\theta_1'^*$ plus the common part of K' and the plane $z=0$ is a continuous curve. Let θ_2' denote a similar set of segments in the plane $z=1$ parallel to the y-axis.

Let H be the common part of K' and the plane $z=1/3$. Let V denote the set of all points P of H such that for some point P' of t', $T(P')$ and $S(P')$ are the projections of P onto the x-axis and the y-axis respectively. Consider the point sets $\delta_1, \delta_2, \delta_3, \cdots$. If P is any point of δ_{i-1} there exists a finite collection W_P such that, in order that w should belong to W_P, it is necessary and sufficient that w should be a set of three arcs $t_{j_1j_2\cdots j_{i-1}k_1}$, $t_{j_1j_2\cdots j_{i-1}k_2}$, and $t_{j_1'j_2'\cdots j_{i-1}'k_3}$ with $k_1\neq k_2$ and for some e, $j_e\neq j_e'$ such that P is a point of a component of $g_{j_1j_2\cdots j_{i-1}k_1}\cdot g_{j_1j_2\cdots j_{i-1}k_2}\cdot g_{j_1'j_2'\cdots j_{i-1}'k_3}$.

For each i, each point P of δ_{i-1}, and each element of the collection W_P, consider two points of H, one whose projections onto the x-axis and the y-axis are points of the images, under T and S respectively, of $\tau^{-1}(P)$ belonging to $t_{j_1j_2\cdots j_{i-1}k_1}$ and $t_{j_1'j_2'\cdots j_{i-1}'k_3}$ respectively, and another with the same projection onto the y-axis whose projection onto the x-axis belongs to $T(t_{j_1j_2\cdots j_{i-1}k_2})$.

Let s_{Pw}' denote the straight line segment joining these two points. Let s_{Pw}'' denote the straight line segment obtained similarly by permuting y and S with x and T respectively in the above expressions. Let R' be the countable set of all such segments s_{Pw}' and s_{Pw}''.

Consider the point sets $\gamma_1, \gamma_2, \gamma_3, \cdots$. If Q is any point of γ_i there exists a finite collection W_Q' such that in order that w' should belong to W_Q', it is necessary and sufficient that w' should be a set of two arcs $t_{j_1j_2\cdots j_{i-1}k_1}$ and $t_{j_1j_2\cdots j_{i-1}k_2}$ with $k_1\neq k_2$ such that Q is a point of a component of $g_{j_1j_2\cdots j_{i-1}k_1}\cdot g_{j_1j_2\cdots j_{i-1}k_2}$. For each i, each point Q of γ_i, and each element of the collection W_Q', consider a point of H whose projections onto the x-axis and the y-axis are points of the images, under T and S respectively, of $\tau^{-1}(Q)$ belonging to $t_{j_1j_2\cdots j_{i-1}k_1}$ and $t_{j_1j_2\cdots j_{i-1}k_2}$ respectively. Let $s_{Qw'}'$ and $s_{Qw'}''$ be straight line segments joining such a point with the points of V having either an x or a y coordinate in common with it. Let $r_{Qw'}'$ and $r_{Qw'}''$ denote the two straight line segments obtained similarly by permuting y and S with x and T respectively in the above expressions. Let R'' be the countable set of all such segments $s_{Qw'}'$, $s_{Qw'}''$, $r_{Qw'}'$, and $r_{Qw'}''$. Denote the elements of $R'+R''$ by $R_1, R_2, R_3, \cdots$. For each i, add to the set K' all segments in the planes $z=[2j-1]/2^i$ for $j=1, 2, \cdots, 2^i-1$, whose projections onto the plane $z=0$ coincide with that of R_i. Let θ denote the set of all such segments. The set ϕ is $\theta+\theta_1'+\theta_2'$.

Let K be $K'+\phi^*$. The continuum K is a continuous curve and there exists an upper semi-continuous collection G of mutually exclusive continua filling up K which with respect to its elements as points is topologically equivalent to M. Each of the continua of G is the sum of one continuum of G' and those segments of ϕ which have both end points in it.

It is clear that K is connected im kleinen at every point of K in the plane $z=0$ or the plane $z=1$, and at every point of the set $\phi^*-K'\cdot\phi^*$. It suffices to show that K is connected im kleinen at every point of K' for which $0<z<1$.

Let $\alpha_1, \alpha_2, \alpha_3, \cdots$ be a sequence of collections of parallelepipeds such that for each i, the collection α_i consists of all rectangular parallelepipeds with one face parallel to the plane $y=0$ and with bases in the planes $z=0$ and $z=1$ whose projection X on the x-axis and projection Y on the y-axis have the property that $T^{-1}(X)$ is some $t_{j_1j_2\cdots j_i}$ and $S^{-1}(Y)$ is some $t_{j_1'j_2'\cdots j_i'}$ for which $g_{j_1j_2\cdots j_i}\cdot g_{j_1'j_2'\cdots j_i'}$ exists.

Let $\omega_1, \omega_2, \omega_3, \cdots$ be a sequence of collections of point sets such that in order that a point set should belong to ω_i it is necessary and sufficient that it should consist of all points on or in the interior of some parallelepiped of α_i. Let $\beta_1, \beta_2, \beta_3, \cdots$ be a sequence of point sets such that for each i, β_i is ω_i^*.

Let β be the closure of that subset of the continuum K' for which $0<z<1$. The point set β is the common part of the sequence of point sets $\beta_1, \beta_2, \beta_3, \cdots$ and therefore $\beta+\theta^*$ is the common part of the sequence of point sets $\beta_1+\theta^*, \beta_2+\theta^*, \cdots$. To show that K is a continuous curve it is sufficient to show that $\beta+\theta^*$ is a continuum, for if so the common part of any element A of ω_i and $\beta+\theta^*$ is the sum of a finite number of continua, as each component of $A\cdot(\beta+\theta^*)$ must contain either a point or an end point of a segment of θ which also contains a point not in A and there exist only a finite number of such segments in θ.

To show that $\beta+\theta^*$ is a continuum it is sufficient to show that each $\beta_i+\theta^*$ is a continuum as the common part of a monotonic sequence of compact continua is a continuum. Because for each i, the ordering U_i of continua of M of the decomposition D_i is a chain, that is, each element intersects the one which follows it in the ordering, the closure of the sum of all elements of ω_i which contain points of V plus those elements of θ defined by use of $\gamma_1, \gamma_2, \gamma_3, \cdots$ is a continuum V_i.

To show that $\beta_i+\theta^*$ is a continuum it is sufficient to show that each element A of ω_i is in the same component of $\beta_i+\theta^*$ as V_i. Let $A(t_{j_1j_2\cdots j_i}, t_{j_1'j_2'\cdots j_i'})$ be that element of ω_i whose projection on the x-axis is $T(t_{j_1j_2\cdots j_i})$ and whose projection on the y axis is $S(t_{j_1'j_2'\cdots j_i'})$. Let C be a component of $g_{j_1j_2\cdots j_i}\cdot g_{j_1'j_2'\cdots j_i'}$. Consider the case for $j_1\neq j_1'$. (If $j_1=j_1'$, the desired result may be obtained by substituting $g_{j_1j_2\cdots j_e}\cdot g_{j_1'j_2'\cdots j_e'}$, with $j_k=j_k'$, $k<e$, $j_e\neq j_e'$, for the set $g_{j_1}\cdot g_{j_1'}$ in what follows.) C is a subset of a component C' of $g_{j_1}\cdot g_{j_1'}$. There exist a point P of C' in γ_1 and a segment of θ from V_i to a point P' of $A(t_{j_1}, t_{j_1'})$ whose image under G is P. P' is in some $A(t_{j_1l_2}, t_{j_1'l_2'})$. Let $c_1, c_2, \cdots, c_k$ be a chain of components of the common parts of C' and elements of the de-

composition D_2 of g_{j_1} such that c_1 is the component of $g_{j_1l_2}\cdot C'$ containing P, and c_k is the component of $g_{j_1j_2}\cdot C'$ containing C. Let $g_{j_1l_3}$ be an element of D_2 of g_{j_1} containing c_2. Let P_1 be a point of $c_1\cdot c_2$ such that there is an element $g_{j_1'm_2'}$ of D_2 of g_{j_1}' containing P_1 for which there is a segment of θ from a point P_1' of $A(t_{j_1l_2}, t_{j_1'm_2'})$ to a point P_1'' of $A(t_{j_1l_3}, t_{j_1'm_2'})$ such that the image of P_1' under G is P_1.

There exists a chain $d_{11}, d_{12}, \cdots, d_{1h_1}$ of components of the common part of c_1 and elements of the decomposition D_2 of g_{j_1}' such that d_{11} containing P is in $g_{j_1'l_2'}$ and d_{1h_1} containing P_1 is in $g_{j_1'm_2'}$. But then there exist points $P_{11}, P_{12}, \cdots, P_{1h_1-1}$ such that P_{1i} is in $d_{1i}\cdot d_{1i+1}$ with d_{1i} in $g_{j_1'l_3'}$ and d_{1i+1} in $g_{j_1'l_4'}$ and there is a segment of θ from a point P_{1i}' whose image under G is P_{1i} of $A(t_{j_1l_2}, t_{j_1'l_3'})$ to a point of $A(t_{j_1l_2}, t_{j_1'l_4'})$. Then $A(t_{j_1l_3}, t_{j_1'm_2'})$ is in the same component of $\beta_2+\theta^*$ as V_2. But by successive reapplications of the above argument, if we use in the next stage P_1 and $t_{j_1l_3}$ as P and $t_{j_1l_2}$ were used, and work along the chain $c_1, c_2, \cdots, c_k$, it follows that $A(t_{j_1j_2}, t_{j_1'j_2'})$ must be in the same component of $\beta_2+\theta^*$ as V_2 for it may be specified that d_{kh_k} contains C and is a subset of $g_{j_1'j_2'}$. By reapplications of the above, if we use P and P_{11} as P and C were used, then P_{11} and P_{12} as P and C were used, $\cdots$, P_{1h_1-1} and P_1 as P and C were used, $\cdots$, it follows that $A(t_{j_1j_2j_3}, t_{j_1'j_2'j_3'})$ must be in the same component of $\beta_3+\theta^*$ as V_3, and finally $A(t_{j_1j_2\cdots j_i}, t_{j_1'j_2'\cdots j_i'})$ must be in the same component of $\beta_i+\theta^*$ as V_i, as was to be shown.

The following example indicates the nature of some of the complexity of the problem of Theorem II. Let H' be a totally disconnected closed subset of the interval $0\leqq x\leqq 1$, $y=0$ in the plane and let H be a continuum consisting of H' and the sum of all circles with centers on the x-axis such that each contains two points of H' and has no point of H' in its interior. Let τ be a continuous transformation of an interval t of end points A and B into H such that there exists a point P of t, not A or B, such that $\tau(AP)$ is a reversibly continuous transformation, carrying the arc AP into the subset of H for which $y\geqq 0$, and $\tau(PB)$ is a reversibly continuous transformation, carrying the arc PB into the subset of H for which $y\leqq 0$.

If a two-dimensional continuum K' and a collection G' are defined as in the argument for Theorem II based on τ, H, and t, there does not exist a continuous curve K in three-dimensional Euclidean space consisting of K' and a countable number of arcs such that each intersects only one element of G', each two which intersect, intersect the same element of G', and if $\epsilon>0$, only a finite number are of diameter greater than ϵ.

THEOREM III. *If M is a completely separable metric locally compact continuous curve, there exists a one-dimensional continuous curve K in three-dimensional Euclidean space and an upper semi-continuous collection G of mutually exclusive compact continua filling up K which with respect to its elements as points is topologically equivalent to M.*

Let $D_1, D_2, D_3, \cdots$ be a sequence δ of compact connected domains in M regarded as space such that, if P is any point of M and D is any domain containing P, there exists an i such that $\overline{D}_i$ is a subset of D. There exists a compact continuous curve C_1 containing D_1. There exists a finite set α_1 of domains of δ covering the boundary of C_1 such that α_1 includes D_2 and α_1^* is connected. For each domain D of α_1 let C be a compact continuous curve containing D such that the collection consisting of C_1 and the elements of the collection β_1 of all C's so obtained (one for each element of α_1) has the X property.

There exists a finite collection α_2 of elements of δ covering the boundary of $\beta_1^*+C_1$ such that $\overline{\alpha_2^*}$ has no point in common with C_1 and such that $C_1+\beta_1^*+\alpha_2^*$ is connected and covers $\sum_1^3 D_i$. For each element D of α_2 there exists a compact continuous curve C containing it and no point of C_1 such that if β_2 is the collection of all such continua (one for each element of α_2) then the collection consisting of C_1 and the elements of β_1 and β_2 has the X property.

There exists a finite collection α_3 of elements of δ covering the boundary of $C_1+\beta_1^*+\beta_2^*$ such that the closure of no element of α_3 has a point in common with $C_1+\beta_1^*$ and the set $C_1+\beta_1^*+\beta_2^*+\alpha_3^*$ is connected and covers $\sum_1^4 D_i$. For each element D of α_3 there exists a compact continuous curve C containing D and no point of $C_1+\beta_1^*$ such that if β_3 is the collection of all such continua (one for each element of α_3) then the collection consisting of C_1 and the elements of β_1, β_2, and β_3 has the X property.

In an analogous fashion, sets $\beta_4, \beta_5, \cdots$ can be defined such that the sequence $C_1, \beta_1, \beta_2, \beta_3, \cdots$ satisfies the conditions that for each i, the collection consisting of C_1 and the elements of $\beta_1+\beta_2+\cdots+\beta_i$ is finite and has the X property, for each $i>2$, β_i^* contains no point of $C_1+\sum_1^{i-2}\beta_j^*$, for each i, $C_1+\sum_1^i\beta_j^*$ is connected, $C_1+\sum_1^\infty\beta_j^*$ covers M, and no element of β_i is an element of β_{i+1}.

Let $C_1, C_2, C_3, \cdots$ be a sequence ω of compact continuous curves such that for each $i>1$, C_i is an element of some β_{k_i}, for each i, each element of β_i is a term of ω, if C_i precedes C_j then the β_{k_i} of which C_i is an element is identical with or precedes the β_{k_j} of which C_j is an element, and for each i, $\sum_1^i C_j$ is connected. Let B_i denote $\sum_1^i C_j$.

For each i, there exists a number N_i such that no continuum C_n for $n>N_i$ intersects any continuum C_j for $j\leqq i$. Then by Lemma 4 of Theorem I there exists a sequence $G_1, G_2, G_3, \cdots$ such that G_1 is the set of continua whose elements are the terms of ω, and for each e, (1) G_e is a countably infinite set of compact continuous curves covering M containing, for each k, only a finite set of continua containing points of B_k, (2) $G_1+G_2+G_3+\cdots+G_e$ has the X property, (3) each continuum of G_e for $e>1$ is of diameter less than $1/e$, and (4) each continuum of G_{e+1} is a subset of some continuum of G_e.

From the sequence $G_1, G_2, G_3, \cdots$ and by means of an argument analogous to that used for Theorem II, it may readily be shown that there exists a sequence K'_{B_1}, K'_{B_2}, K'_{B_3}, $\cdots$ and a sequence K_{B_1}, K_{B_2}, K_{B_3}, $\cdots$ such that (1) for each i, K'_{B_i} is a compact continuum in three-dimensional Euclidean space, K_{B_i} is a one-dimensional compact continuous curve in three-dimensional Euclidean space which contains K'_{B_i}, there exist an upper semi-continuous collection G_{B_i} of mutually exclusive continua filling up K_{B_i} and an upper semi-continuous collection G'_{B_i} of mutually exclusive continua filling up K'_{B_i} such that each collection with respect to its elements as points is topologically equivalent to B_i and the continua K_{B_i} and K'_{B_i} and collections G_{B_i} and G'_{B_i} are obtained by a method analagous to that used in the proof of Theorem II, (2) for each i, $K_{B_{i+1}}$ contains K_{B_i}, every element of G_{B_i} is a subset of an element of $G_{B_{i+1}}$, and no two elements of G_{B_i} are in the same element of $G_{B_{i+1}}$, (3) for each i, every point of K'_{B_i} in the plane $z=1/2$ whose projection on the x-axis lies in the interval $\langle 2j-1, 2j\rangle$ and whose projection on the y-axis lies in the interval $\langle 2j-1, 2j\rangle$ is in the inverse image under the transformation T_{B_i} defined by G'_{B_i} of C_j, (4) for each i, that part of K_{B_i} for which $x\leqq 2i-2$ and $y\leqq 2i-2$ is identical with $K_{B_{i-1}}$, and (5) for each i, no point of K_{B_i} in the plane $z=0$ has an x projection lying in a segment of the x-axis $\langle 2j, 2j+1\rangle$, $j=1, 2, 3, \cdots$, and no point of K_{B_i} in the plane $z=1$ a y projection lying in a segment of the y-axis $\langle 2j, 2j+1\rangle$, $j=1, 2, 3, \cdots$. This last condition is important with reference to the adding of segments in the plane $z=0$ for, in some cases, under the other conditions of this paragraph, the end points of a segment $\langle 2j, 2j+1\rangle$ in the x-axis cannot belong to the same element of G_{B_i}.

The continuous curve $K=K_{B_1}+K_{B_2}+K_{B_3}+\cdots$ contains a continuous curve satisfying the conditions of the theorem. For each integer i, there exists a number j_i such that there is no vertical segment in K either whose x coordinate is less than or equal to i and whose y coordinate is greater than j_i or whose y coordinate is less than or equal to i and whose x coordinate is greater than j_i. For each even integer i, delete from K every point in the plane $z=0$ whose x coordinate is less than or equal to i and whose y coordinate is greater than j_i, and in the plane $z=1$ delete every point whose y coordinate is less than or equal to i and whose x coordinate is greater than j_i.

The resulting continuum is a continuous curve and there does exist an upper semi-continuous collection of compact continua filling it up which with respect to its elements as points is topologically equivalent to M.

The following theorem can be proved by an argument similar to but much shorter than the argument for Theorem III.

THEOREM IV. *If M is a completely separable metric locally compact continuum, there exist a one-dimensional continuum K in three-dimensional Euclidean space and an upper semi-continuous collection G of mutually exclusive compact continua filling up K which with respect to its elements as points is topologically equivalent to M.*

This theorem can also be shown as a consequence of Theorem III.

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