ADJUNCTION OF SUBFIELD CLOSURES TO ORDERED DIVISION RINGS

BY

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1. **Preface.** Let Σ be an ordered division ring, P its prime field (i.e. the field of rationals), P^* the closure of P in its order topology; P^* is then order-isomorphic to the field of real numbers in their natural order. It has been shown by B. H. Neumann [1](1) that Σ can be extended to an ordered division ring $\Sigma(P^*)$ continuing the order of Σ and containing P^* in its centre.

With a few changes in Neumann's proof the following generalisation will be proved: Let F be an arbitrary subfield of the centre of Σ. Then Σ can be extended to an ordered division ring $\Sigma(F^*)$ continuing the order of Σ and containing the closure F^* of F with respect to its order-topology in its centre.

The new tool which is used throughout the paper is a mapping of Σ into a system consisting of the symbols $-\overline{\infty}$ and $+\overline{\infty}$ and an ordered residue class division ring $\overline{D}$ obtained by a peculiar type of fundamental sequence modulo the corresponding null-sequences.

2. **The mapping $\sigma \rightarrow \bar{\sigma}$.** Let Σ be an ordered division ring with centre Z, F a fixed subfield of Z, and Σ^+, F^+ the multiplicative groups of the positive elements of Σ, F respectively.

We can assume the order of the additive semigroup of F^+ to be nonarchimedean(2). Then the archimedean classes of the additive semigroups of Σ^+, F^+ form multiplicative groups $\mathfrak{a}(\Sigma)$, $\mathfrak{a}(F)$ when the product is defined by the product of representatives of the corresponding classes(3). The archimedean classes of the additive semigroups of Σ^+, F^+ will be called *additive archimedean classes* of Σ^+, F^+ respectively. The additive archimedean class of an element $\sigma \in \Sigma^+$ in $\mathfrak{a}(\Sigma)$ will be denoted by $[\sigma]$. Thus $[\sigma]$ consists of those $\tau \in \Sigma^+$ for which there exist positive integers s, t such that $|\sigma| < t\tau$, $\tau < s|\sigma|$.

Let F^* be the closure of F with respect to its order-topology; it can be constructed by means of fundamental sequences and null-sequences in the usual way. Thus F^* is an ordered field containing (an ordered subfield isomorphic to) F such that its additive group is the order-topological completion of the additive group of F. With this t-completion in the sense of Cohen-Goffman(4) a unique ordinal $\xi^* = \xi(F)$ is associated, and for the construction

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(1) Numbers in brackets refer to the bibliography at the end of the paper.

(2) For otherwise the closure F^* of F is order-isomorphic to the field of reals, and F^* can be embedded in Σ according to [1].

(3) Obviously this definition is independent of the choice of the representatives.

(4) Cf. [2].

of F^* it is sufficient to use fundamental well-ordered ξ^*-sequences.

Let $\mathfrak{R}$ be the subring of Σ consisting of all elements $\sigma \in \Sigma$ for which there exists a centre element $\zeta \in F^+$ with $|\sigma| < \zeta$, that is, for which the additive archimedean class $[\sigma]$ of $|\sigma|$ is not greater than all archimedean classes of F^+. We shall call the elements of $\mathfrak{R}$ *finite*(5) and the elements of $\Sigma - \mathfrak{R}$ *infinite*(5).

We shall now deal with certain sequences over $\mathfrak{R}$, similar to those leading to F^*, and start with the following definitions:

A well-ordered ξ^*-sequence $\{\sigma_\alpha\}$ of elements $\sigma_\alpha \in \mathfrak{R}$ is called an *F-fundamental sequence* (abbreviated *FF-sequence*) if for every $\epsilon \in F^+$ there exists an ordinal $\gamma(\epsilon) < \xi^*$ such that $|\sigma_\alpha - \sigma_\beta| < \epsilon$ for $\alpha, \beta > \gamma(\epsilon)$.

An FF-sequence $\{\sigma_\alpha\}$ such that for every $\epsilon \in F^+$ there exists an ordinal $\gamma(\epsilon)$ such that $|\sigma_\alpha| < \epsilon$ for $\alpha > \gamma(\epsilon)$ is called an *F-null-sequence* (abbreviated *FN-sequence*).

The FF-sequences form a ring $\overline{R}$ when addition, subtraction, and multiplication are defined in the usual way. The FN-sequences form a two-sided ideal $\overline{I}$ in $\overline{R}$. The residue class ring $\overline{R}/\overline{I}$ is a division ring $\overline{D}$(6).

Special FF-sequences are those where a fixed element $\rho \in \mathfrak{R}$ is repeated ξ^*-times: $\{\rho\}$, clearly their residue class mod $\overline{I}$ defines an element of $\overline{D}$, and we shall denote it by $\bar{\rho}$. Similarly we shall mark subsets of $\overline{D}$ which correspond to subsets of $\mathfrak{R}$ by means of the mapping $\rho \rightarrow \bar{\rho}$ by a bar "–".

The map $\overline{\mathfrak{R}}$ of $\mathfrak{R}$ in $\overline{D}$ is a division ring. The mapping $F \rightarrow \overline{F}$ is an isomorphism. $\overline{D}$ can be ordered so that the order of $\overline{F}$, induced by $F \cong \overline{F}$, is preserved. $\overline{D}$ contains the closure $\overline{F}^*$ (with respect to the the order-topology) of $\overline{F}$ as a subfield(6).

We shall assume from now on that $\overline{D}$ is ordered with preservation of the order in $\overline{F}$. We shall call the elements of $\mathfrak{R}$ mapped onto $\overline{0}$ *infinitely small*(5); they form a two-sided maximal prime ideal in $\mathfrak{R}$ which we shall denote by $\mathfrak{P}$.

The aim of our paper is to embed Σ in an ordered division ring Σ^* containing (a field order-isomorphic to) F^*. If the mapping $\mathfrak{R} \rightarrow \overline{\mathfrak{R}}$ is an isomorphism, then Σ^* can be taken to be $\overline{D}$. For if no elements $\sigma \in \Sigma^+$ exist with the property $\sigma < \zeta$ for all $\zeta \in F^+$, then also no elements $\tau \in \Sigma^+$ exist with $\tau > \zeta$ for all $\zeta \in F^+$. Therefore all elements of Σ are finite, $\overline{\mathfrak{R}}$ is order-isomorphic to Σ, and $\overline{D}$ contains $\overline{F}^*$ and $\overline{\mathfrak{R}}$. We shall exclude this trivial case in the following by the assumption $\mathfrak{R} \neq \Sigma$.

Then there exist infinitely small nonzero elements. The construction of $\overline{D}$ sets up a natural mapping of Σ into the system obtained from $\overline{D}$ by adding the two symbols $-\overline{\infty}, +\overline{\infty}$ with the usual conventions. The elements $\tau \in \Sigma - \mathfrak{R}$ are mapped onto $\pm\overline{\infty}$ according as $\tau \gtrless 0$.

Our mapping $\sigma \rightarrow \bar{\sigma}$ is a generalisation of the mapping $\sigma \rightarrow \widehat{\ }\sigma$ in [1] since

(5) This definition, of course, is dependent on the fixed field F, but for the sake of brevity we shall omit the qualification "with respect to F."

(6) The proof runs along the usual lines and is omitted. Cf., for instance, [3, chap. IX].

there $^\frown P^*$ can be constructed by means of fundamental sequences instead of by means of Dedekind sections.

If all elements of $\mathfrak{R}$ are mapped onto elements of $\overline{F}^*$, we are confronted with a situation similar to that in [1], and, in fact, the whole construction of Σ^* including the proofs can be taken over easily from [1].

But, in general, this need not be the case. For there may exist elements $\bar{\rho}\in\overline{\mathfrak{R}}$ such that $\bar{\sigma}\ll\bar{\rho}\ll\bar{1}$ holds for all $\bar{\sigma}\ll\bar{1}$, $\bar{\sigma}\in\overline{F}^*$. Then we shall denote the subring of all elements of $\mathfrak{R}$ mapped into $\overline{F}^*$ by $\mathfrak{S}$ and assume $\mathfrak{S}\neq\mathfrak{R}$. The map $\overline{\mathfrak{S}}=\overline{\mathfrak{R}}\cap\overline{F}^*$ is a field and consists of exactly the same additive archimedean classes as F. Finally we shall denote the subring of all elements of $\mathfrak{R}$ mapped onto the centre of $\overline{\mathfrak{R}}$ by $\mathfrak{Z}$. Then $\overline{\mathfrak{S}}\subseteq\overline{\mathfrak{Z}}$ holds.

Now (8.4) and (8.5) in [1] are no longer valid in our case. However, we can still take over the construction of Σ^* in [1] and most of the proofs there. We shall have to supply amendments only when (8.4) or (8.5) are applied in [1].

As in [1], we shall denote by P an arbitrary subfield of $Z\cap\mathfrak{R}$ and by $P_{\max}$ one of the maximal subfields of $Z\cap\mathfrak{R}$. We may assume $\overline{P}_{\max}\neq\overline{F}^*$, otherwise no extension would be required. It is then possible to adjoin to $P_{\max}$ an element $\bar{\theta}\in\overline{F}^*$. We shall have to show that the corresponding adjunction of an element θ to $P_{\max}$, and thus to Σ, leads again to a division ring $\Sigma(\theta)$ and that $\Sigma(\theta)$ can be ordered with preservation of the order in Σ. By the successive adjunctions to Σ of all these θ, according to the classical Steinitz procedure(7), the required extension Σ^* is obtained.

As in [1], in all occurring polynomial domains $\Sigma[x]$, $\mathfrak{R}[x]$, etc., the variable x will be assumed commutative with the coefficients. Finally we shall use the abbreviations:

$$\begin{aligned}
\sigma^\tau &= \tau^{-1}\sigma\tau, && \sigma,\tau\in\Sigma, && && \tau\neq 0,\\
\bar{\sigma}^\tau &= \{\tau^{-1}\sigma\tau\} \bmod \bar{I}, && \sigma\in\mathfrak{R}, && \tau\in\Sigma, && \tau\neq 0,\\
f^\tau(x) &= \tau^{-1}f(x)\tau, && f(x)\in\Sigma[x], && \tau\in\Sigma, && \tau\neq 0,\\
\bar{f}^\tau(x) &= \{\tau^{-1}f(x)\tau\} \bmod \bar{I}, && f(x)\in\mathfrak{R}[x], && \tau\in\Sigma, && \tau\neq 0.
\end{aligned}$$

$\mathfrak{R}$ as well as $\mathfrak{R}[x]$ are invariant under all $\tau\in\Sigma$, hence $\bar{\sigma}^\tau$ and $\bar{f}^\tau(x)$ are defined.

Then the following lemmas will help to replace [1, (8.4)] and [1, (8.5)] here:

2.1. LEMMA. *If $\bar{f}(\bar{x})\in\overline{\mathfrak{R}}[\bar{x}]$, $\bar{\theta}\in\overline{F}^*$, $\bar{f}(\bar{\theta})>\bar{0}$, then $\bar{f}^\tau(\bar{\theta})>\bar{0}$ for every nonvanishing $\tau\in\Sigma$.*

Proof. Let $f(x)\in\mathfrak{R}[x]$ be a fixed image of $\bar{f}(\bar{x})$ and $\{\theta_\alpha\}$, $\theta_\alpha\in F$, be a FF-sequence such that $\{\theta_\alpha\}$ mod $\bar{I}=\bar{\theta}$. From $\bar{f}(\bar{\theta})>\bar{0}$ it follows that there exist an $\epsilon\in F^+$ and an ordinal $\gamma(<\phi)$ such that $f(\theta_\alpha)>\epsilon>0$ for $\phi>\alpha>\gamma$(8). But then $f^\tau(\theta_\alpha)>\epsilon>0$ holds, too, for $\phi>\alpha>\gamma$, i.e., $\bar{f}^\tau(\bar{\theta})>\bar{0}$.

(7) Cf. [4].

(8) This can be shown in the usual way, cf., for instance, [3, chap. IX].

2.2. Corollary. *If* $\bar{f}(\bar{x}) \in \overline{\mathfrak{R}}[\bar{x}]$, $\bar{\theta} \in \bar{F}^*$, $\bar{f}(\bar{\theta}) = \bar{0}$, *then* $\bar{f}^\tau(\bar{\theta}) = \bar{0}$ *for every nonvanishing* $\tau \in \Sigma$.

2.3. Lemma. *If* $\bar{g}(\bar{x}) \in \overline{\mathfrak{R}}[\bar{x}]$ *is irreducible and monic*, $\bar{\theta} \in \bar{F}^*$, $\bar{g}(\bar{\theta}) = \bar{0}$, *then* $\bar{g}(\bar{x}) = \bar{g}^\tau(\bar{x})$ *for every nonvanishing* $\tau \in \Sigma$.

Proof. Suppose $\bar{h}(\bar{x}) = \bar{g}(\bar{x}) - \bar{g}^\tau(\bar{x}) \neq \bar{0}$ for a certain $\tau \in \Sigma$. Then $\bar{h}(\bar{\theta}) = \bar{0}$ holds by (2.2). But this contradicts the irreducibility of $\bar{g}(\bar{x})$ in $\overline{\mathfrak{R}}[\bar{x}]$ since the degree of $\bar{h}(\bar{x})$ is less than the degree of $\bar{g}(\bar{x})$.

2.4. Corollary. *If* $\bar{g}(\bar{x}) \in \overline{\mathfrak{R}}[\bar{x}]$ *is irreducible and monic*, $\bar{\theta} \in \bar{F}^*$, $\bar{g}(\bar{\theta}) = \bar{0}$, *then* $\bar{g}(\bar{x}) \in \overline{\mathfrak{Z}}[\bar{x}]$.

3. **Alterations in Neumann's proof.** Now the sections 9–17 in [1] can be taken over mutatis mutandis. Thus all accents have to be replaced by bars, and $\sigma \rightarrow \bar{\sigma}$ is a mapping into $\bar{D}$, whilst in [1], $\sigma \rightarrow \frown\sigma$ was a mapping into $\frown P^*$. Only the following alterations are necessary:

In section 9 whenever $g(x)$ occurs the additional assumption has to be made that it is monic and irreducible over $\mathfrak{R}[x]$ and that $\bar{g}(\bar{\theta}) = \bar{0}$ holds, which involves no loss of generality. In the proof of (9.5)(9), the lemma (8.5) was applied in order to show that $\frown h'' = \frown h$ holds. Here the corresponding fact $\bar{h}'' = \bar{h}^{\delta^{-1}} = \bar{h}$ follows from $\bar{f}(\bar{x}) = \bar{f}^{\delta^{-1}}(\bar{x})$ and (2.3).

In section 10 the absence of commutativity in our $\overline{\mathfrak{R}}$ provides no special difficulty since $\bar{g}(\bar{x})$, defined as in (10.41), lies in the centre $\overline{\mathfrak{Z}}[\bar{x}]$ of $\overline{\mathfrak{R}}[\bar{x}]$ because of (2.4). The original proof of (10.73) used (8.5), therefore it has to be altered as follows: $p > 0$ implies $\bar{q}(\bar{\theta}) > \bar{0}$ (according to (10.51)), hence $\bar{q}^\sigma(\bar{\theta}) > \bar{0}$ by (2.1). But $\bar{p}^\sigma(\bar{x}) = \bar{g}(\bar{x})^\lambda \bar{q}^\sigma(\bar{x})$ by (2.3), hence $p^\sigma > 0$.

In section 11, instead of (11.4) only a restricted lemma can be proved.

3.1. Lemma. *If* $\sigma \in \mathfrak{S}$, $\rho \in \mathfrak{R}$, *then* $A_\sigma \rho = o(\rho)$.

Proof. We have $\bar{\sigma} = \bar{\sigma}^\rho$, hence $\sigma^\rho - \sigma = o(1)$, or upon left-hand multiplication by ρ: $\sigma\rho - \rho\sigma = o(\rho)$.

Consequently in (11.51), (11.52), and (11.53), σ and τ must now be restricted to belong to $\mathfrak{S}$. Similarly in section 12 the restrictions ξ_1, ξ_2, $\xi \in \mathfrak{S}$ are necessary in all lemmas.

In section 13, instead of (13.2) we have only the following lemma:

3.2. Lemma. *If* $\xi \in \mathfrak{S}$, *then* $(pq)\xi = p(\xi)q(\xi) + o(q(\xi))$.

This follows from (13.21) and (3.1).

(13.3), (13.4), (13.5), and (13.6) have no value for us since (13.7) cannot be generalised in a way suitable for our purposes. But the following weaker lemma will help us:

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(9) Here and in the following numbers of formulae refer either to our paper or to [1, part II]. No misunderstanding is possible since our paper consists only of the sections 1–3, but [1, part II] starts with section 7.

3.3. COROLLARY. *If* $\xi \in \mathfrak{S}$, $p(\xi) \neq o(1)$, *then* $(pq)\xi = p(\xi)q(\xi)(1+o(1))$.

Proof. $p(\xi) \neq o(1)$ implies $(p(\xi))^{-1} = O(1)$. Hence

$$o(q(\xi)) = o((p(\xi))^{-1}p(\xi)q(\xi)) = o(O(p(\xi)q(\xi)) = o(p(\xi)q(\xi))$$

and (3.2) does the rest.

In the sections 14 and 15, θ and ζ belong already to $\mathfrak{S}$ quite naturally. Thus our proofs are not hampered by the previous restrictions that the arguments of all polynomials must belong to $\mathfrak{S}$. In (15.2) and (15.21), $\frown\mathfrak{R}$ has to be replaced by $\overline{\mathfrak{S}}$. Of course the normal extension Φ, as introduced in the proof of (15.2), need not lie in $\overline{F}^*(-1)^{1/2}$. But according to Steinitz [4], for every field K an algebraically closed extension field L can be constructed. Thus if $K = \overline{\mathfrak{S}}$, Φ lies in L. In the proof of (15.5) our lemma (3.2) must be applied twice.

The ordering of a simple algebraic extension $\Sigma(\theta)$ of Σ can be defined as in the sections 16 and 17. All proofs but one remain valid, only the proof of (16.73) has to be altered (because it applied (13.7) in [1]) as follows:

$\bar{f}(\bar{x})$, as defined in (9.21), is irreducible over $\overline{\mathfrak{R}}$. For otherwise $f(x)$ would factorise in $\mathfrak{R}[x]$, in contradiction of (9.8). Put $p_3(x) = \pi_2^{-1}p_1\pi_2(x)$, then $p_3(x) \in \mathfrak{R}[x] - \mathfrak{P}[x]$, according to its definition. Since $p_3(x)$ is of smaller degree than $f(x)$, we have $\bar{p}_3(\bar{\zeta}) \neq \bar{0}$, i.e. $p_3(\zeta) \neq o(1)$. Now (16.73) follows from (3.3).

Finally in the place of (17.5) we have the following result:

3.4. THEOREM. *Every ordered division ring* Σ *can be extended to an ordered division ring* $\Sigma(F^*)$, *containing the closure* F^* *of a chosen subfield of* F *of* Σ (*with respect to the order-topology in* F) *with preservation of the order-relations in* Σ.

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