COMPLEX TAUBERIAN THEOREMS FOR POWER SERIES[1]

BY

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1. **Introduction.** Given a power series [2]

$$f(z) = \sum a_n z^n = (1 - z) \sum s_n z^n \tag{1.1}$$

with radius of convergence 1, we wish to study the relations between the sequence (s) and its associated function $f(z)$ which may be considered as a transformation $T(s)$. Direct (or Abelian) theorems conclude from the sequence (s) to the transformation $T(s)$, while Tauberian theorems infer conclusions from the behavior of $T(s)$ to the behavior of (s) under specified additional conditions (Tauberian conditions).

For the special transformation T which transforms (s) into $f(z)$ according to (1.1) we distinguish two types of Tauberian theorems. Tauberian theorems of real character use assumptions about $f(z)$ where z is on the real axis, and have real Tauberian conditions; for example

$$\lim_{z\to 1-0} f(z) = s \quad \textit{and} \quad a_n \geqq 0 \quad \textit{implies} \quad \lim_{n\to\infty} s_n = s.$$

In this paper we are concerned with Tauberian theorems of complex character in which the assumptions on $f(z)$ are essentially complex.

One of these complex Tauberian theorems was given first by Fatou [3, p. 389]:

THEOREM A. *If the function $f(z)$ defined by* (1.1) *is regular at $z=1$ and $a_n \to 0$ $(n\to\infty)$, then $\sum a_n$ converges.*

Another theorem of this type is due to M. Riesz (see, e.g. [9, p. 64]):

THEOREM B. *If the function $f(z)$ defined by* (1.1) *is regular in the region*

$$S_1: \begin{cases} |z| < R, & R > 1, \\ |\operatorname{arc}(z-1)| > \theta_0, & 0 < \theta_0 < \pi/2, \end{cases}$$

and continuous in $\bar{S}_1$ (the closure of S_1), then $\sum a_n$ converges.

After recalling some well known facts in summability theory we shall relax the assumptions on $f(z)$ in the Theorems A and B and prove a theorem which contains both as special cases. The condition about the behavior of

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(1) The results of this paper are contained in the author's Ph.D. Thesis [4]. Numbers in brackets refer to the bibliography at the end of the paper.

(2) We shall always let $\sum_{n=0}^{\infty} a_n = \sum a_n$.

$f(z)$ in S_1 is thereby localized to a smaller region S_2 in the neighborhood of $z=1$ which is the smallest possible in a certain sense. In §4 other regions are taken instead of S_2.

In §§5 and 6 we assume that $f(z)=\sum a_n z^n$ has a positive radius of convergence and that $z=1$ lies on the boundary of the region of V-summability associated with $f(z)$. In analogy to the above theorems we infer conclusions from the behavior of $f(z)$ in the neighborhood of $z=1$ to the behavior of $V(s)$. In an extended sense these theorems are also complex Tauberian theorems. For V the methods of Euler-Knopp, Borel, and Meyer-König are taken and thus extensions of results of Karamata, Meyer-König, and Obrechkoff are obtained.

2. **The methods** E_p, B, S_α. The methods of Euler-Knopp, Borel, and Meyer-König are useful for the analytic continuation of power series. Assume that a series $\sum a_n$ with partial sums $s_n=\sum_{\nu=0}^{n} a_\nu$ is given and that the associated power series $f(z)=\sum a_n z^n$ has a positive radius of convergence.

a. The method E_p (Euler-Knopp) with fixed parameter p $(0<p<\infty)$ is defined by the triangular matrix (E_p) with elements

$$c_{n\nu}=\frac{1}{2^{pn}}\binom{n}{\nu}(2^p-1)^{n-\nu} \qquad (n,\nu=0,1,2,\cdots;\nu\leqq n). \tag{2.1}$$

The E_p-transformation of the sequence s_n is therefore the new sequence

$$E_p(n;s_\nu)=\frac{1}{2^{pn}}\sum_{\nu=0}^{n}\binom{n}{\nu}(2^p-1)^{n-\nu}s_\nu \qquad (n=0,1,\cdots)$$

and we say that E_p-lim $s_n=s$ (or E_p-$\sum a_n=s$) if $\lim_{n\to\infty}E_p(n;s_\nu)=s$.

For our purposes another definition of the E_p-transformation of the sequence s_n is more suitable. Letting

$$z=\phi_p(w)=\frac{w}{2^p-(2^p-1)w} \tag{2.2}$$

and developing $f(z)$ into powers of w we obtain a power series

$$F(w)=\sum a_n' w^n$$

which is convergent in the neighborhood of $w=0$. It is easy to prove that

$$E_p(n;s_\nu)=\sum_{\kappa=0}^{n}a_\kappa' \quad\text{where}\quad \begin{cases}a_n'=2^{-p}E_p(n-1;a_{\nu+1}) & (n=1,2,\cdots),\\ a_0'=a_0.\end{cases} \tag{2.3}$$

Hence E_p-$\sum a_n$ exists if and only if $\sum a_n'$ converges, and E_p-lim $a_n=0$ is necessary for the existence of E_p-$\sum a_n$. (Here we use the fact that $E_p(n;a_{\nu+1})\to 0$ $(n\to\infty)$ is equivalent to $E_p(n;a_\nu)\to 0$ $(n\to\infty)$.)

If the method E_p is applied to $\sum a_n z^n$ for different values of z, one obtains a region $\mathfrak{G}_{E_p}$ in the z-plane in the interior of which the power series is E_p-summable to the value $f(z)$, whereas it is certainly not summable for any z which lies outside of $\overline{\mathfrak{G}_{E_p}}$. On the boundary of $\mathfrak{G}_{E_p}$ no general statement can be made and the situation is somewhat similar to the situation on the boundary of the circle of convergence of a power series. $\mathfrak{G}_{E_p}$ can be constructed from the singularities of $f(z)$; we shall use that $f(z)$ is necessarily regular in $\mathfrak{K}_{E_p} = \mathfrak{K}((2^p-1)/(2^{p+1}-1))$ if $z=1$ is on the boundary of $\mathfrak{G}_{E_p}$. (We denote by $\mathfrak{K}(a)$ the region $|z-a|<1-a$ for $0<a<1$.) $\mathfrak{K}_{E_p}$ is the map of $|w|<1$ under the transformation $z=\phi_p(w)$.

b. Two methods of Borel are known. In the case of the "exponential method" we let

$$B(x;\, s_\nu) = e^{-x} \sum \frac{s_n x^n}{n!} \qquad (x \geqq 0)$$

where the sum exists for $x \geqq 0$ since $\sum a_n z^n$ has a positive radius of convergence; we say that B-lim $s_n = s$ (or B-$\sum a_n = s$) if $\lim_{x\to\infty} B(x;\, s_\nu) = s$.

The second method is called Borel's "integral method" and is often more suitable to power series. With the given series $\sum a_n$ we associate the function $\phi(t) = \sum a_n t^n/n!$ for $t \geqq 0$ and let

$$B'(x;\, s_\nu) = \int_0^x e^{-t}\phi(t)dt \qquad (x \geqq 0);$$

we say that B'-lim $s_n = s$ (or B'-$\sum a_n = s$) if $\lim_{x\to\infty} B'(x;\, s_\nu) = s$.

Concerning the relations between these two methods it is known that

$$(2.4) \qquad B\text{-lim } s_n = s \quad \text{implies} \quad B'\text{-lim } s_n = s,$$

but not conversely. However, since

$$(2.5) \qquad B(x;\, s_\nu) = B(x;\, a_\nu) + B'(x;\, s_\nu),$$

the converse of (2.4) is true if and only if $B(x;\, a_\nu) \to 0\ (x \to \infty)$. The relations (2.4) and (2.5) also imply that B-lim $a_n = 0$ is necessary for the existence of B-$\sum a_n$.

The region $\mathfrak{G}_B$ of Borel-summability of the power series $\sum a_n z^n$ is defined analogously to $\mathfrak{G}_{E_p}$ in **a**, and is the same for both methods B and B'. If $z=1$ lies on the boundary of $\mathfrak{G}_B$, then $f(z)$ is necessarily regular in $\mathfrak{K}_B = \mathfrak{K}(1/2)$.

c. The succession of the methods E_p and B is continued in a certain sense by the method of Meyer-König [10, p. 272]). This method depends on a parameter $\alpha(0<\alpha<1)$ and is defined by the matrix (S_α) with elements

$$c_{n\nu} = (1-\alpha)^{n+1}\binom{n+\nu}{n}\alpha^\nu \qquad (n,\, \nu = 0, 1, \cdots).$$

The S_α-transformation of the sequence s_n

$$S_\alpha(n; s_\nu) = (1-\alpha)^{n+1}\sum\binom{n+\nu}{n}\alpha^\nu s_\nu \qquad (n = 0, 1, \cdots)$$

exists if and only if $s_\nu = O(\nu^{-n}\alpha^{-\nu})$ for $\nu\to\infty$ and all fixed $n=0, 1, \cdots$; one then says that (S_α) is applicable to the sequence s_n. This obviously implies the regularity of $f(z)$ in $|z| <\alpha$, and the regularity of $f(z)$ in $|z| \leqq\alpha$ is sufficient for the applicability of (S_α) to the sequence s_n.

Later we use an alternative definition of the S_α-transformation. Assume that (S_α) is applicable to the sequence s_n so that $f(z)$ is regular in $|z| <\alpha$; assume in addition that $f(z)$ is regular at $z=\alpha$. Letting

$$z = \frac{\alpha}{1-(1-\alpha)w} \tag{2.6}$$

and developing $f(z)$ into powers of w we obtain a power series

$$F(w) = \sum a_n' w^n$$

which is convergent in the neighborhood of $w=0$. One finds that

$$S_\alpha(n; s_\nu) = \sum_{\kappa=0}^{n} a_\kappa' \quad \text{where} \quad \begin{cases} a_n' = \dfrac{\alpha}{1-\alpha}S_\alpha(n; a_{\nu+1}) & (n = 1, 2, \cdots), \\ a_0' = a_0 + \dfrac{\alpha}{1-\alpha}S_\alpha(0; a_{\nu+1}). \end{cases} \tag{2.7}$$

Hence S_α-$\sum a_n$ exists if and only if $\sum a_n'$ converges, and S_α-lim $a_n=0$ is necessary for the existence of S_α-$\sum a_n$.

In the application to power series the situation here is slightly more complicated than in **a** and **b**. The region of S_α-summability has not been investigated in full but it is known that the regularity of $f(z)$ in $\mathfrak{R}_{S_\alpha} = \mathfrak{R}(1/(2-\alpha))$ is necessary for the existence of S_α-$\sum a_n$ (again assuming the regularity of $f(z)$ at $z=\alpha$).

3. **On two theorems of Fatou and M. Riesz.** The starting point for our investigations on the circle of convergence of the power series

$$f(z) = \sum a_n z^n \qquad \left(\limsup_{n\to\infty} |a_n|^{1/n} = 1\right) \tag{3.1}$$

is the Theorems A and B of the introduction. Szász showed [1, pp. 485–486] that in Theorem B the region S_1 can be replaced by another region which has a finite number of corners on $|z| =1$ instead of one as in the case of S_1(3). We show in Theorem 1 that the assumptions on $f(z)$ can be localized completely to a neighborhood of $z=1$ if the condition $a_n\to 0$ $(n\to\infty)$ is added.

Let S_2 be the region

(3) Such a region can be expressed by $\bigcap_1^n\{z_\kappa\cdot S_1\}$ with $|z_\kappa| =1$ $(\kappa=1, 2, \cdots, n)$.

$$S_2 = S_2(\delta_0, \theta_0): \quad \begin{cases} |z-1| < \delta_0, & \delta_0 > 0, \\ |\operatorname{arc}(z-1)| > \theta_0, & 0 < \theta_0 < \pi/2. \end{cases}$$

THEOREM 1. *Let* (3.1) *be regular and bounded in* S_2, *and* $a_n \to 0$ $(n \to \infty)$. *Then* $\sum a_n = s$ *if* A-$\sum a_n = s$, *i.e. if* $\lim_{z \to 1-0} f(z) = s$.

REMARKS. a. Theorem 1 contains the Theorems A and B and also Szász's extension of Theorem B. The conditions $a_n \to 0$ $(n \to \infty)$ and A-$\sum a_n = s$ are both necessary for $\sum a_n = s$, together they are sufficient for $\sum a_n = s$ if $f(z)$ fulfills the above conditions.

b. Let $K = (e^{i\phi_0}, e^{-i\phi_0})$ be a closed subarc of $|z| = 1$ within $|z-1| < \delta_0$. Under the assumptions of Theorem 1 the series $\sum a_n z^n$ converges for $z = 1$ and Theorem A assures its convergence for every other z on K. We extend Theorem 1 by proving the uniform convergence of (3.1) on K.

THEOREM 2. *Under the assumptions of Theorem* 1 *the series* (3.1) *converges uniformly on* K.

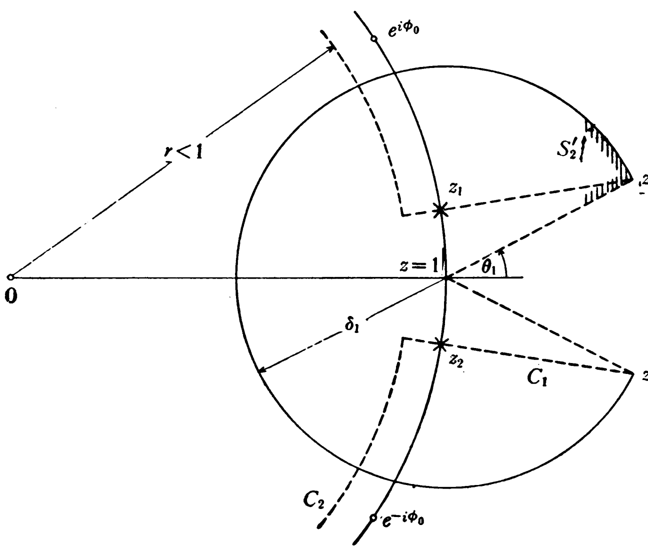

FIG. 1

Proof. By a theorem on bounded functions (see for example [13, p. 65]) one obtains first that $f(z)$ is continuous in $\overline{S_2'}$ where $S_2' = S_2(\delta_1, \theta_1)$ for every δ_1, θ_1 with $0 < \delta_1 < \delta_0$ and $\theta_0 < \theta_1 < \pi/2$. We may assume that $\lim f(z) = 0$ for $z \to 1$ in S_2' and that δ_1 is chosen such that

$$(3.2) \qquad |f(z)| < \epsilon \cdot (\pi \cos \theta_1)/8$$

for $z \in S_2'$ and a given $\epsilon > 0$. Draw the path $C = C_1 + C_2$ as indicated in Fig. 1. (C_2 is the part of C in $|z| < 1$.)

To prove the uniform convergence of $\sum a_n e^{in\phi}$ for $|\phi| \leqq \phi_0$, take first all ϕ with $|\phi| \leqq \phi_1 = (1/2)$ arc z_1. If $\sigma_n(\phi)$ are the arithmetic means of $s_n(\phi) = \sum_{\nu=0}^{n} a_\nu e^{i\nu\phi}$, then

$$| s_n(\phi) - \sigma_n(\phi) | = \frac{1}{n+1} \left| \sum_{\nu=1}^{n} a_\nu \nu e^{i\nu\phi} \right| = \frac{1}{n+1} \left| \frac{1}{2\pi i} \left\{ \sum_{\nu=1}^{n} \nu e^{i\nu\phi} \int_{C_1} \frac{f(z)dz}{z^{\nu+1}} + \int_{C_2} f(z) \sum_{\nu=1}^{n} \frac{\nu e^{i\nu\phi}}{z^{\nu+1}} dz \right\} \right|$$
$$= \frac{1}{2\pi(n+1)} | A_n^{(1)}(\phi) + A_n^{(2)}(\phi) |.$$

Estimation of $A_n^{(1)}$: For $\nu = 1, 2, \cdots$

$$\left| \int_1^{z_1'} \frac{f(z)dz}{z^{\nu+1}} \right| < \epsilon \cdot \frac{\pi \cos \theta_1}{8} \int_0 \frac{d\tau}{|1 + \tau e^{i\theta_1}|^{\nu+1}}$$
$$< \epsilon \cdot \frac{\pi \cos \theta_1}{8} \int_0^\infty \frac{d\tau}{(1 + \tau \cos \theta_1)^{\nu+1}} = \frac{\pi}{8\nu} \cdot \epsilon,$$

and similarly for the other parts of C_1. Hence

$$(3.3) \qquad | A_n^{(1)}(\phi) | < \frac{n\pi}{2} \cdot \epsilon \qquad (n = 1, 2, \cdots),$$

uniformly for $|\phi| \leqq \phi_1$.

Estimation of $A_n^{(2)}$: Evaluating first the finite sum occurring in the expression for $A_n^{(2)}$ we obtain

$$A_n^{(2)}(\phi) = e^{i\phi} \int_{C_2} f(z) \frac{z^{n+1} - (n+1)ze^{in\phi} + ne^{i\phi(n+1)}}{z^{n+1}(z - e^{i\phi})^2} dz.$$

In this integral we have first

$$\int_{C_2} \frac{f(z)dz}{(z - e^{i\phi})^2} = o(n),$$

and we shall prove that also

$$(3.4) \qquad n \int_{C_2} \frac{f(z)dz}{z^n (z - e^{i\phi})^2} = o(n),$$

always uniformly for $|\phi| \leqq \phi_1$. Assume for the moment that (3.4) is true. Then

$$(3.5) \qquad | A_n^{(2)}(\phi) | < \frac{n\pi}{2} \cdot \epsilon \qquad (n > n_1),$$

uniformly for $|\phi| \leqq \phi_1$, and therefore $|s_n(\phi) - \sigma_n(\phi)| < \epsilon/2$ for $n > n_1$ and

$|\phi| \leqq \phi_1$. Using now the fact [15, pp. 94–95] that $\sigma_n(\phi)$ converges uniformly to $f(e^{i\phi})$ for $|\phi| \leqq \phi_1$, we get

$$|s_n(\phi) - f(e^{i\phi})| \leqq |s_n(\phi) - \sigma_n(\phi)| + |\sigma_n(\phi) - f(e^{i\phi})| < \epsilon$$

for $n > n_2$ and all ϕ in $|\phi| \leqq \phi_1$. Since for ϕ with $\phi_1 \leqq |\phi| \leqq \phi_0$ the uniform convergence of $\sum a_n e^{in\phi}$ follows from an extension of Theorem A due to M. Riesz [15, p. 90], the proof of Theorem 2 is completed provided (3.4) is true.

For the proof of (3.4) we apply a method of M. Riesz [15], choosing the numbers $b_1 = b_1(\phi)$ and $b_2 = b_2(\phi)$ such that $H(z_1) = H(z_2) = 0$ for

$$H(z) = H(z, \phi) = \frac{1}{(z - e^{i\phi})^2} + b_1 + b_2 z.$$

One finds that $b_1(\phi)$ and $b_2(\phi)$ are bounded for $|\phi| \leqq \phi_1$. Therefore using (3.3) and $a_n \to 0$ $(n \to \infty)$ we obtain

$$\begin{aligned}\int_{C_2} \frac{f(z)H(z)dz}{z^n} &= \int_{C_2} \frac{f(z)dz}{z^n(z - e^{i\phi})^2} + b_1 \int_C \frac{f(z)dz}{z^n} + b_2 \int_C \frac{f(z)dz}{z^{n-1}} \\ &\qquad - b_1 \int_{C_1} \frac{f(z)dz}{z^n} - b_2 \int_{C_1} \frac{f(z)dz}{z^{n-1}} \\ &= \int_{C_2} \frac{f(z)dz}{z^n(z - e^{i\phi})^2} + o(1),\end{aligned}$$

so that it remains now to show that

$$\lim_{n \to \infty} \int_{C_2} \frac{f(z)H(z)dz}{z^n} = 0, \tag{3.6}$$

uniformly for $|\phi| \leqq \phi_1$. But by Theorem A the series $\sum a_n z^n$ converges for z_1 and z_2 and therefore uniformly on C_2 so that

$$\int_{C_2} \frac{f(z)H(z)dz}{z^n} = \sum_\nu a_\nu \int_{C_2} z^{\nu - n} H(z) dz$$

which is a matrix transformation of the zero-sequence a_ν. We have to show

$$\lim_{n \to \infty} c_{n\nu} = 0 \qquad (\nu = 0, 1, \cdots, \text{fixed}), \tag{3.7}$$

$$\limsup_{n \to \infty} \sum_\nu |c_{n\nu}| < \infty, \tag{3.8}$$

uniformly for $|\phi| \leqq \phi_1$, where

$$c_{n\nu} = c_{n\nu}(\phi) = \int_{C_2} z^{\nu - n} H(z, \phi) dz.$$

Integrating twice by parts we get for $\nu \neq n-1, \nu \neq n-2$

$$c_{n\nu} = -\frac{1}{\nu - n + 1}\left\{\frac{z^{\nu-n+2}}{\nu - n + 2}\, \frac{\partial H(z)}{\partial z}\Bigg|_{z_1}^{z_2} - \frac{1}{\nu - n + 2}\int_{C_2} z^{\nu-n+2}\, \frac{\partial^2 H(z)}{\partial z^2}\, dz\right\}.$$

Since $H(z) = H(z, \phi)$ is regular in the finite z-plane except for $z = e^{i\phi}$ where $|\phi| \leqq \phi_1$, we can replace the path C_2 by the arc: $|\phi| \geqq \text{arc } z_1$ of the unit circle. For z on this arc the function $H(z)$ and its first and second derivatives with respect to z are bounded, uniformly for $|\phi| \leqq \phi_1$, by a constant L_1, say. Therefore

$$|c_{n\nu}| \leqq L_1 \frac{2 + 2\pi}{(\nu - n + 1)(\nu - n + 2)} \qquad (\nu \neq n - 1; \nu \neq n - 2),$$

uniformly for $|\phi| \leqq \phi_1$, which proves (3.7). But obviously $c_{n,n-1}$ and $c_{n,n-2}$ are uniformly bounded for $|\phi| \leqq \phi_1$, by a constant L_2, say, so that

$$\begin{aligned} \sum_{\nu} |c_{n\nu}| &\leqq 2L_2 + L_1 \sum_{\nu\neq n-1, \nu\neq n-2} \frac{2 + 2\pi}{(\nu - n + 1)(\nu - n + 2)} \\ &\leqq 2L_2 + 2L_1(2 + 2\pi) \end{aligned}$$

which proves (3.8) and therefore completes the proof of Theorems 1 and 2.

4. **Weaker assumptions on** $f(z)$. Until now our assumptions on $f(z)$ concerned its behavior in the regions S_1 and S_2, whose boundaries have an osculation of first order with $|z| = 1$ at $z = 1$. The question arises what can be said in the case that S_1 and S_2 are substituted by the regions S_3 and S_4, whose boundaries have an osculation of second order with $|z| = 1$ at $z = 1$:

$$S_3: \begin{cases} |z| < R, & R > 1 \\ |z| < 1 + c_0\phi^2, & c_0 > 0 \end{cases}; \quad S_4: \begin{cases} |z - 1| < \delta_0, & \delta_0 > 0 \\ |z| < 1 + c_0\phi^2, & c_0 > 0 \end{cases} \quad (z = |z| \cdot e^{i\phi}).$$

In the neighborhood of $z = 1$ the boundaries of S_3 and S_4 behave like a circle which touches $|z| = 1$ exteriorly at $z = 1$.

First we remark that *it is impossible to substitute* S_1 *by* S_3 *in Theorem* B *or* S_2 *by* S_4 *in Theorem* 1. For there exist power series which define a function $f(z)$ regular in S_3 and continuous in $\overline{S_3}$ and which are not even V-summable for $z = 1$ [5, p. 331]; by V we denote in this paragraph any of the methods E_p $(0 < p < \infty)$, B, S_α $(0 < \alpha < 1)$. We shall show, however, that the following modifications of Theorems B and 1 are true.

THEOREM 3. *If* (3.1) *is regular in* S_3 *and continuous in* $\overline{S_3}$, *then* $\sum a_n$ *converges provided that* V-$\sum a_n$ *exists*([4]).

This is a $V \rightarrow K$-Theorem ("K" standing for "convergence") under complex Tauberian conditions.

([4]) Theorem 3 and its proof also hold if "continuous in $\overline{S_3}$" is replaced by "bounded in S_3."

Proof. The assumptions on $f(z)$ imply $a_n = O(1/n^{1/2})$ [5, p. 327], which is a Tauberian condition for V-summability.

We now prove the localization of Theorem 3.

THEOREM 4. *Let* (3.1) *be regular and bounded in* S_4, *and* $a_n \to 0$ $(n \to \infty)$. *Then* $\sum a_n = s$ *if* $V\text{-}\sum a_n = s$(5).

Proof. It is sufficient to show

$$\lim_{n\to\infty} \sum_{\nu=n}^{n+[\epsilon_n n^{1/2}]} a_\nu = 0 \qquad \text{for any positive zero-sequence } \epsilon_n, \tag{4.1}$$

because this is a Tauberian condition for V-summability. (For E_p and B see for example [6, p. 312]; for S_α see [4, p. 36].) To prove (4.1) we construct a path similar to the one in Fig. 1, except that the part $(z_2', 1, z_1')$ of C_1 is substituted by a segment of $|z| = 1 + c_1\phi^2$ $(0 < c_1 < c_0)$; the parts (z_2, z_2') and (z_1, z_1') remain rectilinear as in Fig. 1. Then with $k = [\epsilon_n n^{1/2}]$ we have

$$\sum_{\nu=n}^{n+k} a_\nu = \frac{1}{2\pi i}\left[\sum_{\nu=n}^{n+k} \int_{C_1} \frac{f(z)dz}{z^{\nu+1}} + \int_{C_2} f(z) \sum_{\nu=n}^{n+k} z^{-\nu-1}dz\right].$$

As proved in [5], $\int_{C_1} f(z)dz/z^{\nu+1} = O(1/\nu^{1/2})$ $(\nu \to \infty)$, and the second integral equals

$$\int_{C_2} \frac{f(z)dz}{z^n(z-1)} - \int_{C_2} \frac{f(z)dz}{z^{n+k+1}(z-1)}.$$

Both of these integrals tend to zero for $n \to \infty$, which is shown in the same way as (3.4) for $\phi = 0$. Hence

$$\sum_{\nu=n}^{n+k} a_\nu = O\left(\sum_{\nu=n}^{n+k} \frac{1}{\nu^{1/2}}\right) + o(1) = O\left([\epsilon_n n^{1/2}]\cdot\frac{1}{n^{1/2}}\right) + o(1) = o(1) \quad (n \to \infty),$$

which proves (4.1) and therefore Theorem 4.

5. **Tauberian theorems asserting E_p- and S_α-summability.** Consider now a power series

$$f(z) = \sum a_n z^n \tag{5.1}$$

with positive radius of convergence on the boundary of its region of E_p- and S_α-summability. We investigate the question from what assumptions about $f(z)$ in the neighborhood of $z=1$ we can derive the summability of $\sum a_n$ by means of the methods E_p and S_α. The following assumptions will be frequently used:

(5) Comparing Theorem 4 with Theorem 1 it is noted that the assumptions on $f(z)$ are relaxed in Theorem 4; but since $V\text{-}\sum a_n = s$ implies $A\text{-}\sum a_n = s$ if $\limsup |a_n|^{1/n} = 1$, the assumptions on $\sum a_n$ are strengthened.

(*) $\qquad f(z)$ is regular in $\mathfrak{R}_{E_p} = \mathfrak{R}\left(\dfrac{2^p - 1}{2^{p+1} - 1}\right) \qquad$ for $p > 0$;

(**) $\qquad f(z)$ is regular in $|z| < \alpha$ and in $\mathfrak{R}_{S_\alpha} = \mathfrak{R}\left(\dfrac{1}{2 - \alpha}\right) \qquad$ for $0 < \alpha < 1$.

As to the necessity of these assumptions for the E_p- and S_α-summability of $\sum a_n$ see §2, **a** and **c**.

Corresponding to Theorem 1 we have

THEOREM 5. *Assume that* (5.1) *fulfills the condition* (*) [*or* (**)] *and is regular and bounded in* S_2, *furthermore* E_p-lim $a_n = 0$ [*or* S_α-lim $a_n = 0$]. *Then* E_p-$\sum a_n = s$ [*or* S_α-$\sum a_n = s$] *if* $\lim_{z\to 1-0} f(z) = s$.

Proof. We restrict ourselves to the proof of the case of E_p-summability. According to §2, **a** the question is whether the series $\sum a_n' w^n$ representing $F(w) = f(\phi_p(w))$ with $\phi_p(w) = w/(2^p - (2^p - 1)w)$ converges for $w = 1$. We shall show that for $F(w) = \sum a_n' w^n$ all the assumptions of Theorem 1 are fulfilled.

(a) The image of the region S_2 in the z-plane under $w = \phi_p^{-1}(z)$ is a region in the w-plane whose boundary has an exterior osculation of order one with $|w| = 1$ at $w = 1$. Hence $F(w)$ (which by (*) is regular in $|w| < 1$) is regular and bounded in some region S_2 in the w-plane.

(b) We have $a_n' \to 0$ ($n \to \infty$) since E_p-lim $a_n = 0$ (§2, **a**).

(c) Finally $\lim_{w\to 1-0} F(w) = \lim_{z\to 1-0} f(z) = s$.

Hence, by Theorem 1, $\sum a_n' = s$, i.e. E_p-$\sum a_n = s$.

If $f(z)$ is regular at the point $z = 1$ on the boundary of the region of summability, Theorem 5 yields analogues to Theorem A of the introduction.

The proof of Theorem 5 is based upon the following idea. Suppose that certain assumptions on $F(w) = \sum a_n' w^n$ allow one to draw conclusions about $\sum a_n'$, and we "transform" these assumptions on $f(z)$ by $z = \phi_p(w)$. Then from the "transformed" assumptions on $f(z)$ we can conclude to $\sum a_n'$ and therefore to E_p-$\sum a_n$, since E_p-$\sum a_n$ behaves like $\sum a_n'$. Following are two more examples of this general principle.

THEOREM 6. *Assume that* (5.1) *fulfills the condition* (*) [*or* (**)] *and is continuous in* $\overline{\mathfrak{R}_{E_p}}$ [*or* $\overline{\mathfrak{R}_{S_\alpha}}$]. *Then* $C_\epsilon E_p$-$\sum a_n$ [*or* $C_\epsilon S_\alpha$-$\sum a_n$] *exists for every* $\epsilon > 0$.

REMARK. The case where $f(z)$ is regular in $|z| < 1$ and continuous in $\overline{\mathfrak{R}_{E_p}}$ [or $\overline{\mathfrak{R}_{S_\alpha}}$] was treated by Meyer-König [11, p. 352]. He proved that then $E_p C_\epsilon$-$\sum a_n$ [or $S_\alpha C_\epsilon$-$\sum a_n$] exists for $\epsilon = 1, 2, \cdots$. These results are contained in Theorem 6, for since the matrices (E_p) and (C_ϵ) are Hausdorff matrices, $(C_\epsilon E_p) = (E_p C_\epsilon)$; on the other hand it is proved that for $\epsilon = 1, 2, \cdots$ the methods $C_\epsilon S_\alpha$ and $S_\alpha C_\epsilon$ are equivalent [12, p. 450].

Proof of Theorem 6. We restrict ourselves to the case of E_p-summability.

The assumptions on $f(z)$ imply that $F(w)=\sum a_n' w^n$ is regular in $|w|<1$ and continuous in $|w|\leqq 1$ whence by a known result of M. Riesz [15, pp. 94–95] we obtain that C_ϵ-$\sum a_n'$ exists for every $\epsilon>0$, i.e. $C_\epsilon E_p$-$\sum a_n$ exists for every $\epsilon>0$.

If in Theorem 6 the continuity of $f(z)$ for $z\to 1$ is sufficiently strong, we can derive the E_p- [or S_α-] summability of $\sum a_n$.

THEOREM 7. *Assume that $f(z)$ fulfills the condition* (*) [*or* (**)] *and that*

$$f(z) = s + o((1-z)^\eta) \qquad (\eta > 0), \tag{5.2}$$

uniformly for $z\to 1$ in $\mathfrak{R}_{E_p}$ [or $\mathfrak{R}_{S_\alpha}$]. Then E_p-$\sum a_n = s$ [or S_α-$\sum a_n=s$] if E_p-lim $a_n=0$ [*or* S_α-lim $a_n=0$].

The last condition is certainly fulfilled under the stronger assumption that $f(z)$ is continuous in $\overline{\mathfrak{R}_{E_p}}$ [or $\overline{\mathfrak{R}_{S_\alpha}}$], since this means that $F(w)$ is continuous in $|w|\leqq 1$, so that $a_n'\to 0$, i.e. E_p-lim $a_n=0$ [or S_α-lim $a_n=0$].

Proof. For the function $F(w)=\sum a_n' w^n$ we have

$$F(w) = f(\phi_p(w)) = s + o((1-\phi_p(w))^\eta) = s + o((1-w)^\eta) \qquad (\eta>0),$$

uniformly for $w\to 1$ in $|w|<1$ since $1-\phi_p(w)=(2^p/(2^p-(2^p-1)w))(1-w)$ $\sim(1-w)$ for $w\to 1$; furthermore $a_n'\to 0$ since E_p-lim $a_n=0$. Hence by a known theorem [17, p. 220] the series $\sum a_n'$ converges, i.e. E_p-$\sum a_n$ exists, and its value is s. The proof for the S_α-case is similar.

Assuming that there exists an analogue to Theorem 7 for B-summability (see Theorem 7′, §6) we obtain the following

COROLLARY. *Let $f(z)=\sum a_n z^n$ be regular in $|z|<1$ and*

$$f(z) = s + o((1-z)^\eta) \qquad (\eta>0),$$

uniformly for $z\to 1$ in $\mathfrak{R}_{E_p}$ [or $\mathfrak{R}_B$, or $\mathfrak{R}_{S_\alpha}$]. Then E_p-$\sum a_n=s$ [or B-$\sum a_n=s$ or S_α-$\sum a_n=s$].

For the proof one notes that E_p-lim $a_n=0$ [or S_α-lim $a_n=0$] since $a_n'\to 0$ is implied by the boundedness of $f(z)$ in $\mathfrak{R}_{E_p}$ [or $\mathfrak{R}_{S_\alpha}$]. We shall prove in Theorem 9 that B-lim $a_n=0$ is a consequence of the boundedness of $f(z)$ in $\mathfrak{R}_B$.

The above corollary has some relationship to a theorem which Hardy and Littlewood have stated without proof [7, p. 53]:

Let $f(z)=\sum a_n z^n$ be regular in $|z|<1$ and $f(z)=s+o((1-z)^\eta)$ $(\eta>0)$, uniformly for $z\to 1$ in some circle touching $|z|=1$ interiorly at $z=1$. Then B-$\sum a_n=s$.

Obviously this result and the corollary overlap, i.e. neither one is included in the other(6).

We can combine the above corollary with Theorem 3:

(6) For a series $\sum a_n$ with lim sup $|a_n|^{1/n}=1$ we have $E_p\to B\to S_\alpha$.

Let $\mathfrak{R}_1$ and $\mathfrak{R}_2$ be two circular regions whose bounding circles touch $|z|=1$ at $z=1$ exteriorly and interiorly, respectively. Assume that $f(z)=\sum a_n z^n$ is regular in $\mathfrak{R}_1$, continuous in $\overline{\mathfrak{R}_1}$ and strongly continuous for $z\to 1$ in $\mathfrak{R}_2$ (i.e., $f(z)=s+o((1-z)^\eta)$ $(\eta>0)$, uniformly for $z\to 1$ in $\mathfrak{R}_2$). Then $\sum a_n = s$.

For the proof note that $\mathfrak{R}_2$ is one of the regions of the set $\mathfrak{R}_{E_p}$, $\mathfrak{R}_B$, $\mathfrak{R}_{S_\alpha}$.

6. Tauberian theorems asserting *B*-summability. In this last section we shall deal with a power series

$$f(z) = \sum a_n z^n \tag{6.1}$$

with positive radius of convergence, for which $z=1$ lies on the boundary of its

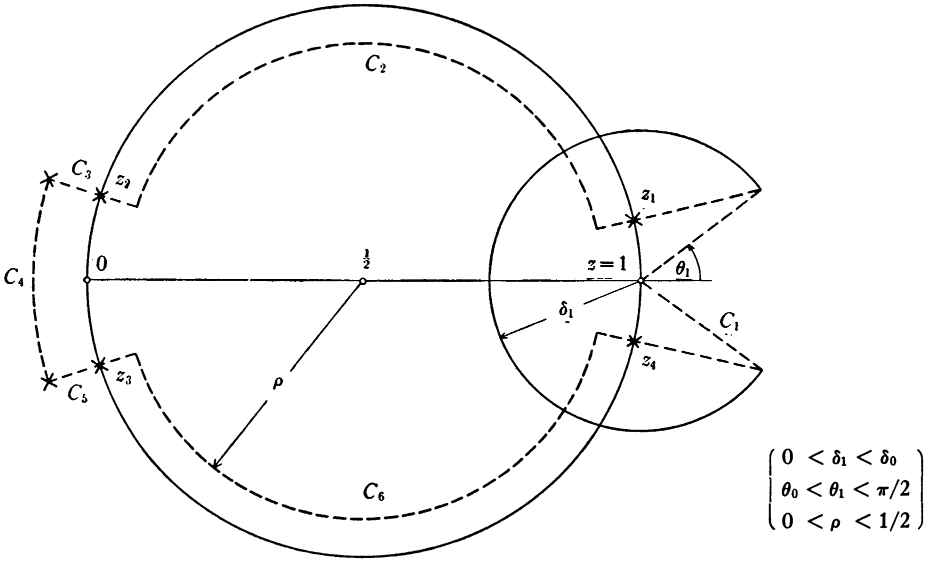

FIG. 2

region of B-summability. This implies the regularity of $f(z)$ in $\mathfrak{R}_B$. The section will be divided into three parts. In part A we prove the analogue to Theorem 1, deriving the B-summability of $\sum a_n$ from assumptions on $f(z)$ in the neighborhood of $z=1$. Also given are sufficient conditions on $f(z)$ which imply B-lim $a_n=0$. Part B contains a necessary and sufficient condition for the existence of B-$\sum a_n$ and analogues to Theorems 6 and 7 are derived. Finally in part C a theorem of Obrechkoff is discussed.

PART A.

THEOREM 8. *Let* (6.1) *be regular and bounded in S_2, and B-lim $a_n=0$. Then B-$\sum a_n = s$ if $\lim_{z\to 1-0} f(z)=s$.*

Proof. Since $z=1$ lies on the boundary of the region of B'-summability, B'-$\sum a_n z^n$ exists for $0<z<1$, and we have for these values of z

$$f(z) = B'\text{-}\sum a_n z^n = \int_0^\infty e^{-t}\sum \frac{a_n(tz)^n}{n!}\,dt = \frac{1}{z}\int_0^\infty e^{-t/z}\phi(t)dt$$
$$= (\zeta+1)\int_0^\infty e^{-t\zeta}\psi(t)dt = (\zeta+1)I(\zeta) \quad \left(\zeta = \frac{1}{z} - 1\right),$$

where we let $\psi(t) = e^{-t}\phi(t) = e^{-t}\sum a_n\ (t^n/n!) = B(t;\, a_n)$ for $t \geqq 0$. The integral $I(\zeta)$ exists for $\zeta > 0$ and has the limit s for $\zeta \to 0$, so that by a Tauberian theorem for Laplace-integrals (see for example [6, p. 164]) it is sufficient to show that $\int^x \psi(t)dt$ is a slowly oscillating function. We shall show that

$$\lim_{x\to\infty} \int_x^{x+\epsilon(x)x} \psi(t)dt = 0 \tag{6.2}$$

for any positive function $\epsilon(x)$ tending to zero as $x \to \infty$. For (6.2) implies that $I(0) = s$, i.e. that $B'\text{-}\sum a_n = s$, which is equivalent to $B\text{-}\sum a_n = s$ since B-lim $a_n = 0$.

We now prove (6.2). Denote by $C = \sum_{i=1}^{6} C_i$ the path as indicated in Fig. 2, and let $|f(z)/z| < M$ for z on C. (Let $f(1) = s$; the points z_2 and z_3 should be chosen such that $f(z)$ is regular on $C - C_1$, thereafter they are fixed.) Then for $t > 0$

$$\begin{aligned} B(t;\, a_n) = \psi(t) &= \frac{1}{2\pi i} e^{-t}\sum\left(\int_C \frac{f(z)dz}{z^{n+1}}\,\frac{t^n}{n!}\right) \\ &= \frac{1}{2\pi i} e^{-t}\int_C \frac{f(z)}{z}\sum\frac{(t/z)^n}{n!}\,dz = \frac{1}{2\pi i}\int_C \frac{f(z)}{z}\,e^{-t(1-1/z)}dz, \end{aligned} \tag{6.3}$$

so that

$$\begin{aligned} \int_x^{x+\epsilon(x)x}\psi(t)dt &= \frac{1}{2\pi i}\int_x^{x+\epsilon(x)x}\int_{C_1+\cdots+C_6}\frac{f(z)}{z}\,e^{-t(1-1/z)}dzdt \\ &= \frac{1}{2\pi i}\,[I_1(x) + \cdots + I_6(x)]. \end{aligned}$$

Divide C_1 into its four rectilinear segments. Then we have for instance

$$\left|\int_1^{1+\delta_1 e^{i\theta_1}} \frac{f(z)}{z}\,e^{-t(1-1/z)}dz\right| < M\int_0^\infty e^{-\sigma t y}dy = \frac{M}{\sigma t},$$

since for z on $(1,\, 1+\delta_1 e^{i\theta_1})$ we have $\Re(1-1/z) \geqq \sigma|1-z|$ for a constant $\sigma > 0$. With similar estimations for the other parts of C_1 we obtain

$$\int_{C_\cdot} \frac{f(z)}{z}\,e^{-t(1-1/z)}dz = O\left(\frac{1}{t}\right) \qquad (t \to \infty). \tag{6.4}$$

and hence

$$I_1(x) = \int_x^{x+\epsilon(x)x} O\left(\frac{1}{t}\right) dt = O\left(\frac{1}{x}\right) \cdot \epsilon(x)x = o(1) \qquad (x \to \infty).$$

For the estimation of $I_3(x)$ and $I_5(x)$ note that for instance on C_3 we have $\Re(1-1/z) \geqq \tau |z_2 - z|$ for a constant $\tau > 0$, so that by similar computations as in the case of $I_1(x)$ we obtain $|I_3(x)| + |I_5(x)| = o(1)$ $(x \to \infty)$. For z on C_4 we have $\Re(1-1/z) \geqq \eta > 0$, so that

$$\left| \int_{C_4} \frac{f(z)}{z} e^{-t(1-1/z)} dz \right| < 2\pi M e^{-t\eta} = O\left(\frac{1}{t}\right) \qquad (t \to \infty),$$

whence by the same reasoning as above $I_4(x) = o(1)$ $(x \to \infty)$.

Finally we estimate $I_2(x)$ and $I_6(x)$.

$$\begin{aligned} I_2(x) &= \int_x^{x+\epsilon(x)x} \int_{C_2} \frac{f(z)}{z} e^{-t(1-1/z)} dz dt = \int_{C_2} \int_x^{x+\epsilon(x)x} \frac{f(z)}{z} e^{-t(1-1/z)} dt dz \\ &= \int_{C_2} \frac{f(z)}{z-1} e^{-x(1-1/z)} dz - \int_{C_2} \frac{f(z)}{z-1} e^{-x(1+\epsilon(x))(1-1/z)} dz, \end{aligned}$$

and similarly for $I_6(x)$. Hence it is sufficient to show that

$$\lim_{x \to \infty} \int_{C_2 + C_6} \frac{f(z)}{1-z} e^{-x(1-1/z)} dz = 0. \tag{6.5}$$

For this purpose we choose the constants b_1, b_2, b_3, b_4 such that

$$zH(z) = \frac{1}{1-z} + \frac{b_1}{z} + \frac{b_2}{z} \cdot e^{(1-1/z)} + \frac{b_3}{z} \cdot e^{2(1-1/z)} + \frac{b_4}{z} \cdot e^{3(1-1/z)}$$

vanishes at z_1, z_2, z_3, z_4. Then we get

$$\begin{aligned} &\int_{C_2+C_6} zH(z) f(z) e^{-x(1-1/z)} dz \\ &= \int_{C_2+C_6} \frac{f(z)}{1-z} e^{-x(1-1/z)} dz + b_1 \int_C \frac{f(z)}{z} e^{-x(1-1/z)} dz + \cdots \\ &\qquad + b_4 \int_C \frac{f(z)}{z} e^{-(x-3)(1-1/z)} dz \\ &\qquad - b_1 \int_{C_1+C_3+C_4+C_5} \frac{f(z)}{z} e^{-x(1-1/z)} dz - \cdots \\ &\qquad - b_4 \int_{C_1+C_3+C_4+C_5} \frac{f(z)}{z} e^{-(x-3)(1-1/z)} dz. \end{aligned} \tag{6.6}$$

Using (6.4) and the corresponding estimations for the paths C_3, C_4, and C_5, we see that the last four terms in (6.6) tend to zero for $x \to \infty$, while B-lim $a_n = 0$ implies by (6.3) that the second to fifth terms tend to zero for $x \to \infty$. Hence it remains to show that

$$\lim_{x \to \infty} \int_{C_2+C_6} zH(z)f(z)e^{-x(1-1/z)}dz = 0. \tag{6.7}$$

Now the function $f(z)$ has the representation

$$f(z) = \frac{1}{z}\int_0^\infty e^{-t/z}\phi(t)dt \qquad (z \in \mathfrak{R}_B),$$

where the Laplace-integral converges also at the regular points z_1, z_2, z_3, z_4 (see [16, pp. 18–19](7)) and therefore uniformly on C_2+C_6 by a familiar Abelian theorem on Laplace-integrals. The integral in (6.7) becomes therefore

$$\int_0^\infty \int_{C_2+C_6} H(z)e^{(t-x)(1-1/z)}e^{-t}\phi(t)dzdt = \int_0^\infty \psi(t)\int_{C_2+C_6} H(z)e^{(t-x)(1-1/z)}dzdt,$$

which is an integral-transformation of the function $\psi(t)$ which tends to zero for $t \to \infty$. This transform tends to zero for $x \to \infty$ if for any fixed $t_1, t_2 > 0$

$$\lim_{x \to \infty} \int_t^{t_2} | c(x, t) | dt = 0 \quad \text{and} \quad \limsup_{x \to \infty} \int_0^\infty | c(x, t) | dt < \infty, \tag{6.8}$$

where

$$c(x, t) = \int_{C_2+C_6} H(z)e^{(t-x)(1-1/z)}dz \qquad (t \geqq 0, x \geqq 0).$$

Finally the relations (6.8) are proved in the same manner as (3.7) and (3.8) in §3, first integrating twice by parts and then estimating $c(x, t)$. This concludes the proof of Theorem 8.

It should be noted that Theorem 8 combines two theorems analogous to the Theorems A and B of the introduction. We mention the following case.

COROLLARY. *Assume that $z=1$ lies on the boundary of the region of B-summability of $f(z) = \sum a_n z^n$ and that $f(z)$ is regular at $z=1$. Then B-$\sum a_n$ exists if B-lim $a_n = 0$.*

Sometimes it is useful to have "B-lim $a_n = 0$" substituted by an assumption on $f(z)$.

(7) This theorem of M. Riesz states: *In $J(w) = \int_0^\infty e^{-tw}\phi(t)dt$ let $\int_0^x \phi(t)dt = o(e^{xc})$ $(c > 0)$ $(x \to \infty)$ so that $J(w)$ is regular for $\Re(w) > c$. If $J(w)$ is regular at w_0 with $\Re(w_0) = c$, then $\int_0^\infty e^{-tw_0}\phi(t)dt$ converges.* Here we put $w = 1/z$, $c = 1$, and $w_0 = 1$; the assumption $e^{-x}\int_0^x \phi(t)dt \to 0$ $(x \to \infty)$ is clearly fulfilled since B-lim $a_n = 0$ and therefore $\phi(t) = o(e^t)$ $(t \to \infty)$.

THEOREM 9. *Let $f(z)=\sum a_n z^n$ be regular in $|z|<r\,(r>0)$ and in $\mathfrak{K}_B$ and assume that $f(z)$ belongs to the class H^1 of the circle $\mathfrak{K}_B$, i.e. that*

$$\int_{-\pi}^{+\pi}\left|f\left(\frac{1}{2}+Re^{i\theta}\right)\right|d\theta \leqq K \qquad \text{for } 0<R<\frac{1}{2}. \tag{6.9}$$

Then B-$\lim a_n=0$.

Proof. Again we have

$$2\pi i B(x;a_n)=\int_C \frac{f(z)}{z}e^{-x(1-1/z)}dz \qquad (C=C_1+\cdots+C_4).$$

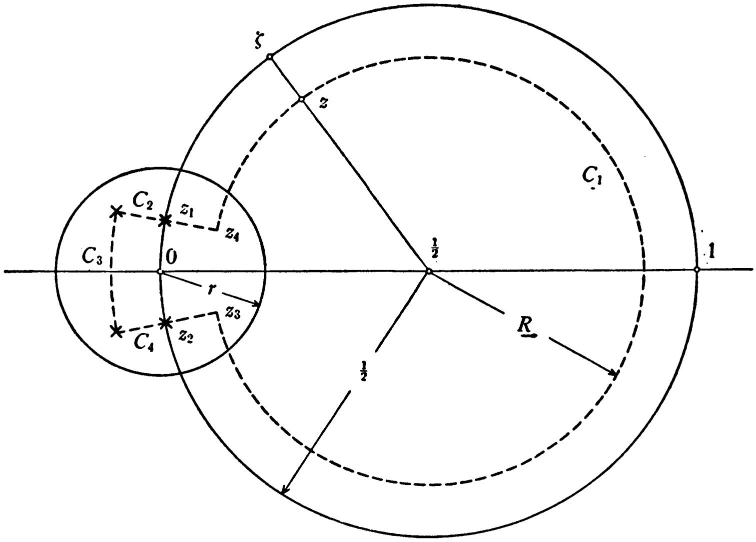

FIG. 3

Given an $\epsilon>0$, we choose C_2 and C_4 so short that

$$\int_{C_2+C_4}\left|\frac{f(z)}{z}\right||dz|<\frac{\epsilon}{3}\cdot$$

Then we have

$$\left|\int_{C_2+C_4}\frac{f(z)}{z}e^{-x(1-1/z)}dz\right|\leqq\int_{C_2+C_4}\left|\frac{f(z)}{z}\right||dz|<\frac{\epsilon}{3} \qquad (x\geqq 0).$$

Furthermore

$$\left|\int_{C_3}\frac{f(z)}{z}e^{-x(1-1/z)}dz\right|=O(e^{-x\eta})<\frac{\epsilon}{3} \qquad (x\geqq x_1),$$

since for z on C_3 we have $\Re(1-1/z) \geqq \eta > 0$. Finally we estimate

$$(6.10) \qquad I(x) = \int_{C_1} \frac{f(z)}{z} e^{-x(1-1/z)} dz.$$

Let x be fixed for the moment and put $z(\theta) = 1/2 + Re^{i\theta}$, $\zeta(\theta) = 1/2 + (1/2)e^{i\theta}$, $z_4 = z(\alpha)$. Then $\lim_{R \to 1/2} f(z(\theta))$ exists for almost all θ in $(-\pi, +\pi)$ and represents there a Lebesgue-integrable function $f(\zeta(\theta))$ such that

$$(6.11) \qquad \int_{-\pi}^{+\pi} | f(z(\theta)) - f(\zeta(\theta)) | d\theta \to 0 \qquad \text{for } R \to \frac{1}{2}$$

(see for example [18, p. 162]). Therefore in

$$I(x) = \int_{-\alpha}^{+\alpha} \frac{f(z(\theta)) - f(\zeta(\theta))}{z(\theta)} e^{-x(1-1/z(\theta))} dz(\theta) + \int_{-\alpha}^{+\alpha} \frac{f(\zeta(\theta))}{z(\theta)} e^{-x(1-1/z(\theta))} dz(\theta)$$
$$+ \left(\int_{z_4}^{z_1} + \int_{z_2}^{z_3} \right) \frac{f(z)}{z} e^{-x(1-1/z)} dz$$

the first and third terms tend to zero as $R \to 1/2$, while the second term tends to

$$\int \frac{f(\zeta)}{\zeta} e^{-x(1-1/\zeta)} d\zeta,$$

for $R \to 1/2$, the integral being taken as Lebesgue-integral over the arc: $(z_2, 1, z_1)$ of $|\zeta - 1/2| = 1/2$. Substituting $w = 1/\zeta - 1$, we get therefore

$$I(x) = \int_{-w_0}^{+w_0} f^*(w) e^{wx} dw$$

where w is purely imaginary and $f^*(w)$ is Lebesgue-integrable on the finite section $\langle -w_0, +w_0 \rangle$ of the imaginary axis. By the Riemann-Lebesgue theorem,

$$| I(x) | < \epsilon/3 \qquad (x \geqq x_2),$$

which completes the proof of Theorem 9.

It is a well known fact that a power series $f(z) = \sum a_n z^n$ with radius of convergence 1 is B-summable at $z = 1$ if $f(z)$ is regular at $z = 1$. If we combine Theorems 8 and 9 we obtain the following sharper result:

A power series $f(z) = \sum a_n z^n$ with radius of convergence 1 is B-summable at $z = 1$ if $f(z)$ is regular and bounded in S_2 and $\lim_{z \to 1-0} f(z)$ *exists.*

The result of the author [5, p. 331] mentioned previously in §3 shows furthermore that herein *the region S_2 cannot be replaced by a region whose boundary has an osculation of second order with $|z| = 1$ at $z = 1$.*

PART B. We give first a necessary and sufficient condition for B-summability.

THEOREM 10. *Let $f(z)=\sum a_n z^n$ be regular in $|z|<r$ $(r>0)$ and in $\mathfrak{K}_B$ and assume that $f(z)$ belongs to the class H^1 of the circle $\mathfrak{K}_B$, i.e. that (6.9) is fulfilled. Then B-$\sum a_n = s$ if and only if*

$$\lim_{x\to\infty}\frac{1}{\pi}\int_{-\tau}^{+\tau}\frac{\sin xt}{t}\,f\left(\frac{1}{1+it}\right)dt = s \tag{6.12}$$

for every fixed $\tau>0$, the integral being taken as Lebesgue-integral.

REMARK. Karamata has proved this theorem in the case $r=1$ and under the assumption that $f(z)$ is bounded in $\mathfrak{K}_B$ [8, pp. 156–157], but his method is not applicable under the more general assumptions of Theorem 10.

Proof. We have

$$2\pi i B(x; s_n) = \int_{C_1+\cdots+C_4}\frac{f(z)}{z(1-z)}\,e^{-x(1-1/z)}dz = I_1(x)+\cdots+I_4(x),$$

where $C=\sum_{i=1}^4 C_i$ is again the path of Fig. 3. As in the proof to Theorem 9 one finds that for a given $\epsilon>0$

$$|I_2(x)|+|I_3(x)|+|I_4(x)|<\epsilon/2 \qquad (x\geqq x_1). \tag{6.13}$$

Consider the auxiliary function

$$H(x) = \int_{C_1}\frac{f(z)}{z(1-z)}\,e^{x(1-1/z)}dz.$$

On the segments (z_4, z_1) and (z_2, z_3) of C_1 we have $\Re(1-1/z)\leqq 0$ and therefore

$$\left|\left(\int_{z_4}^{z_1}+\int_{z_2}^{z_3}\right)\frac{f(z)}{z(1-z)}\,e^{x(1-1/z)}dz\right| < \frac{\epsilon}{8} \qquad (x\geqq 0)$$

if only $|z_4-z_1|<\delta=\delta(\epsilon)$, and since on the remaining part of C_1 the estimation $\Re(1-1/z)\leqq-\eta<0$ holds,

$$|H(x)|<\epsilon/4 \qquad (x\geqq x_2). \tag{6.14}$$

We now estimate

$$I_1(x) = \int_{C_1}\frac{f(z)}{z}\left[\frac{e^{-x(1-1/z)}-e^{x(1-1/z)}}{1-z}\right]dz + H(x) = I(x)+H(x)$$

by a method similar to the one used in the proof of Theorem 9. The bracketed term is regular for all $z\neq 0$ and therefore $I(x)$ can be treated as $I(x)$ in (6.10). Hence

$$I(x) = \int\frac{f(\zeta)}{\zeta}\left[\frac{e^{-x(1-1/\zeta)}-e^{x(1-1/\zeta)}}{1-\zeta}\right]d\zeta,$$

the integral being taken as Lebesgue-integral along the arc: $(z_2, 1, z_1)$ of

$|\zeta-1/2|=1/2$. Letting $\zeta=1/(1+it)$ (t real), we get

$$I(x) = 2i\int_{-T}^{+T} \frac{\sin xt}{t} f\left(\frac{1}{1+it}\right)dt,$$

where $z_2=1/(1+iT)$. By the Riemann-Lebesgue theorem,

$$\left|2\left(\int_{-T}^{-\tau} + \int_{+\tau}^{+T}\right)\right| < \frac{\epsilon}{4} \qquad (x \geqq x_3), \tag{6.15}$$

and hence from (6.13)–(6.15) it follows that for $x \geqq \max(x_1, x_2, x_3)$

$$\left|2\pi i B(x; s_n) - 2i\int_{-\tau}^{+\tau} \frac{\sin xt}{t} f\left(\frac{1}{1+it}\right)dt\right| < \epsilon,$$

which proves Theorem 10.

We prove now the analogues to Theorems 6 and 7 in the case of B-summability.

THEOREM 6′. *Assume that* (6.1) *is continuous in* $\overline{\mathfrak{R}_B}$. *Then* $C_\epsilon B\text{-}\sum a_n$ *exists for every* $\epsilon>0$.

REMARK. Since $C_\epsilon B$ and BC_ϵ are equivalent methods [2, p. 45], Theorem 6′ contains the result of Karamata [8, p. 157] who proved the BC_ϵ-summability of $\sum a_n$ under the assumption that $f(z)$ is regular in $|z|<1$ and continuous in $\overline{\mathfrak{R}_B}$.

Proof. For z in $\mathfrak{R}_B$ we have $f(z) = (1/z)\int_0^\infty e^{-t/z}\phi(t)dt$, and by a theorem of Riesz [16, p. 20] the integral is C_ϵ-summable for $z=1$, i.e. $C_\epsilon B'\text{-}\sum a_n=s$, provided that

$$\lim_{x\to\infty} \frac{e^{-x}}{x^\epsilon}\int_0^x (x-t)^\epsilon\phi(t)dt = 0. \tag{6.16}$$

But by Theorem 9 we have B-lim $a_n=0$, i.e. $\phi(t)=\epsilon(t)\cdot e^t$, where $\epsilon(t)\to 0$ $(t\to\infty)$. It is easily shown that the integral transformation with the generating function $c(x, t)=e^{t-x}x^{-\epsilon}(x-t)^\epsilon$ $(0<t\leqq x, \epsilon>0)$ transforms every function tending to zero for $t\to\infty$ into one tending to zero for $x\to\infty$, so that (6.16) holds. Now the relation $B(x; s_n)=B(x; a_n)+B'(x; s_n)$ implies $C_\epsilon B(x; s_n) = C_\epsilon B(x; a_n)+C_\epsilon B'(x; s_n)$, and the last two terms tend to zero and to s respectively. This proves Theorem 6′.

If the continuity of $f(z)$ for $z\to 1$ in $\mathfrak{R}_B$ is sufficiently strong, we obtain

THEOREM 7′. *Assume that* (6.1) *belongs to the class* H^1 *of the circle* $\mathfrak{R}_B$, *i.e. that* (6.9) *is fulfilled,*

$$f(z) = s + o((1-z)^\eta) \qquad (\eta > 0),$$

uniformly for $z\to 1$ *in* $\mathfrak{R}_B$. *Then* $B\text{-}\sum a_n=s$.

Proof. Without loss of generality we may assume $s=0$, so that for the limit function $f(\zeta)$ existing almost everywhere on $|\zeta-1/2|=1/2$ we have $|f(\zeta)|=|f(1/(1+it))|\leqq A|t|^{\eta}$ for $|t|<t_0$ where $A>0$ is a constant. Given an $\epsilon>0$, we may choose $\tau<t_0$ in (6.12) so small that $\int_0^\tau t^{\eta-1}dt<\epsilon\pi/2A$. (If (6.12) holds for some $\tau>0$ it holds for all $\tau>0$ by the Riemann-Lebesgue theorem.) For this τ we obtain

$$\left|\frac{1}{\pi}\int_{-\tau}^{+\tau}\frac{\sin xt}{t}f\left(\frac{1}{1+it}\right)dt\right|\leqq\frac{2A}{\pi}\int_0^\tau\frac{dt}{t^{1-\eta}}<\epsilon$$

for all $x\geqq 0$, so that (6.12) is fulfilled.

PART C. This section is concluded with the discussion of a theorem of Obrechkoff [14, p. 1813].

Suppose we have given a power series $f(z)=\sum a_nz^n$ with positive radius of convergence. Let $z=1$ be a singular point of $f(z)$ lying in the interior of L where L is a rectilinear part of the Borel-polygon associated with $f(z)$(8). Assume that the singularity of $f(z)$ at $z=1$ is such that in the region $S_2=S_2(\delta_0,\theta_0)$ the function $f(z)$ is regular and $|1-z|^{\delta}|f(z)|$ is bounded for some $\delta(0<\delta<1)$.

The theorem of Obrechkoff states that under these assumptions the series $\sum a_nz_0^n$ is B-summable for every regular point $z_0=R_0e^{i\phi_0}$ in the interior of L.

Obrechkoff's proof of this theorem is valid only under restricted conditions. If the circle K: $|z-z_0/2|=R_1(R_1>R_0/2)$ is drawn and $A_1=1+a_1e^{i\theta_0}$ and $A_2=1+a_2e^{-i\theta_0}$ are the two points lying on K with $|\text{arc } (z-1)|=\theta_0$, his proof depends on the fact that a_1 and a_2 tend to zero if R_1 approaches $R_0/2$. This, however, is true if and only if the circle $|z-z_0/2|<R_0/2$ is contained in the region $|\text{arc } (z-1)|>\theta_0$, i.e. if and only if

$$|\text{arc } z_0|+\theta_0\leqq\pi/2. \tag{6.17}$$

Therefore the theorem of Obrechkoff remains valid if either z_0 is close enough to $z=1$, or if θ_0 may be chosen small enough such that (6.17) holds. (Note that arc $(z_0-1)=\pm\pi/2$ since $z=1$ is in the interior of L; therefore $|\text{arc } z_0|<\pi/2$.)

But if (6.17) holds, the region $|z-z_0/2|<R_0/2$ is contained in the region $|\text{arc } (z-1)|>\theta_0$. Therefore

$$\overline{\lim}\sup|z-1|^{\delta}|f(z)|\leqq M\quad\text{for}\quad z\to 1\ \text{in}\left|z-\frac{z_0}{2}\right|<\frac{R_0}{2}\qquad(0<\delta<1),$$

whilst $f(z)$ remains bounded for the other part of $|z-z_0/2|<R_0/2$; this is because z_0 is an interior point of L and therefore $z=1$ is the only singularity of $f(z)$ on $|z-z_0/2|=R_0/2$. Theorem 9 assures now that B-lim $a_nz_0^n=0$ and hence by Theorem 8 or its corollary the existence of B-$\sum a_nz^n$ follows. This

(8)By "interior of L" we mean that $z=1$ is not a corner of the Borel-polygon.

is a new proof of Obrechkoff's theorem in its modified form.

Added January 28, 1953. It has been investigated by Garten and Karamata [Math. Zeit. vol. 40 (1936) pp. 756–759 and vol. 45 (1939) pp. 635–641] under what conditions on a_n

$$B-\sum a_n = s \text{ implies } B-\sum \alpha_n = s \quad (\alpha_0 = a_0 + a_1, \alpha_n = a_{n+1}, n=1, 2, \cdots). \tag{6.18}$$

Their restrictions on the series [$a_n = o(n^k)$ ($n \to \infty$, k fixed) and $a_n = o(e^{n^\rho})$ ($n \to \infty$, $\rho < 1/3$), respectively] are such that the associated function $f(z) = \sum a_n z^n$ is necessarily regular in $|z| < 1$. It should be noted that if $f(z)$ does not fulfill this condition, Theorem 9 may be applicable, since B-lim $a_{n+1} = 0$ is necessary and sufficient for the validity of (6.18) (see [6, p. 183]).

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