

SIMULTANEOUS PARTITIONINGS OF TWO SETS⁽¹⁾

BY

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1. Introduction and definitions. In [5]⁽²⁾ R. H. Bing introduced the concept of partitioning a set. This concept has been applied by him and others as a means of obtaining many notable results [1; 2; 3; 4; 5; 6; 7; 8; 9]. In this paper an extension of this concept to the case of two sets is considered.

We shall consider only subsets of a metric space, particularly continuous curves and similar sets. A set M is partitionable if for each $\epsilon > 0$, there exists a finite collection $G = \{g_1, \dots, g_k\}$ of disjoint open connected subsets of M , such that $\sum_{i=1}^k g_i$ is dense in M and $\delta(g_i) < \epsilon$ for each i . The mesh of G equals $\max \delta(g_i)$, $1 \leq i \leq k$. Each g_i is an element of G , written $g_i \in G$. G is called an ϵ -partitioning of M . If each element of G has property S , G is called an S , ϵ -partitioning of M . $\{G_i\}$ is a decreasing sequence of S -partitionings of M if (1) for each i , G_i is an S -partitioning of M and each element of G_i is contained in an element of G_{i-1} (that is, G_i is a refinement of G_{i-1}), and (2) the limit, as i increases without limit, of the mesh of G_i is 0. Suppose H and G are two partitionings of a set M , G a refinement of H . Let $h \in H$ and let g_1, g_2, \dots, g_k be all the elements of G which are contained in h . g_i is called an interior element of G if $\bar{g}_i \subset h$. Otherwise g_i is a border element of G . G is a core refinement of H if for each $h \in H$ the sum of the closures of all interior elements of G contained in h is connected and intersects the closure of each border element of G contained in h .

Let M and N be two partitionable sets such that $N \subset M$. If $G = \{g_1, g_2, \dots, g_k\}$ is a partitioning of M such that $G' = \{g_1N, g_2N, \dots, g_kN\}$ constitutes a partitioning of N , then G is called a simultaneous partitioning of M and N . If g_i and g_iN both have property S for each i , G is a simultaneous S -partitioning of M and N . If $\delta(g_i) < \epsilon$ for each i , G is a simultaneous S , ϵ -partitioning of M and N . $\{G_i\}$ is a decreasing sequence of simultaneous core partitionings of M and N if the following conditions are satisfied. (1) $\{G_i\}$ is a decreasing sequence of S -partitionings of M , and if $g \in G_i$ then gN has property S . (2) G_i is a core refinement of G_{i-1} for each i . (3) Let $g \in G_{i-1}$ and let $g_1, g_2, \dots, g_k, g_{k+1}, g_{k+2}, \dots, g_{k+n}$ be the elements of G_i which are contained in g and which intersect N where $\bar{g}_i \subset g$ if $i \leq k$ and $\bar{g}_i \not\subset g$ if $i > k$. Then $\sum_{i=1}^k \text{Cl}(g_iN)$ is connected and intersects $\text{Cl}(g_iN)$ for each i from $k+1$ to

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(²) Numbers in brackets refer to the Bibliography at the end of the paper.

$k+n$. The question arises, what results regarding partitionings have analogies in the case of simultaneous partitionings?

2. Preliminary results. First let us recall some facts which have been previously established. An important one which characterizes partitionable sets is: Theorem A. A set M is partitionable if and only if it has property S. This means that if we wish to partition a set M into elements which themselves can be further partitioned, then these elements must have property S. Thus a fundamental result (Theorem B) is that if M is a partitionable set and ϵ is an arbitrary positive number, then there exists an S, ϵ -partitioning of M . A consequence of this is that if M is partitionable, then there exists a decreasing sequence of S -partitionings of M (Theorem C). Of course many other results have been obtained concerning partitionings, and many applications of this concept have been found.

LEMMA 1. *Let M be a continuous curve, U an open partitionable connected subset of M , and H_1, H_2, \dots, H_k a finite number of connected disjoint closed subsets of U with property S. Then there exists a finite collection $\{W_1, W_2, \dots, W_k\}$ of disjoint connected open subsets of U with property S, whose sum is dense in U , with $W_i \supset H_i$ for each i .*

Proof. Let G_1 be an $S, 1$ -partitioning of U . Let $D = \sum_{i=1}^k H_i$. Let $\{p_i\}$, $i=1, 2, \dots, n$, be a collection of points obtained by selecting a point from each element of G_1 . Let A_1, A_2, \dots, A_n be n arcs in U such that $A_i \supset p_i$ for each i , each A_i intersects exactly one component of D , and $A_i A_j = 0$ if A_i and A_j intersect different components of D . For each i from 1 to k , let $S_1(H_i)$ equal H_i plus all the above arcs which intersect H_i .

Let $S'_1(H_i) = \{x | x \in S_1(H_i) \text{ and } \rho(x, \text{bdy } U) \geq 1\}$. There exists a positive number d_1 such that $\rho(S_1(H_i), S'_1(H_j)) > d_1$ if $i \neq j$. About each point of $S'_1(H_i)$ there exists an open connected set with property S containing p of diameter less than $\min(1, d_1/3)$. The closure of each such set thus is contained in U . A finite number of these sets cover $S'_1(H_i)$. Let $T'_1(H_i)$ equal the sum of their closures. Let $T_1(H_i) = T'_1(H_i) \cup S_1(H_i)$. Do this for each i from 1 to k . Then each $T_1(H_i)$ is a closed connected partitionable subset of U and $T_1(H_i) \cdot T_1(H_j) = 0$ if $i \neq j$.

Let G_2 be an $S, 1/2$ -partitioning of U which is a refinement of G_1 . Let $D_1 = \sum_{i=1}^k T_1(H_i)$. Proceed in a manner exactly similar to that described in the above two paragraphs with only the added condition that each new arc must be contained in an element of G_1 . Thus sets $T_2(H_i)$, $i=1, 2, \dots, k$ are obtained which are each closed partitionable connected subjects of U and such that $T_2(H_i) \cdot T_2(H_j) = 0$ if $i \neq j$.

Proceed similarly to obtain sets $T_n(H_i)$, $i=1, 2, \dots, k$; $n=3, 4, \dots$, using S -partitionings G_n of mesh less than $1/2^{n-1}$. Define $W_i = \sum_{n=1}^{\infty} T_n(H_i)$ for each i . All the conclusions are clearly satisfied except possibly that each W_i has property S. To prove this, recall $W_i = \sum_{n=1}^{\infty} T_n(H_i)$. $T_{n+1}(H_i)$ is

obtained from $T_n(H_i)$ by adding a finite number of arcs, each of diameter less than $1/2^{n-2}$ and a finite number of connected sets with property S each of diameter less than $1/(2^{n-2})$. Hence any point in $T_{n+1}(H_i)$ can be joined to $T_n(H_i)$ by a connected set in $T_{n+1}(H_i)$ of diameter less than $1/(2^{n-3})$. Given $\epsilon > 0$ take $\epsilon/3$. Since $\sum_{n=1}^{\infty} 1/2^n = 1$, there exists a positive integer N such that each point of W_i can be joined to $T_N(H_i)$ by a connected set in W_i of diameter less than $\epsilon/3$. Since $T_N(H_i)$ has property S it is the sum of a finite number of connected sets each of diameter less than $\epsilon/3$, say C_1, C_2, \dots, C_m . Let C'_i equal C_i plus all points of W_i which can be joined to it by a connected set of diameter less than $\epsilon/3$ which lies in W_i . Then the diameter of C'_i is less than ϵ and $\sum_{i=1}^n C'_i = W_i$, thus proving that W_i has property S.

LEMMA 2. *Let U be an open partitionable connected subset of a continuous curve M , and V a closed partitionable connected subset of U . Let H and K be two sets in U such that $\rho(H, K) > 0$. Then H and K can be expanded into larger closed subsets of U , H' , and K' , such that $\rho(H', K') > 0$, and $H', K', H'V, K'V$ all have property S.*

Proof. Let $\rho(H, K) = \epsilon > 0$. Let G (G') be an $S, \epsilon/6$ -partitioning of U (V). Let H' equal H plus the closures (relative to U) of all elements of G whose closures intersect H . Define K' similarly. Let H'' equal H' plus the closures (relative to V) of all elements of G' whose closures intersect H' . Define K'' similarly. Then $\rho(H'', K'') > 0$ and $H'', K'', H''V$, and $K''V$ all have property S.

THEOREM 1. *Let U be an open connected partitionable subset of a continuous curve M , V a closed partitionable subset of U , and H and K two subsets of U such that $\rho(H, K) > 0$. Then there exists a collection $\{W_1, W_2, \dots, W_n\}$ of open disjoint connected subsets of U with property S whose sum is dense in U , such that no W_i intersects both H and K but each W_i intersects either H or K , and such that each W_iV is either void or an open connected subset of V with property S, and $\sum_{i=1}^n W_iV$ is dense in V .*

Proof. We may clearly suppose that H and K are closed subsets of U . It is also no loss in generality to assume that $(H+K)$ intersects each component of V . To prove this, let C be a component of V which contains no point of $(H+K)$. Then there exists an arc A in U intersecting C and one of the sets H and K but lying at a positive distance from the other set. A may be added to the set which it intersects. This procedure may be repeated until each component of V contains a point of $(H+K)$. Also by Lemma 2 we may assume that H and K are closed subsets of U such that that H, K, HV , and KV all have property S. Let $\{H_i\}, i=1, 2, \dots, n, \{K_i\}, i=1, 2, \dots, m, \{A_i\}, i=1, 2, \dots, r, \{B_i\}, i=1, 2, \dots, s$, be the components of H, K, HV , and KV respectively.

Let G_1 be an $S, 1$ -partitioning of V . Let $\{p_i\}, i=1, 2, \dots, n_1$, be a col-

lection of points obtained by selecting a point from each element of G_1 . Let $\{Z_i\}$, $i=1, 2, \dots, n_1$, be a collection of arcs in V such that Z_i contains p_i for each i , each Z_i intersects exactly one component of $(HV+KV)$, and $Z_i Z_j=0$ if Z_i and Z_j intersect different components of $(HV+KV)$. Let A'_i (or B'_i or H'_i or K'_i) equal A_i (or B_i or H_i or K_i) plus all the arcs Z_i which intersect A_i (or B_i or H_i or K_i). Let $H' = \sum_{i=1}^{n_1} H'_i$ and $K' = \sum_{i=1}^{m_1} K'_i$.

Let G'_1 be an S , 1-partitioning of U . Let $\{q_i\}$, $i=1, 2, \dots, m_1$, be the set of points obtained by selecting a point from each of the elements of G'_1 which does not intersect V . Join q_i to some component of $(H'+K')$ by an arc entirely in $U-V$ except possibly for an end point if this is possible. If not, join q_i to V by an arc intersecting V at only one point, say y , and then join y by an arc in V to some component of $(\sum_{i=1}^r A'_i + \sum_{i=1}^s B'_i)$. This can be done in such a way that we have a finite family of arcs N_1, N_2, \dots, N_{m_1} , such that N_i contains q_i for each i , each N_i intersects exactly one component of $(H'+K')$ and at most one component of $(\sum_{i=1}^r A'_i + \sum_{i=1}^s B'_i)$, and such that $N_i N_j=0$ if N_i and N_j intersect different components of $(H'+K')$ or different components of $(\sum_{i=1}^r A'_i + \sum_{i=1}^s B'_i)$. For each i from 1 to r let $S_1(A_i)$ equal A'_i plus the intersection of V with all the above arcs which intersect A'_i . Similarly define $S_1(B_i)$ for each i from 1 to s . For each i from 1 to n (m) let $S_1(H_i)$ ($S_1(K_i)$) equal H'_i (K'_i) plus all the above arcs which intersect H'_i (K'_i).

Let $E_1 = \{x | x \in (\sum_{i=1}^r S_1(A_i) + \sum_{i=1}^s S_1(B_i)) \text{ and } \rho(x, \text{bdy } U) \geq 1\}$. This is a compact set. Let $\{D_i\}$, $i=1, 2, \dots, r+s$, be the components of $(\sum_{i=1}^r S_1(A_i) + \sum_{i=1}^s S_1(B_i))$. Let d_1 be a positive number such that $\rho(E_1 D_i, D_j) > d_1$ if $i \neq j$. Around each point p of E_1 there exists an open connected subset of V containing p with property S , of diameter less than $\min(1/2, d_1/3)$. The closure of each such set thus lies in V . A finite number of such sets cover E_1 . Let $T_1(A_i)$ equal $S_1(A_i)$ plus the closures of all these finite number of sets which intersect $S_1(A_i)$, for each i from 1 to r . Define $T_1(B_i)$ similarly for each i from 1 to s . Then each $T_1(A_i)$ and each $T_1(B_i)$ is a connected partitionable closed subset of V , and no two of these sets intersect.

Let $F_1 = \{x | x \in (\sum_{i=1}^n S_1(H_i) + \sum_{i=1}^m S_1(K_i)) \text{ and } \rho(x, \text{bdy } U) \geq 1\}$. Again F_1 is compact. Let $\{D'_i\}$, $i=1, 2, \dots, n+m$, be the components of $(\sum_{i=1}^n S_1(H_i) + \sum_{i=1}^m S_1(K_i))$. Let d'_1 be a positive number such that $\rho(F_1 D'_i, D'_j) > d'_1$ if $i \neq j$. About each point p of F_1 there exists an open partitionable connected subset of U containing p of diameter less than $\min(1/2, d'_1/3)$, whose closure lies in U and does not intersect $V - (\sum_{i=1}^r T_1(A_i) + \sum_{i=1}^s T_1(B_i))$. A finite number of such sets cover F_1 . For each i from 1 to n let $T_1(H_i)$ equal $S_1(H_i)$ plus the closures of all these finite number of sets which intersect $S_1(H_i)$, plus all components of $(\sum_{i=1}^r T_1(A_i) + \sum_{i=1}^s T_1(B_i))$ which intersect $S_1(H_i)$. Define $T_1(K_i)$ similarly for each i from 1 to m . Note that each $T_1(H_i)$ and each $T_1(K_i)$ is a closed connected

partitionable subset of U and no two of these sets intersect.

Let G_2 be an $S, 1/2$ -partitioning of V which is a refinement of G_1 . Proceed in exactly the same manner as in the second paragraph of this proof to obtain sets $T'_1(A_i), i=1, 2, \dots, r, T'_1(B_i), i=1, 2, \dots, s, T'_1(H_i), i=1, 2, \dots, n,$ and $T'_1(K_i), i=1, 2, \dots, m,$ with only the added restriction that each arc concerned must be contained in some element of G_1 and hence be of diameter less than 1.

Let G'_2 be an $S, 1/2$ -partitioning of U which is a refinement of G'_1 . Proceed in a manner similar to that described in the third paragraph of this proof to obtain sets $S_2(A_i), S_2(B_i), S_2(H_i), S_2(K_i),$ with only the added condition that if M_i is one of the arcs concerned then $M_i V$ is contained in the closure of some element of G_1 and $M_i - V$ is contained in some element of G'_1 .

Let $E_2 = \{x | x \in (\sum_{i=1}^r S_2(A_i) + \sum_{i=1}^s S_2(B_i)) \text{ and } \rho(x, \text{bdy } U) \geq 1/2\}$. Let $\{X_i\}, i=1, 2, \dots, r+s,$ be the components of $(\sum_{i=1}^r S_2(A_i) + \sum_{i=1}^s S_2(B_i))$. Let d_2 be a positive number such that $d_2 < \rho(X_i, X_j E_2)$ if $i \neq j$. Let $\eta_2 = \min(1/4, d_2/3)$. About each point p of E_2 there exists an open connected partitionable subset of V of diameter less than η_2 containing p . A finite number of these sets cover E_2 . Let $T_2(A_i)$ equal $S_2(A_i)$ plus the closures of all these finite number of sets which intersect $S_2(A_i)$, for each i from 1 to r . Define $T_2(B_i)$ similarly for each i from 1 to s .

Let $\{Y_i\}, i=1, 2, \dots, n+m,$ be the components of $(\sum_{i=1}^n S_2(H_i) + \sum_{i=1}^m S_2(K_i))$ and let d'_2 be a positive number such that $d'_2 < \rho(Y_i, Y_j F_2)$ if $i \neq j$ where $F_2 = \{x | x \in (\sum_{i=1}^n S_2(H_i) + \sum_{i=1}^m S_2(K_i)) \text{ and } \rho(x, \text{bdy } U) \geq 1/2\}$. Let $\eta'_2 = \min(1/4, d'_2/3)$. About each point p of F_2 there exists an open connected partitionable subset of U which contains p and is of diameter less than η'_2 , whose closure lies in U and does not intersect $V - (\sum_{i=1}^r T_2(A_i) + \sum_{i=1}^s T_2(B_i))$. A finite number of these sets cover F_2 . For each i from 1 to n , let $T_2(H_i)$ equal $S_2(H_i)$ plus the closures of all these finite number of sets which intersect $S_2(H_i)$ plus all components of $(\sum_{i=1}^r T_2(A_i) + \sum_{i=1}^s T_2(B_i))$ which intersect $S_2(H_i)$. Define $T_2(K_i)$ similarly for each i from 1 to m .

Thus we obtain sets $T_2(A_i), i=1, 2, \dots, r, T_2(B_i), i=1, 2, \dots, s, T_2(H_i), i=1, 2, \dots, n, T_2(K_i), i=1, 2, \dots, m,$ all of which are closed partitionable connected subsets of U . Moreover each $T_2(A_i)$ is a component of $(\sum_{i=1}^r T_2(A_i) + \sum_{i=1}^s T_2(B_i))$ as is each $T_2(B_i)$. Also each $T_2(H_i)$ is a component of $(\sum_{i=1}^n T_2(H_i) + \sum_{i=1}^m T_2(K_i))$, as is each $T_2(K_i)$.

This procedure is repeated using S -partitionings $G_3, G_4, \dots, G_3, G'_4, \dots,$ each G_i (or G'_i) being a refinement of G_{i-1} (or G'_{i-1}), and each G_i (or G'_i) being of mesh less than $1/2^{i-1}$. Define $R_i = \sum_{j=1}^{\infty} T_j(H_i), i=1, 2, \dots, n, S_i = \sum_{j=1}^{\infty} T_j(K_i), i=1, 2, \dots, m, Q_i = \sum_{j=1}^{\infty} T_j(A_i), i=1, 2, \dots, r, L_i = \sum_{j=1}^{\infty} T_j(B_i), i=1, 2, \dots, s.$

Then the sets R_i and S_i are each open subsets of U since each point of R_i for example is an interior point of some $T_n(H_i)$. Also each of these sets is connected and their sum Y is dense in U and has property S for the same

reasons that each W_i in Lemma 1 has property S . Each R_i and each S_i is a component of Y and hence has property S also. Similarly each Q_i and each L_i is an open connected subset of V with property S . Also $(\sum_{i=1}^r Q_i + \sum_{i=1}^s L_i)$ is dense in V and actually $\sum_{i=1}^r Q_i = (\sum_{i=1}^n R_i) \cdot V$ and $\sum_{i=1}^s L_i = (\sum_{i=1}^m S_i) \cdot V$ so that each Q_i (L_i) is a closed subset of some R_j (S_j).

By Lemma 1, there exist sets $W_1, W_2, \dots, W_{r_1}, r_1 \geq r$, in $\sum_{i=1}^n R_i$ such that $W_i \cdot V = Q_i$ for $1 \leq i \leq r$ and $W_i \cdot V = 0$ if $i > r$, $\sum_{i=1}^{r_1} W_i$ is dense in $\sum_{i=1}^n R_i$ and each W_i is an open connected subset of U with property S . Similarly there exist sets $W_{r_1+1}, \dots, W_{r_1+s_1}, s_1 \geq s$, such that $W_{r_1+i} \cdot V = L_i$ for $1 \leq i \leq s$ and $W_{r_1+i} \cdot V = 0$ if $i > s$, $\sum_{i=1}^{s_1} W_{r_1+i}$ is dense in $\sum_{i=1}^m S_i$ and each W_{r_1+i} is an open connected subset of U with property S .

3. The fundamental theorems.

THEOREM 2. *Let M be a compact partitionable set and N a closed partitionable subset. Then for every $\epsilon > 0$ there exists a simultaneous S, ϵ -partitioning of M and N .*

Proof. It may easily be shown that it is sufficient to prove the theorem for the case when M and N are both connected (and hence continuous curves).

Given $\epsilon > 0$ take $\epsilon/4$. Pick a finite set of points, $\{p_i\}, i=1, 2, \dots, n$, such that if $p \in M$ then $D(p, p_i) < \epsilon/4$ for some i . Define $A_i = \{x \mid x \in M \text{ and } D(x, p_i) \leq \epsilon/4\}$, and $B_i = \{x \mid x \in M \text{ and } D(x, p_i) \geq \epsilon/2\}$. Then the sets A_i and B_i are at positive distance apart for each i . In particular $\rho(A_1, B_1) > 0$. By Theorem 1 there exist disjoint open connected subsets of M , say V_1, V_2, \dots, V_m , such that no V_i intersects both A_1 and B_1 but each V_i intersects $(A_1 + B_1)$, $\sum_{i=1}^m V_i$ is dense in M and has property S , each $V_i \cdot N$ is either void or an open connected subset of N with property S and $\sum_{i=1}^m V_i \cdot N$ is dense in N . Clearly if $V_i \cdot A_1 \neq 0, \delta(V_i) < \epsilon$. Let X_1 (Y_1) be the sum of those V_i which intersect A_1 (B_1).

Let V_n be an arbitrary component of Y_1 . Then $V_n \cdot A_1 = 0$. Consider $V_n \cdot A_2$ and $V_n \cdot B_2$. If $V_n \cdot N$ is not void these two sets satisfy the conditions of Theorem 1 for H and K with $U = V_n$ and $V = V_n \cdot N$ and so we can apply the theorem again. If $V_n \cdot N = 0$, it can be S, ϵ -partitioned (Theorem 4, p. 1104 of [5]). The procedure can be repeated until the components of Y_1 are exhausted. This gives us a simultaneous S -partitioning G of Y_1 and $Y_1 \cdot N$. Let X_2 equal the sum of all elements of G which are of diameter less than ϵ . Let Y_2 equal the sum of the remaining elements of G . Proceeding similarly we obtain sets $X_3, Y_3, \dots, X_{n-1}, Y_{n-1}$. Then the components of $(Y_{n-1} + \sum_{i=1}^{n-1} X_i)$ are the elements of a simultaneous S, ϵ -partitioning of M and N .

The following results may be obtained in the same manner and will be used several times.

THEOREM 3. *Let M and N be two continuous curves such that $N \subset M$ and let U be an open partitionable connected subset of M such that UN is connected*

and partitionable. Then for every $\epsilon > 0$ there exists a simultaneous S, ϵ -partitioning of U and UN .

THEOREM 4. *Let M be a compact partitionable set and N a closed partitionable subset. Then there exists a sequence $\{G_i\}$ of simultaneous S -partitionings of M and N such that G_i is a refinement of G_{i-1} of mesh less than $1/i$.*

4. Remarks and examples. Theorem 2 would follow immediately (using Lemma 1) from Theorem B if for every S, ϵ -partitioning G of M and every element g of G which intersects N , gN had property S . However this is not the case. Very simple examples can be found where G is an S -partitioning of M , $g \in G$, and gN does not have property S . These and other examples show that it is not possible to prove Theorem 2 by working with M only or with N only at first, but rather the sets N and M must be considered simultaneously as was done. These examples can easily be constructed and so are not included.

A COUNTER EXAMPLE. One might wonder if the restriction that N must be contained in M could be removed. That is, consider the following theorem.

FALSE STATEMENT D. *Let M and N be two arbitrary continuous curves, ϵ an arbitrary positive number, and U an open connected partitionable subset of $(M+N)$ such that UM and UN are both connected and partitionable. Then there exists an S, ϵ -partitioning $G = \{g_1, g_2, \dots, g_k\}$ of U such that $\{g_i M\}$, $i = 1, 2, \dots, k$, and $\{g_i N\}$, $i = 1, 2, \dots, k$, are S -partitionings of UM and UN respectively.*

It can easily be shown that a consequence of this theorem is that if p is an arbitrary point of $(M+N)$ and ϵ an arbitrary positive number, there exists an open connected set U such that $p \in U$, $\delta(U) < \epsilon$, and UM and UN are both connected and partitionable. However consider a straight line L in euclidean 3-space and points p and p_1 on L . Let p_2 be the mid-point of $[p, p_1]$ and in general p_i be the mid-point of $[p, p_{i-1}]$. Consider a sequence $\{q_i\}$ of points defined as follows. Points q_1, q_2, \dots, q_6 are the points, $p_{6k}, p_{6k-3}, p_{6k+2}, p_{6k-1}, p_{6k+4}, p_{6k+1}$, respectively, with $k=1$. The points q_7 to q_{12} are the points $p_{6k}, p_{6k-3}, p_{6k+2}, p_{6k-1}, p_{6k+4}, p_{6k+1}$, respectively, with $k=2$, and so on. Let $\{A_i\}$ be a sequence of arcs such that (1) the end points of A_i are q_i and q_{i+1} , (2) each pair of these arcs are disjoint except possibly for a common end point, and (3) $\rho(x, [q_i, q_{i+1}]) < 1/2^i$ if $x \in A_i$. Let $N = p + \sum_{i=1}^{\infty} A_i$ and let $M = [p, p_1]$. Then there do not exist arbitrarily small open subsets U of $(M+N)$ containing p such that UM and UN are both connected. The proof is omitted. This shows that Theorem D is indeed a false theorem and thus that the results of this paper are not valid for two arbitrary continuous curves.

5. Some additional theorems.

THEOREM 5. *Let p be an arbitrary point of a compact partitionable set M and let N be a closed partitionable subset of M and let ϵ be an arbitrary positive*

number. Then there exists in M an open partitionable connected set U such that $p \in U$, $\delta(U) < \epsilon$, and UN is either void or connected and partitionable.

Proof. (a) If $p \in (M - N)$ the proof is left to the reader. (b) Suppose $p \in N$. Given ϵ , take $\epsilon/2$. By Theorem 2 there exists a simultaneous S , $\epsilon/2$ -partitioning G of M and N . If p belongs to an element of G , that element serves as the set U . Otherwise let $\{g_i\}$, $i = 1, 2, \dots, k$, be the elements of G such that $p \in \text{Cl}(g_i)$. Let V be the interior of $\sum_{i=1}^k \text{Cl}(g_i)$. If VN is connected we are done. If not VN has a finite number of components and we may apply Lemma 1.

THEOREM 6. Let M be a compact partitionable set, N a closed partitionable subset, p an arbitrary point of M , and δ and ϵ any two positive numbers such that $\delta < \epsilon$. Then there exists an open set U such that U and UN both have property S and $A \subset U$ and $\rho(U, B) > 0$, where $A = \{x \mid x \in M \text{ and } D(x, p) \leq \delta/2\}$ and $B = \{x \mid x \in M \text{ and } D(x, p) \geq \epsilon/2\}$. (If B is void omit the conclusion $\rho(U, B) > 0$.)

Proof. If $B = 0$, let $U = M$. Otherwise let $\gamma = (\epsilon - \delta)/2$. Let G be a simultaneous S -partitioning of M and N of mesh less than $\gamma/3$. Let U equal the interior of the sum of the closures of all elements of G whose closures intersect A . This set is of the required type. U clearly has property S . UN also has property S because it is the sum of sets gN where $\bar{g}A \neq 0$, plus a subset of the limit points of this sum. The other conclusions of Theorem 6 are clearly satisfied.

THEOREM 7. Let M be a compact partitionable set and N a closed partitionable subset of M . Then there exists a decreasing sequence $\{G_i\}$ of simultaneous S -partitionings of M and N such that if g and h are two elements of G_i for which $gN \neq 0$, $hN \neq 0$, and $\rho(gN, hN) > 0$, then $\rho(g, h) > 0$.

Proof. We prove the theorem by showing that if partitionings G_1, G_2, \dots, G_{n-1} have been chosen so that the conclusions of Theorem 7 are satisfied for them, then G_n can be chosen of the required type.

Let H be any simultaneous S , $1/n$ -partitioning of M and N which is a refinement of G_{n-1} . Let $\{h_i\}$, $i = 1, 2, \dots, k$, be the elements of H which intersect N and $\{h_i\}$, $i = k+1, \dots, k+t_1$, be the remaining elements. Consider h_1 . If $\rho(h_1, h_i) = 0$ only if (a) $i > k$ or (b) $i \leq k$ and $\rho(h_1N, h_iN) = 0$, pass on to h_2 . However if, for some $i \leq k$, $\rho(h_1, h_i) = 0$ but $\rho(h_1N, h_iN) > 0$, let h_m be one such h_i for which $\rho(h_1N, h_iN)$ is a minimum. Consider a simultaneous S -partitioning H' of M and N which is a refinement of H of mesh less than $(1/4) \cdot \rho(h_1N, h_mN)$. For each i from 1 to k let h'_i equal the interior of the sum of the closures of all elements of H' whose closures intersect h_iN and let $\{h'_i\}$, $i = k+1, \dots, k+t_2$, be the remaining elements of H' . Then if $i \leq k$, $\rho(h'_i, h'_i) > 0$ unless $\rho(h'_iN, h'_iN) = 0$.

Now omitting h'_1 and considering only the remaining elements of $H'' = \{h'_i\}, i=1, 2, \dots, k+t_2$, treat h'_2 in a similar manner to the way in which h_1 was treated. After at most $k-1$ steps a partitioning is obtained which satisfies the conditions on G_n .

In fact this theorem may be strengthened so that if δ equals the minimum of $\rho(g, h)$ where g and h vary over all elements of G_i which intersect N and are such that $\rho(gN, hN) > 0$, then the diameter of g' is less than δ/H where H is an arbitrary positive integer, and g' is an arbitrary element of G_i which does not intersect N .

THEOREM 8 (CORE PARTITIONINGS). *Let M be a compact partitionable set and N a closed partitionable subset. Then there exists a decreasing sequence of simultaneous core partitionings $\{G_i\}$ of M and N .*

Proof. Let G_1 be a simultaneous $S, 1$ -partitioning of M and N , let $g \in G_1$ and let G'_1 be the partitioning of g consisting of the single element g . If $gN=0$, it has previously been shown [7 and 8, Theorem 3, p. 1119] that there exists a core refinement of G'_1 of mesh less than $1/2$. Thus suppose $gN \neq 0$.

Let H be a simultaneous $S, 1/2$ -partitioning of g and gN . Let T be a connected set in g such that $TN \neq 0$, T intersects every element of H , and $\rho(M-g, T) > 0$. If TN is not a connected set in gN which intersects each element h of H for which $hN \neq 0$, a set S may be obtained from T by adding a finite number of dendrons in gN chosen in such a way that SN is a connected set in gN which intersects each element h of H for which $hN \neq 0$. Note that $\rho(M-g, S) = \delta > 0$.

Let H' be a simultaneous S -partitioning of g and gN which is a refinement of H of mesh less than $\delta/3$ and which satisfies the following condition. If $h' \in H'$, $h'N \neq 0$, and $\rho(h', S) = 0$, then $\rho(h'N, S) = 0$ also. To show that H' exists we may consider an arbitrary simultaneous S -partitioning K of g and gN which is a refinement of H of mesh less than $\delta/3$. If there exists an element k of K such that $kN \neq 0$, $\rho(k, S) = 0$, and $\rho(kN, S) = \gamma > 0$, it is merely necessary to consider a simultaneous S -partitioning of k and kN of mesh less than $\gamma/3$. If this procedure is repeated for all elements of K similar to k , then the collection of all sets thus obtained plus all the unchanged elements of K constitutes a partitioning satisfying the conditions on H' .

Let C equal the sum of the closures of all elements of H' whose closures intersect S , say $C = \sum_{i=1}^k \text{Cl}(h'_i)$ and let $D = \sum_{i=1}^k \text{Cl}(h'_i N)$. Then C and D are both connected sets in g . C is connected because for each i from 1 to k , h'_i has a limit point in S . D is connected because for each i from 1 to k for which $h'_i N \neq 0$, $h'_i N$ has a limit point in SN .

Consider the border elements h_1, h_2, \dots, h_m of H which are contained in g . Consider an arbitrary component K of $(\sum_{i=1}^m h_i) - C$. Thus K will be contained in some element of H . If $KN=0$, leave K alone. If $KN \neq 0$, KN has a

finite number of components, X_1, X_2, \dots, X_n , and K may be partitioned into open disjoint connected partitionable subsets W_1, W_2, \dots, W_n such that $W_i \supset X_i$ for each i (Lemma 1). Treat each component of $(\sum_{i=1}^m h_i) - C$ in this manner. Let $\{R_i\}, i=1, 2, \dots, r$, and $\{S_i\}, i=1, 2, \dots, s$, be the sets thus obtained which have a boundary point in common with bdy g , where $R_i N = 0$ and $S_i N$ is nonvoid and connected. These sets plus all the sets obtained by the same process whose closures lie in g , plus all elements of H' which are contained in $(C + (g - \sum_{i=1}^m h_i))$, constitute a simultaneous $S, 1/2$ -partitioning G of g and gN . The border elements of G are $\{R_1, \dots, R_r, S_1, \dots, S_s\}$ and all the remaining elements are interior elements (border and interior elements with respect to G_1).

Suppose E equals the sum of the closures of all the interior elements of G , defined above, say $E = \sum_{i=1}^r \text{Cl}(g_i)$ and let $F = \sum_{i=1}^r \text{Cl}(g_i N)$. Then E and F are both connected, as we shall now prove. Suppose E is not connected. Let E' be a component of E which does not intersect C . Suppose E' contains an element g' of G which is contained in a border element h of H . If $g' N = 0$, g' is a component of $h - C$ and hence g' has a limit point in C , since h is connected. If $g' N \neq 0$, $g' N$ is a component of $(h - C)N$ and hence has a limit point in CN because hN is connected. In each case E' intersects C , contrary to assumption. E' cannot contain an element of H' which is contained in an interior element h of H for in this case E' would contain h and hence again intersect C , a contradiction. Hence E is connected. We may proceed in exactly the same manner to prove that F is connected. The same arguments show that if g'' is an arbitrary border element of G , then E contains a limit point of g'' and F contains a limit point of $g'' N$ unless the latter set is void.

Once G has been defined, this shows how the other elements of G_1 may be treated. By dealing with each of them in the same manner, we obtain a simultaneous $S, 1/2$ -partitioning G_2 of M and N of the required type. The method is general and so the proof is complete.

6. Unanswered questions. Consider the following two theorems which may possibly be true, and if so, might lead to further results.

CONJECTURE E. *Let M and N be two continuous curves such that $N \subset M$, U a connected, uniformly locally connected (ulc), open subset of M such that UN is ulc and connected, and ϵ an arbitrary positive number. Then there exists a simultaneous ϵ -partitioning G of U and UN , such that if $g \in G$ then g and gN both are ulc.*

This is a stronger result than Theorem 2 since for subsets of a compact metric space ulc implies property S but not conversely.

CONJECTURE F. *Let M and N be two continuous curves such that $N \subset M$, and ϵ an arbitrary positive number. Then there exists a simultaneous S, ϵ -partitioning G of M and N such that if $g \in G$ then $\bar{g}N = \text{Cl}(gN)$.*

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