ON INTERPOLATION TO A GIVEN ANALYTIC FUNCTION BY ANALYTIC FUNCTIONS OF MINIMUM NORM

BY

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We shall consider here the following problem. *Let the region R_1 of the z-plane contain the points*

$$\begin{array}{llll} & \beta_{11}, & & \\ (0.1) \qquad & \beta_{21}, & \beta_{22}, & \\ & \beta_{31}, & \beta_{32}, & \beta_{33}, \\ & \cdots & \cdots, & \end{array}$$

and let the function $f(z)$ be analytic in these points. To study the convergence to $f(z)$ of the sequence of functions $g_n(z)$; here $g_n(z)$ is analytic throughout R_1, coincides with $f(z)$ in the points $\beta_{n1}, \beta_{n2}, \cdots, \beta_{nn}$, and among all functions with these two properties has the least norm in R_1. This problem has been previously studied [6; 7] where norm is [lub $|g_n(z)|$, z in R_1], and is now to be studied (§1) where norm is measured by a surface integral over R_1, or (§2) a parametric integral over the boundary of R_1, or (§3) a line integral over the boundary of R_1. If the norm is measured by the integral of the square of the modulus, we obtain by this method an expansion of $f(z)$ in a series of orthogonal functions, an expansion whose convergence properties we study (§4) in some detail. The asymptotic behavior of these orthogonal functions themselves and of their zeros is investigated in §5.

1. Interpolation by functions of minimum norm, surface integrals. If R_1 is a given region, we define $\mathcal{L}^q(R_1)$ $(0<q<\infty)$ as the class of functions $F(z)$ each analytic in R_1 with $\iint_{R_1}|F(z)|^q dS<\infty$, and define $\mathcal{L}^\infty(R_1)$ as the class of functions $F(z)$ each analytic and bounded in R_1; otherwise expressed, the *norm* of $F(z)$ in R_1 in these respective cases is $[\iint_{R_1}|F(z)|^q dS]^{1/q}$ and its limit $(q\to\infty)$ lub $[|F(z)|$, z in $R_1]$. We define $\mathcal{L}_n^q(R_1)$ as the subclass of $\mathcal{L}^q(R_1)$ consisting of those functions of $\mathcal{L}^q(R_1)$ which coincide with the given $f(z)$ in the points $\beta_{n1}, \beta_{n2}, \cdots, \beta_{nn}$. The functions of class $\mathcal{L}_n^q(R_1)$ form a normal family in R_1, and standard methods show that there exists at least one such function $F_n(z)$ of minimum norm. The function $F_n(z)$ is unique if $1<q<\infty$, and also if $q=\infty$ and R_1 is simply connected.

If S is any point set, we denote its closure by $\overline{S}$. With the generic notation

$$V_\sigma^q(F) = \left[\iint_{R_\sigma} |F(z)|^q dS\right]^{1/q},$$

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where R_σ is to be defined, our main result can now be formulated:

THEOREM 1.1. *Let R_1 be a finite region whose boundary C_1 consists of a finite number of mutually disjoint Jordan curves. Let R_0 be a point set whose boundary C_0 consists of a finite number of mutually disjoint Jordan curves, such that $\overline{R}_0$ lies in R_1 and separates no point of $R_1-\overline{R}_0$ from C_1. Suppose the points* (0.1) *not necessarily distinct lie in $\overline{R}_0$, and that*

$$(1.1)\qquad \lim_{n\to\infty} \left|(z-\beta_{n1})(z-\beta_{n2})\cdots(z-\beta_{nn})\right|^{1/n} = e^{V_1(z)}$$

uniformly on any closed bounded set in the complement of $\overline{R}_0$. Let $V_2(z)$ be the function harmonic in R_1, continuous in $\overline{R}_1$, equal to $V_1(z)$ on C_1. Suppose the function $V(z)\equiv V_1(z)-V_2(z)$ is continuous and equal to $\gamma(<0)$ on C_0. With the notation $\phi(z)\equiv 1-V(z)/\gamma$, we denote generically by C_σ the locus $\phi(z)=\sigma$, $0\leqq\sigma\leqq 1$, in $\overline{R}_1-R_0$, and by R_σ the point set consisting of $\overline{R}_0$ plus those points of $R_1-\overline{R}_0$ for which $0<\phi(z)<\sigma$.

Let the given function $f(z)$ be analytic throughout R_ρ but not throughout any $R_{\rho'}$, $0<\rho<\rho'<1$. Then for fixed q, $0<q\leqq\infty$, the sequence of extremal functions $F_n(z)$ of class $\mathcal{L}_n^q(R_1)$ converges to $f(z)$ throughout R_ρ, uniformly on any closed subset of R_ρ, and we have for $0<t\leqq\infty$

$$(1.2)\qquad \limsup_{n\to\infty}\left[N_\sigma^t(f-F_n)\right]^{1/n} = e^{\gamma(\rho-\sigma)},\qquad 0\leqq\sigma<\rho;$$

we also have for $0<t\leqq\infty$ if $\rho\leqq\sigma<1$ and for $0<t\leqq q$ if $\sigma=1$

$$(1.3)\qquad \limsup_{n\to\infty}\left[N_\sigma^t(F_n)\right]^{1/n} = e^{\gamma(\rho-\sigma)}.$$

For $q=\infty$, Theorem 1.1 has already been established (loc. cit.) for $t=\infty$, and follows at once for $0<t<\infty$. We henceforth denote the extremal functions $F_n(z)$ for $q=\infty$ by $f_n(z)$, and shall employ the latter as comparison functions in our proof. We shall also use the following:

LEMMA 1.1. *Let R_0 and R_1 satisfy the conditions of Theorem 1.1, let $F(z)$ be an arbitrary function of class $\mathcal{L}^q(R_1)$, $0<q\leqq\infty$, and let T be any closed set interior to R_σ, $0\leqq\sigma\leqq 1$. Then we have*

$$|F(z_0)| \leqq K_\sigma N_\sigma^q(F),\qquad z_0 \text{ in } T$$

where the constant K_σ depends on q, σ, and T, but not on z_0 or $F(z)$.

Lemma 1.1 is a consequence of the principle of maximum modulus if $q=\infty$ and otherwise is not difficult to prove [5, p. 96].

To proceed with the proof of the theorem for $q<\infty$, let ϵ (>0) be given. By (1.3) for $q=\infty$, $t=\infty$, we have for n sufficiently large

$$[\text{lub } |f_n(z)|,\ z \text{ in } R_1] \leqq e^{[\gamma(\rho-1)+\epsilon]n},$$

whence for the extremal function $F_n(z)$

$$N_1^q(F_n) \leqq N_1^q(f_n) \leqq e^{[\gamma(\rho-1)+\epsilon]n} N_1^q(1), \tag{1.4}$$

and by the Hölder inequality

$$N_1^t(F_n) \leqq N_1^q(F_n)\, N_1^t(1)/N_1^q(1) \leqq e^{[\gamma(\rho-1)+\epsilon]n} N_1^t(1), \qquad 0 < t < q.$$

Thus the first member of (1.3) is not greater than the second member of (1.3) for $\sigma=1$.

To continue with the proof of Theorem 1.1 we introduce points $\alpha_{n1}, \alpha_{n2}, \cdots, \alpha_{nn}$, $n=1, 2, \cdots$, uniformly distributed on C_1 with respect to the conjugate function of $V(z)$ as parameter. It is then known [5, pp. 167–168; 7, pp. 47–48; 9, pp. 271–272] that we have

$$\lim_{n\to\infty} \left| (z-\alpha_{n1})(z-\alpha_{n2}) \cdots (z-\alpha_{nn}) \right|^{1/n} = e^{V_2(z)},$$

uniformly on any closed subset of R_1, from which there follows by (1.1)

$$\begin{aligned} &\lim_{n\to\infty} \left| r_n(z) \right|^{1/n} = e^{V(z)}, \\ &r_n(z) \equiv \frac{(z-\beta_{n1}) \cdots (z-\beta_{nn})}{(z-\alpha_{n1}) \cdots (z-\alpha_{nn})}, \end{aligned} \tag{1.5}$$

uniformly on any closed subset of $R_1 - \overline{R}_0$. A consequence of (1.5) for $0<\sigma<\sigma_1<1$ and for arbitrary ϵ (>0) is for n sufficiently large

$$\begin{aligned} &\left| r_n(z) \right| \leqq e^{[\gamma(1-\sigma)+\epsilon]n}, && z \text{ on } C_\sigma, \\ &\left| r_n(z) \right| \geqq e^{[\gamma(1-\sigma_1)-\epsilon]n}, && z \text{ on } C_{\sigma_1}. \end{aligned} \tag{1.6}$$

By use of the triangle inequality, by (1.4) and Lemma 1.1 we have

$$\left| F_n(z) - f_n(z) \right| \leqq K_1 e^{[\gamma(\rho-1)+\epsilon]n}, \qquad z \text{ on } C_{\sigma_1}.$$

But the function $[F_n(z)-f_n(z)]/r_n(z)$ has only removable singularities in the points β_{nk}, so by the principle of maximum modulus for the region R_{σ_1} and by (1.6) we have

$$\left| F_n(z) - f_n(z) \right| \leqq K_2 e^{[\gamma(\rho-\sigma+\sigma_1-1)+3\epsilon]n}, \qquad z \text{ in } R_\sigma. \tag{1.7}$$

Choose $0<\rho\leqq\sigma<1$. By (1.3) for the case $q=t=\infty$ already established we have

$$\left| f_n(z) \right| \leqq K_3 e^{[\gamma(\rho-\sigma)+\epsilon]n}, \qquad z \text{ in } R_\sigma,$$

whence by (1.7)

$$\left| F_n(z) \right| \leqq K_4 e^{[\gamma(\rho-\sigma+\sigma_1-1)+3\epsilon]n}, \qquad z \text{ in } R_\sigma,$$

so by the arbitrariness of σ_1 (<1) and ϵ (>0) the first member of (1.3) is not greater than the second member of (1.3).

We turn now to the case $0<\sigma<\rho$. If ϵ (>0) is arbitrary we have by (1.2) as already established for $q=t=\infty$

$$|f(z)-f_n(z)| \leqq K_5 e^{[\gamma(\rho-\sigma)+\epsilon]n}, \qquad z \text{ in } R_\sigma,$$

whence from (1.7)

$$|f(z)-F_n(z)| \leqq K_6 e^{[\gamma(\rho-\sigma+\sigma_1-1)+3\epsilon]n}, \qquad z \text{ in } R_\sigma; \tag{1.8}$$

thus the first member of (1.2) is not greater than the second member. For the case $\sigma=0$ we obtain this same conclusion from the case $\sigma>0$ by integrating the tth power of both members of (1.8) over R_0, taking the superior limit of the tnth root of the first member, and then allowing σ to approach zero. That $F_n(z)$ converges to $f(z)$ throughout R_ρ, uniformly on any closed subset of R_ρ, follows from (1.2) with $t=\infty$.

The fact that the first member of (1.2) is not less than the second member of (1.2) for $\sigma>0$ and that the first member of (1.3) is not less than the second member of (1.3) can now be proved by standard methods [7, p. 50], assuming the contrary, using Lemma 1.1 to study the sequence of functions involved on an auxiliary level locus C_{σ_1} near C_σ, $\sigma_1<\sigma$, and applying the two-constant theorem in $R_{\sigma_1}-\overline{R}_0$ to show that $f(z)$ is analytic throughout some $R_{\rho'}$, $\rho'>\rho$. The case $\sigma=0$ in (1.2) is exceptional here, but can be treated similarly by using an auxiliary set of Jordan curves C_0' interior to R_0 instead of the locus C_{σ_1}; when C_0' approaches C_0, the analog $\phi'(z)$ of $\phi(z)$ approaches [4] the function $\phi(z)$ uniformly throughout $\overline{R}_1-R_0$, and the reasoning already outlined applies in essence.

Theorem 1.1, whose proof is now complete, can be generalized by inserting positive continuous weight functions in the integrals of (1.2) and (1.3). Several corollaries of Theorem 1.1 can be readily proved by the reader:

COROLLARY 1.1. *For any sequence of functions* $\{F_n^*(z)\}$ *of the respective classes* $\mathcal{L}_n^q(R_1)$ *we have for* $q\leqq t\leqq\infty$

$$\limsup_{n\to\infty}\,[N_\sigma^t(F_n^*)]^{1/n} \geqq e^{\gamma(\rho-\sigma)}, \qquad \rho<\sigma\leqq 1.$$

COROLLARY 1.2. *Equality in* (1.2) *and* (1.3) *holds for every subsequence* $\{F_{n_k}(z)\}$ *with* $\limsup_{k\to\infty}(n_{k+1}-n_k)<\infty$.

COROLLARY 1.3. *For any sequence of functions* $\{F_{n_k}^*(z)\}$ *of the respective classes* $\mathcal{L}_{n_k}^q(R_1)$ *with* $\limsup_{k\to\infty}(n_{k+1}-n_k)<\infty$ *we have for* $q\leqq t\leqq\infty$

$$\limsup_{n_k\to\infty}\,[N_\sigma^t(F_{n_k}^*)]^{1/n_k} \geqq e^{\gamma(\rho-\sigma)}, \qquad \rho<\sigma\leqq 1.$$

COROLLARY 1.4. *Let the function* $f(z)$ ($\not\equiv 0$) *of Theorem* 1.1 *be analytic throughout* R_1. *Then the extremal functions* $F_n(z)$ *converge to* $f(z)$ *throughout* R_1, *uniformly on any closed subset of* R_1, *and we have for all* t (>0)

$$\limsup_{n\to\infty} [N_\sigma^t(f - F_n)]^{1/n} \leqq e^{\gamma(1-\sigma)}, \qquad 0 \leqq \sigma < 1,$$

and for $0 < t \leqq q$

$$\limsup_{n\to\infty} [N_1^t(F_n)]^{1/n} = 1.$$

2. **Interpolation by functions of minimum norm, parametric integrals.** We now modify the problem considered in §1 by replacing the surface integrals by integrals over the loci C_σ, $0 \leqq \sigma \leqq 1$, taken with respect to $\psi(z)$ as parameter, where $\psi(z)$ is the conjugate of $\phi(z)$ in $R_1 - \overline{R}_0$.

We employ the geometric situation and notation of Theorem 1.1. Let q be given, $0 < q \leqq \infty$. The function $G(z)$ is said to belong to the class $\mathfrak{M}^q$ if $G(z)$ is analytic in R_1 and if the norms

$$N_\sigma^q(G) \equiv \left[\int_{C_\sigma} |G(z)|^q d\psi(z)\right]^{1/q}, \qquad 0 \leqq \sigma < 1, \tag{2.1}$$

are bounded, where $N_\sigma^\infty(G)$ is interpreted as $[\max |G(z)|, z \text{ on } C_\sigma]$. The functions $G(z)$ of class $\mathfrak{M}^q$ have two important properties [11]:

A. For all points ζ of C_1, with the possible exception of a set of measure zero (with respect to ψ), $\lim G(z)$ exists when z in R_1 approaches ζ along a level curve of $\psi(z)$. Moreover for these limit values $\int_{C_1} |G(\zeta)|^q d\psi$ exists in the sense of Lebesgue, and $(0 < q < \infty)$

$$\lim_{\sigma\to 1} N_\sigma^q(G) = N_1^q(G),$$

where $N_1^q(G)$ is defined by the analog of (2.1). With $q = \infty$ analogous (Fatou) properties are well known.

B. The function $\log N_\sigma^q(G)$ is a convex function of σ, $0 \leqq \sigma \leqq 1$, in the sense

$$N_\sigma^q(G) \leqq [N_1^q(G)]^\sigma [N_0^q(G)]^{1-\sigma}.$$

Let $\mathfrak{M}_n^q$ denote that subclass of $\mathfrak{M}^q$ consisting of those functions which coincide with the given function $f(z)$ in the points $\beta_{n1}, \cdots, \beta_{nn}$. We prove later the existence of a function $G_n(z)$ of class $\mathfrak{M}_n^q$ of minimum norm.

THEOREM 2.1. *Under the hypothesis of Theorem* 1.1 *on* C_0, C_1, $f(z)$, ρ, *and the points* β_{nk}, *for given* q $(0 < q \leqq \infty)$ *the extremal functions* $G_n(z)$ *converge to* $f(z)$ *throughout* R_ρ, *uniformly on any closed subset of* R_ρ, *and we have for all* $t(>0)$ *with the notation* (2.1)

$$\limsup_{n\to\infty} [N_\sigma^t(f - G_n)]^{1/n} = e^{\gamma(\rho-\sigma)}, \qquad 0 \leqq \sigma < \rho, \tag{2.2}$$

and for all t (>0) *if* $\rho \leqq \sigma < 1$ *and for* $0 < t \leqq q$ *if* $\sigma = 1$,

$$\limsup_{n\to\infty} [N_\sigma^t(G_n)]^{1/n} = e^{\gamma(\rho-\sigma)}. \tag{2.3}$$

We shall need the following analog of Lemma 1.1:

LEMMA 2.1. *Let C_0 and C_1 satisfy the hypothesis of Theorem* 1.1, *let $G(z)$ be an arbitrary function of class $\mathfrak{M}^q$, and let T be an arbitrary closed subset of R_σ, $0<\sigma\leqq 1$, where no critical point of $\phi(z)$ lies on C_σ $(0<\sigma<1)$. Then we have*

$$|G(z_0)| \leqq K_\sigma N_\sigma^q(G), \qquad z_0 \text{ in } T, \tag{2.4}$$

where the constant K_σ depends on q, σ, and T, but not on z_0 or $G(z)$.

If $q=\infty$, the lemma follows from the maximum principle, so henceforth in the proof we assume $0<q<\infty$; we assume also without loss of generality $\sigma=1$. Under a smooth conformal map of R_1 onto a region bounded by a finite number of disjoint analytic Jordan curves the functions $\phi(z)$ and $\psi(z)$ are invariant, and hence N_σ^q is also invariant. Thus we may and do assume in the proof of Lemma 2.1 that C_1 consists of a finite number of mutually disjoint analytic Jordan curves; it follows that $\phi(z)$ and $\psi(z)$ are harmonic on C_1, and $\phi(z)$ has no critical points on C_1. Indeed, we suppose that $\phi(z)$ has no critical points also on the set $R_1-\overline{R}_{1-\eta}$, where η (>0) is suitably chosen, and suppose T interior to $R_{1-\eta}$.

For $1-\eta<\sigma<1$ the function $|G(z)|^q$ is subharmonic in R_σ, and we can write

$$|G(z_0)|^q \leqq \frac{1}{2\pi}\int_{C_\sigma} |G(z)|^q \frac{\partial g}{\partial \nu}\, ds, \qquad z_0 \text{ in } T, \tag{2.5}$$

where $g(z, z_0)$ is Green's function for the region R_σ with pole in z_0, and ν is the inner normal for R_σ. The normal derivatives $\partial g/\partial\nu$ are uniformly bounded for z_0 in T and for $1-\eta<\sigma<1$, as the reader may show. Moreover, the directional derivatives $\partial\phi/\partial\nu=-\partial\psi/\partial s$ are uniformly bounded from zero on C_σ, $1-\eta<\sigma<1$. From (2.5) we can write

$$|G(z_0)| \leqq K_1 N_\sigma^q(G),$$

where K_1 is independent of σ, $1-\eta<\sigma<1$. Approach of σ to unity now yields (2.4) for $\sigma=1$, so Lemma 2.1 is established.

We are now in a position to establish the existence of an extremal function of class $\mathfrak{M}_n^q$. It follows from Lemma 2.1 that a set of functions of class $\mathfrak{M}_n^q$ all of norm less than M $(<\infty)$ is normal in R_1, and the equation $G^k(z)\to G^0(z)$ uniformly on any closed set interior to R_1 implies

$$N_\sigma^q(G^0) \leqq \liminf N_\sigma^q(G^k), \qquad 0\leqq\sigma<1,$$

where the $G^k(z)$ are in $\mathfrak{M}_n^q$. We choose the $G^k(z)$ with

$$N_1^q(G^k) \to \inf \, [N_1^q(G), G \text{ in } \mathfrak{M}_n^q] = M_n,$$

and with $G^k(z) \to G^0(z)$ uniformly on any closed set interior to R_1. For arbitrary ϵ (>0) and for k sufficiently large we have $N_1^q(G^k) < M_n + \epsilon$, whence by Lemma 2.1 with $\sigma = 1$ and $T = C_0$

$$N_0^q(G^k) \leqq K_2(M_n + \epsilon).$$

Thus by property B

$$N_\sigma^q(G^k) \leqq (M_n + \epsilon)^\sigma [K_2(M_n + \epsilon)]^{1-\sigma}$$

so for σ sufficiently near unity we have

$$N_\sigma^q(G^k) \leqq M_n + 2\epsilon.$$

The corresponding inequality holds for G^0:

$$N_\sigma^q(G^0) \leqq M_n + 2\epsilon.$$

Hence G^0 is of class $\mathfrak{M}_n^q$, and we have

$$N_1^q(G^0) \leqq M_n,$$

but the strong inequality is impossible.

The extremal function $G_n(z)$ of class $\mathfrak{M}_n^q$ is unique if $1 < q < \infty$, for if two functions of class $\mathfrak{M}_n^q$ have the same norm, half their sum has a smaller norm.

The proof of Theorem 2.1 now follows directly the proof of Theorem 1.1 by using Lemma 2.1 (which applies for all σ (>0) with at most a finite number of exceptions) instead of Lemma 1.1. The impossibility of inequality instead of equality in (2.2) and (2.3) is proved using property B instead of the two-constant theorem.

3. **Interpolation by functions of minimum norm, line integrals.** We now modify the problem considered in §1 by replacing the surface integrals by line integrals over the loci C_σ, $0 \leqq \sigma \leqq 1$, with respect to arc length. With the hypothesis of Theorem 1.1, let the components of C_1 be rectifiable Jordan curves. For given q, $1 < q < \infty$, the function $H(z)$ is said to belong to the class $\mathcal{E}^q$ if $H(z)$ is analytic in R_1 and can be represented there by an integral

$$H(z_0) \equiv \frac{1}{2\pi i} \int_{C_1} \frac{H_1(z)dz}{z - z_0}, \qquad z_0 \text{ in } R_1, \tag{3.1}$$

where $H_1(z)$ is of class $\mathfrak{L}^q$, namely $H_1(z)$ is measurable and $|H_1(z)|^q$ is Lebesgue integrable with respect to arc length on C_1. The function $H_1(z)$ in (3.1) is not uniquely determined. The class $\mathcal{E}^q$ is similar to a class which was first introduced by Garabedian [3].

For each n let $\mathfrak{L}_n^q$ denote the subclass of $\mathfrak{L}^q$ consisting of all functions

$H_{n1}(z)$ corresponding to functions $H_n(z)$ in the representation (3.1) which coincide with the given $f(z)$ in the points $\beta_{n1}, \beta_{n2}, \cdots, \beta_{nn}$. The class $\mathcal{L}_n^q$ contains all polynomials which interpolate to $f(z)$ in these points, hence is nonempty. There exists a function $H_{n1}^*(z)$ of class $\mathcal{L}_n^q$, essentially unique on C_1, such that the norm

$$(3.2) \qquad N_1^q(H_{n1}^*) = \left[\int_{C_1} |H_{n1}^*(z)|^q |dz|\right]^{1/q}$$

is a minimum among all functions of class $\mathcal{L}_n^q$. This existence follows [3] by the completeness and uniform convexity [Clarkson, 2] of the class $\mathcal{L}^q$, and by the closure and convexity of the subclass $\mathcal{L}_n^q$. The uniqueness of $H_{n1}^*(z)$ follows easily; compare §2. We denote by $H_n^*(z)$ the function of class $\mathcal{E}^q$ which corresponds to $H_{n1}^*(z)$ by the analog of (3.1). Our main result, with notation N_σ^q analogous to (3.2), is

THEOREM 3.1. *Let C_0, C_1, β_{nk}, $f(z)$, and ρ satisfy the conditions of Theorem 1.1, and suppose in addition that C_0 and C_1 are rectifiable. Then for fixed q, $1<q<\infty$, the functions $H_n^*(z)$ of class $\mathcal{E}^q$ converge to $f(z)$ throughout R_ρ, uniformly on any closed subset of R_ρ, and we have for all t (>0)*

$$(3.3) \qquad \limsup_{n\to\infty} \left[N_\sigma^t(f - H_n^*)\right]^{1/n} = e^{\gamma(\rho-\sigma)}, \qquad 0 \leqq \sigma < \rho,$$

$$(3.4) \qquad \limsup_{n\to\infty} \left[N_\sigma^t(H_n^*)\right]^{1/n} = e^{\gamma(\rho-\sigma)}, \qquad \rho \leqq \sigma < 1,$$

and for $0<t\leqq q$

$$(3.5) \qquad \limsup_{n\to\infty} \left[N_1^t(H_{n1}^*)\right]^{1/n} = e^{\gamma(\rho-1)}.$$

The lemmas of §§1 and 2 have an analog here:

LEMMA 3.1. *Under the conditions of Theorem 3.1 on C_0 and C_1, let $H(z)$ be an arbitrary function of class $\mathcal{E}^q$ and $H_1(z)$ a function of class $\mathcal{L}^q$ corresponding to $H(z)$ in (3.1). If T is any closed set interior to R_σ, we have*

$$|H(z_0)| \leqq K_\sigma N_\sigma^q(H), \qquad z_0 \text{ in } T,\ 0 \leqq \sigma < 1,$$

$$|H(z_0)| \leqq K_1 N_1^q(H_1), \qquad z_0 \text{ in } T,\ \sigma = 1,$$

where the constant K_σ depends on q, σ, and T, but not on z_0, $H(z)$, or $H_1(z)$.

Lemma 3.1 follows by the Hölder inequality from Cauchy's integral formula for $0\leqq\sigma<1$ and from (3.1) for $\sigma=1$.

If the components of C_1 in Theorem 3.1 are analytic Jordan curves, the function $f_n(z)$ (notation of §1) is of class $\mathcal{E}^q$, for limit values of $f_n(z)$ bounded in their totality exist almost everywhere on C_1, and $f_n(z)$ is represented in R_1 by the corresponding Cauchy integral. The functions $f_n(z)$ can then be used as comparison functions, and the proof of Theorem 1.1 essentially applies in the present situation.

To prove Theorem 3.1 in the case that the components of C_1 are arbitrary rectifiable Jordan curves, we replace the points α_{nk} of §1 by suitably chosen new points α'_{nk} on a set of analytic Jordan curves C_1' exterior to R_1 but geometrically near C_1. The comparison functions $f_n(z)$ of §1 are replaced by rational functions $R_n(z)$ interpolating to $f(z)$ in the β_{nk} and with poles in the α'_{nk}. The curves C_1' can be so chosen [7, pp. 48–49] that the asymptotic properties of the α'_{nk} and of the $R_n(z)$ differ as little as we please from the asymptotic properties of the α_{nk} and of the $f_n(z)$ respectively. With these modifications, the method of proof of Theorem 1.1 is essentially valid to establish Theorem 3.1.

4. **Series of interpolation.** In the respective situations of Theorems 1.1, 2.1, and 3.1, with $q=2$ and β_{nk} independent of n, the general theory of orthogonal functions can be used; each theorem mentioned leads to a unique formal expansion of $f(z)$ which can be defined by interpolation to $f(z)$ in the points β_k. For definiteness we restrict ourselves in our detailed discussion to the situation and method of §1.

We assume for the present that each β_k lies in R_1, but do not assume (1.1). Let $\phi_n(z)$ be the function of class $\mathcal{L}^2(R_1)$ satisfying the conditions of interpolation

$$\phi_n(\beta_1) = \phi_n(\beta_2) = \cdots = \phi_n(\beta_{n-1}) = 0, \qquad \phi_n(\beta_n) = 1, \tag{4.1}$$

and which minimizes $N_1^2(\phi_n)$ over the class $\mathcal{L}^2(R_1)$. If the points $\beta_1, \beta_2, \cdots, \beta_n$ are not all distinct, equations (4.1) and later statements require special interpretation, as is customary in the theory of interpolation. Let $\phi_n^*(z)$ be defined as $\phi_n(z)/N_1^2(\phi_n)$. In the important case that all β_k are identical, these functions were introduced by Bergman [1]; in the case that the sequence $\{\beta_n\}$ approaches a limit, these functions were studied by Walsh and Davis [10]; existence and uniqueness of the $\phi_n^*(z)$ follow from §1; the fact that the $\phi_n^*(z)$ are mutually orthogonal over R_1 follows readily [1; 10; compare 12], as does the fact that $\phi_n^*(z)$ is orthogonal to any function of class $\mathcal{L}^2(R_1)$ which vanishes in the points $\beta_1, \beta_2, \cdots, \beta_n$. Of course $\phi_n^*(z)$ also is an extremal function, namely the function of class $\mathcal{L}^2(R_1)$ of norm unity which vanishes in $\beta_1, \beta_2, \cdots, \beta_{n-1}$ whose value in β_n is positive and maximum.

An arbitrary function $F(z)$ of class $\mathcal{L}^2(R_1)$ possesses two formal expansions in terms of the $\phi_n^*(z)$; the one is

$$F(z) \sim a_1\phi_1^*(z) + a_2\phi_2^*(z) + \cdots, \quad a_n = \iint_{R_1} F(z)\bar{\phi}_n^*(z)dS, \tag{4.2}$$

where the coefficients a_n are found by the usual orthogonal function (Fourier) method; the other is a series of interpolation

$$F(z) \sim b_1\phi_1^*(z) + b_2\phi_2^*(z) + \cdots, \tag{4.3}$$

where b_1 is determined by setting formally $z=\beta_1$ in (4.3), then b_2 is determined by setting formally $z=\beta_2$ in (4.3), etc. *These two formal expansions are identical*, independently of the completeness of the set of functions $\phi_n^*(z)$; for the proof compare the corresponding discussion for harmonic functions [12]. If the functions $\phi_n^*(z)$ form a complete set, for which it is sufficient that the points β_k have at least one limit point interior to R_1, the formal expansion (4.2) converges to $F(z)$ throughout R_1, uniformly on any closed subset of R_1; if the functions $\phi_n^*(z)$ do not form a complete set, the formal expansion (4.2) converges throughout R_1, uniformly on any closed subset of R_1, to the function of minimum norm which coincides with $F(z)$ in all the points β_k. If values $F(\beta_k)$ are given without the hypothesis of existence of a function $F(z)$ of class $\mathcal{L}^2(R_1)$ which takes those values in the respective points β_k, the formal expansion (4.3) still exists, and a necessary and sufficient condition for the existence of such a function $F(z)$ is the convergence of the series $\sum |b_n|^2$, where the b_n are defined as in (4.3); if this condition is satisfied, the series (4.3) converges in the mean (of order two) in R_1, thus converges throughout R_1, uniformly on any closed subset of R_1, and to the function of class $\mathcal{L}^2(R_1)$ of least norm which takes on the prescribed values in the points β_k. In particular the sum of the first n terms of the series in (4.3) is the function of $\mathcal{L}^2(R_1)$ of least norm which takes on the prescribed values in the points $\beta_1, \beta_2, \cdots, \beta_n$.

We now return to the hypothesis (1.1), and shall prove the validity of an interpolation series expansion under suitable conditions for functions not necessarily of class $\mathcal{L}^2(R_1)$.

THEOREM 4.1. *Let R_0, R_1, C_0, C_1, and the points β_k (independent of n) satisfy the conditions of Theorem* 1.1. *If $f(z)$ is analytic throughout R_σ, then the expansion*

$$f(z) = \sum_{n=1}^{\infty} a_n\phi_n^*(z), \tag{4.4}$$

where the a_n are determined formally by interpolation in the points β_k, is valid in R_σ, uniformly on any closed subset of R_σ.

If $f(z)$ is analytic throughout R_ρ, $0<\rho<1$, but not throughout any $R_{\rho'}$, $\rho'>\rho$, then we have

$$\limsup_{n\to\infty} |a_n|^{1/n} = e^{\gamma(\rho-1)}, \tag{4.5}$$

and if $f(z)$ is analytic throughout R_1, the first member of (4.5) *is not greater than unity. Conversely, if the second member of* (4.4) *is given with* (4.5) *valid, $0<\rho<1$, the series converges throughout R_ρ, uniformly on any closed subset of R_ρ, to a function $f(z)$ analytic throughout R_ρ but not analytic throughout any $R_{\rho'}$, $\rho'>\rho$; this function $f(z)$ has* (4.4) *as its formal expansion found by inter-*

polation in the points β_n. Likewise if the a_n are given with the first member of (4.5) *not greater than unity, the series in* (4.4) *converges throughout R_1, uniformly on any closed subset of R_1, to a function $f(z)$ analytic throughout R_1; this function $f(z)$ has* (4.4) *as its formal expansion found by interpolation in the points β_n.*

If functional values $f(\beta_n)$ are given without assuming the existence of $f(z)$ other than in the points β_n, a formal development (4.4) *exists; equation* (4.5), *with $0<\rho<1$, is a necessary and sufficient condition for the existence of a function $f(z)$ taking on the prescribed values in the points β_n, analytic interior to R_ρ but not analytic interior to any $R_{\rho'}$, $\rho'>\rho$; likewise that the first member of* (4.5) *be not greater than unity is a necessary and sufficient condition for the existence of a function $f(z)$ taking on the prescribed values in the points β_n, analytic throughout R_1.*

We have already remarked that (notation of §1)

$$F_n(z) \equiv \sum_{k=1}^{n} a_k \phi_k^*(z),$$

so the validity of (4.4) follows from Theorem 1.1.

To study the numbers a_n, we write

$$a_n = \iint_{R_1} F_n(z)\bar{\phi}_n^*(z)dS, \quad |a_n| \leqq N_1^2(F_n),$$

by Schwarz's inequality. It follows from (1.3) and Corollary 1.4 that the first member of (4.5) is not greater than the second member, even if $\rho=1$.

To prove equality in (4.5) we need to consider the asymptotic behavior of the $\phi_n^*(z)$. By Lemma 1.1 the functions $|\phi_n^*(z)|$ are uniformly bounded by some K_1 on any closed subset T of R_1. If $r_n(z)$ has the meaning of §1, for given ϵ (>0) and for $0<\sigma<\sigma_1<1$, we have for n sufficiently large

$$|r_{n-1}(z)| \geqq e^{[\gamma(1-\sigma_1)-\epsilon](n-1)}, \qquad z \text{ on } C_{\sigma_1},$$

$$|r_{n-1}(z)| \leqq e^{[\gamma(1-\sigma)+\epsilon](n-1)}, \qquad z \text{ on } C_\sigma.$$

The function $\phi_n^*(z)/r_{n-1}(z)$ is analytic in R_1 when suitably defined in the points β_k, so by the maximum principle

$$|\phi_n^*(z)| \leqq K_1 e^{[\gamma(\sigma_1-\sigma)+2\epsilon](n-1)}, \qquad z \text{ in } \overline{R}_\sigma.$$

Thus we may write ($\epsilon\to 0$, $\sigma_1\to 1$)

$$\limsup_{n\to\infty} [\max |\phi_n^*(z)|, z \text{ on } \overline{R}_\sigma]^{1/n} \leqq e^{\gamma(1-\sigma)}, \qquad 0<\sigma<1. \tag{4.6}$$

Since $f(z)$ is analytic throughout no $R_{\rho'}$, $\rho'>\rho$, the series in (4.4) can converge uniformly throughout no $R_{\rho'}$, so we deduce from (4.6) and from the part of (4.5) already proved the validity of (4.5).

The remainder of Theorem 4.1 follows without difficulty by continued use of (4.6). Theorem 4.1 was previously established [10] by a somewhat different method for the case that β_k approaches a point of R_1.

Theorem 4.1 considers series of interpolation which are series of orthogonal functions according to the orthogonality of §1; the corresponding discussions for orthogonality as measured in §§2 and 3 presents no difficulty; proofs of the precise analogs of Theorem 4.1 are left to the reader.

5. **Asymptotic behavior of orthonormal functions and of their zeros.** Methods previously used by the present authors [13] in the study of zeros of extremal polynomials apply also in the study of the extremal functions $\phi_n^*(z)$ and their analogs.

If the function $U(z)$ is harmonic in a region R, and if each of the functions $h_n(z)$ is locally single-valued and analytic in R except perhaps for branch points, with $|h_n(z)|$ single-valued in R, we say that $U(z)$ is a *harmonic majorant* of the sequence $\{h_n(z)\}$ in R if for every continuum Q (not a single point) in R we have

$$\limsup_{n\to\infty} [\max |h_n(z)|, z \text{ on } Q] \leqq [\max e^{U(z)}, z \text{ on } Q]; \tag{5.1}$$

and $U(z)$ is an *exact harmonic majorant* of the sequence $\{h_n(z)\}$ in R if (5.1) is replaced by *equality* of the two members, for every continuum Q (not a single point) in R.

Again we devote our attention to the situation of §1, but the methods and results appy equally to the situations of §§2 and 3.

THEOREM 5.1. *Under the conditions of Theorem* 4.1 *the function* $U(z) \equiv \gamma[1-\phi(z)]$ *is an exact harmonic majorant of the sequence* $[\phi_n^*(z)]^{1/n}$ *and of every subsequence in* $R_1-\overline{R}_0$.

Equality holds in (4.6) for every σ, $0<\sigma<1$; for suppose the strong inequality to hold for some σ, say

$$|\phi_n^*(z)| \leqq K_1 e^{\gamma n(1-\sigma_1)}, \qquad z \text{ on } C_\sigma,\ 0<\sigma_1<\sigma; \tag{5.2}$$

then we choose ρ, $\sigma_1<\rho<\sigma$, and set $a_n=e^{\gamma n(\rho-1)}$. The series $\sum a_n\phi_n^*(z)$ converges to a function analytic throughout $\overline{R}_\sigma$, in contradiction to Theorem 4.1. It now follows [8, Corollary 2 to Theorem 4] that $U(z)$ is an exact harmonic majorant of the sequence $[\phi_n^*(z)]^{1/n}$ in $R_1-\overline{R}_0$.

If $U(z)$ is not an exact harmonic majorant for every subsequence of the sequence $h_n(z)$, the strong inequality must hold [8, Corollary to Theorem 1] in (5.1) for some subsequence of the $h_n(z)$, for every Q in R. If $\gamma[1-\phi(z)]$ is not an exact harmonic majorant in $R_1-\overline{R}_0$ for every subsequence of the $[\phi_n^*(z)]^{1/n}$, inequality (5.2) holds for some subsequence $\phi_{n_k}^*(z)$, and we reach a contradiction as before by setting $a_{n_k}=e^{\gamma n_k(\rho-1)}$, $a_n=0$ if $n\neq n_k$. Theorem 5.1 is established.

We now state two immediate results [8, Theorem 16 and Remark 1; compare also 13].

THEOREM 5.2. *Under the conditions of Theorem* 4.1 *let* R *be any subregion of* $R_1-\overline{R}_0$, *and let* N_n *indicate the number of zeros of* $\phi_n^*(z)$ *in* R. *Then we have*

$$\lim_{n\to\infty} N_n/n = 0.$$

THEOREM 5.3. *Under the conditions of Theorem* 4.1 *let* R *be a subregion of* $R_1-\overline{R}_0$ *containing no limit points of the totality of zeros of the* $\phi_n^*(z)$. *Then in* R *we have*

$$\lim_{n\to\infty} \left| \phi_n^*(z) \right|^{1/n} = e^{\gamma[1-\phi(z)]},$$

uniformly on any closed subregion of R.

Theorems 5.1 and 5.3 are of significance in the study of the divergence of the series of Theorem 4.1.

THEOREM 5.4. *With the hypothesis of Theorem* 4.1, *the series in* (4.4), *where* (4.5) *is valid with* $0<\rho<1$, *diverges at an arbitrary point of* $R_1-\overline{R}_\rho$ *which is not a limit point of the totality of zeros of the* $\phi_n^*(z)$, *and this series converges uniformly on no continuum* (*not a single point*) *in* $R_1-\overline{R}_\rho$.

The first part of Theorem 5.4 follows at once from (4.5) and Theorem 5.3; the latter part follows by methods used in detail elsewhere [13, remark subsequent to equation (15)].

The function $\phi_n^*(z)$ vanishes by definition in the points $\beta_1, \beta_2, \cdots, \beta_{n-1}$, but in the case of multiply connected regions R_1 there may presumably exist other, *nontrivial*, zeros of $\phi_n^*(z)$ in R_1. Theorem 5.2 contains some information on the numbers of these nontrivial zeros. We proceed to indicate a further result, which applies to the norms not of §1 but to those of §§2 and 3; equation (1.1) is not assumed.

THEOREM 5.5. *With the hypothesis of Theorem* 1.1 *on* R_1, *each* β_{nk} *independent of* n, *and* $N_1^2(G)$ *defined by* (2.1), *let* $\chi_n(z)$ *be the function of class* $\mathfrak{M}^2$ *with* $\chi_n(\beta_1)=\chi_n(\beta_2)=\cdots=\chi_n(\beta_{n-1})=0$, $\chi_n(\beta_n)=1$, *whose norm* $N_1^2(\chi)$ *is least. A circle whose interior lies in* R_1 *and contains* β_n *can contain no nontrivial zero of* $\chi_n(z)$. *Consequently any circle whose interior lies in* R_1 *and contains all the* β_n *contains no nontrivial zero of any* $\chi_n(z)$. *If we have* $\beta_n\to\beta_0$, β_0 *in* R_1, *any circle whose interior lies in* R_1 *and contains* β_0 *can contain no nontrivial zero of* $\chi_n(z)$ *for* n *sufficiently large.*

Let α be a nontrivial zero of $\chi_n(z)$ in R_1. The function $(z-\beta_n)\chi_n(z)/(z-\alpha)$ is analytic throughout R_1 when suitably defined for $z=\alpha$, vanishes in all the points $\beta_1, \beta_2, \cdots, \beta_n$, and is of class $\mathfrak{M}^2$. Consequently (compare §4) this function is orthogonal to $\chi_n(z)$ on C_1:

$$\int_{C_1} \overline{\chi}_n(z) \frac{(z-\beta_n)\chi_n(z)}{z-\alpha}\, d\psi = 0.$$

We make the substitution $z'=1/(z-\beta_n)$, $\alpha'=1/(\alpha-\beta_n)$, whence for the image C_1' of C_1 $(\alpha'\neq 0)$

$$\int_{C_1'} |\chi_n(z)|^2 \frac{d\psi}{\alpha' - z'} = 0.$$

Thus α' is a position of equilibrium in the field of force due to a spread of nonnegative matter over C_1' which repels according to the law of inverse distance, so [9, §7.1, Theorem 1 and §7.2, Theorem 1] α' lies in the convex hull of C_1'. Consequently if the exterior of any circle contains no point of C_1', that exterior contains no point α'. Interpretation of this conclusion in the original z-plane yields the theorem.

This theorem is not invariant under *arbitrary* one-to-one conformal map of R_1. That is to say, an arbitrary such map of R_1 which carries C_1 into a set of mutually disjoint Jordan curves does not necessarily transform a circle in R_1 into a circle. The conclusion of the theorem applies not merely to a circle, but to any Jordan curve which is the image of a circle under a one-to-one conformal map of R_1 which transforms C_1 into a set of mutually disjoint Jordan curves.

Theorem 5.5 extends at once to the situation of §3, as does the supplementary remark regarding the image of a circle, now with the condition that the transformation of R_1 shall transform C_1 into a set of mutually disjoint *rectifiable* Jordan curves. Theorem 5.5 is of obvious significance in connection with the analogs of Theorems 5.3 and 5.4.

Theorem 5.5 does not extend without fundamental modification of proof or conclusion to the situation of §1. The writers plan to return to this question.

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