DEGREES OF COMPUTABILITY[1]

BY

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Introduction. In the theory of recursive functions, several decision problems have been proved unsolvable. A decision problem arises when we are confronted with a set N of mathematical objects and a subset S of N. We desire a mechanical procedure which will operate on elements of N, determining if they are elements of S.

Now, we can operate on a mathematical object using such an effective procedure only if the object is represented for us by an expression, i.e., by a finite sequence on a fixed alphabet. Of course, two different expressions on the same alphabet may represent the same object, and there may even be no effective procedure for determining whether two such expressions represent the same object. Furthermore, there may be no effective procedure for determining whether a given expression on the alphabet represents an element of N.

However, we shall consider that we have solved the decision problem for S considered as a subset of N and with respect to a given notation for the elements of N, if we have devised an effective procedure with which we can determine, when there is given an expression on the alphabet which, in fact, represents an element of N, whether that element belongs to S.

The familiar device of Gödel numbering enables us to replace consideration of expressions by consideration of non-negative integers. Thus, in dealing with decision problems for subsets of N, we have an assignment to each subset S of N, of a partial relation P_S[2]. The domain of P_S is the set of integers corresponding to expressions representing elements of N, and P_S will therefore not be everywhere defined in cases where some expressions do not represent elements of N.

Our purpose, in this work, will be the study of the relations, P_S, for various sets N. We shall wish to determine, for particular sets S, whether P_S is recursive, and more generally, determine what the position of P_S is in the Kleene hierarchy[3]. We shall also be interested in obtaining general results

giving information about P_S in terms of information about S. In Part II we shall study the case that N is the set of computable real numbers, and in Part III we shall study the case that N is the set of recursive functions, and the case that N is the set of partial recursive functions, restricting our attention in the latter case to situations in which all the elements of S are recursive. In all cases, the notation chosen to represent elements of N will be a natural one.

As we have seen, consideration of decision problems leads us naturally to consider the place of partial relations in the Kleene hierarchy. However, the Kleene hierarchy is defined only for total relations and, indeed, it is not clear what is meant by quantification of partial relations. Part I, in addition to fixing our terminology and point of view, will be chiefly devoted to a theory of quantification of partial relations.

We shall generally omit the word "partial" when dealing with partial functions and relations. A knowledge of elementary recursive function theory will be assumed. We assume familiarity with the methods, notation and results of Part III of [5]. In particular, we shall use, without definition or comment, the functions U and S_n^m and the relations $T_n^{P_1, \cdots, P_m}$. We shall use "μ" as the minimalization operator.

Part III is independent of Part II with the exception of Lemma II.6. Theorems, lemmas, and definitions will be numbered consecutively, beginning anew with each part.

PART I

I.1. DEFINITION. An *n-ary relation* is a map on a subset of the set of n-tuples of nonnegative integers into the set whose elements are truth and falsehood An *n-ary function* is a map on a subset of the set of n-tuples of nonnegative integers into the nonnegative integers. An n-ary relation or function is *total* if it is defined for all n-tuples. An n-ary relation P is an extension of an n-ary relation Q, in symbols $Q \subset P$, if P is defined and agrees with Q wherever Q is defined. P is a completion of Q if P is total and is an extension of Q. Similarly we define the notions of extension and completion for functions. If P is an n-ary relation, $\sim P$ is the relation whose domain of definition is that of P, but which never agrees with P.

Note that $\sim$ is a map of the set of relations onto itself.

Unless explicitly stated otherwise, we shall use the usual notations and conventions in connection with relations. In particular we use substitution and strong disjunction and conjunction as defined by Kleene in [5] with the standard notation. In Part I we shall distinguish the notation for a relation or function from that for a value of the relation or function at some argument by use of Church's λ notation([4]). We shall do this in Parts II and III only when it is essential for clarity. We shall also freely use the theory of

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[4] See [1].

partial recursive functions and relations without additional explanation.

We shall use the equality sign in its usual sense according to which an equation "$a = b$" has meaning if and only if both "a" and "b" have meaning and is true if and only if "a" and "b" represent the same object. Thus "$1/0 = 0$" is meaningless but "$(\lambda x)(1/x) = (\lambda x)(x)$" is meaningful and false.

I.2. DEFINITION. An *existential quantifier* is a map E of the set of relations into itself such that:

(i) The image of every n-ary relation is an $(n \dot- 1)$-ary relation.

(ii) If for relations P and P', $P \subset P'$ then $EP \subset EP'$.

(iii)([5]) If P is a total n-ary relation then EP is also total and:

If $n > 0$, then for all $(x_1, \cdots, x_{n-1})$, $(EP)(x_1, \cdots, x_{n-1})$ is true if and only if for some x, $P(x_1, \cdots, x_{n-1}, x)$ is true. If $n = 0$, then $EP = P$.

I.3. THEOREM. *If E and E' are existential quantifiers and P is a relation, then the relations EP and $E'P$ agree on the intersection of their domains of definition, or equivalently they have a common extension.*

Proof. Let Q be any completion of P. By (ii) of I.2 we have $EP \subset EQ$, $E'P \subset E'Q$, but by (iii) of I.2, $E'Q = EQ$. Q.E.D.

I.4. COROLLARY. *If for existential quantifiers, E and E', and a relation P, the relations EP and $E'P$ have the same domain, then $EP = E'P$.*

Theorem I.3 leads us to define an ordering amongst existential quantifiers.

I.5. DEFINITION. $E \subset E'$, if for all relations P, $EP \subset E'P$.

I.6. THEOREM. *Under the ordering $\subset$, the existential quantifiers are partially ordered and form a complete distributive lattice.*

Proof. That $\subset$ defines a partial ordering is trivial to verify. To show that it defines a complete lattice, let S be any set of existential quantifiers; define an existential quantifier F, such that for each n-ary relation P the domain of FP is the union of the domains of the EP for $E \in S$, and such that $(FP)(x_1, \cdots, x_{n-1}) \leftrightarrow (EP)(x_1, \cdots, x_{n-1})$ for every element E of S for which EP is defined at $(x_1, \cdots, x_{n-1})$. The existence of such an F is given by I.3. F is easily seen to be an existential quantifier and the least upper bound of S. Let F' be defined by letting, for each P, $F'P$ be the restriction of FP to the intersection of the domains of the EP, $E \in S$. F' is an existential quantifier and the greatest lower bound of S, showing that the system is a complete lattice.

The distributivity may be easily verified by means of I.4. Q.E.D.

Since the collection of existential quantifiers is a complete lattice, it has a least upper bound E_0, which we shall have frequent occasion to use. For

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([5]) The meaning of (iii) amounts to the intuitive assertion that E yields the expected results when applied to total relations.

any relation P and existential quantifier E, E_0P is an extension of EP. Thus in one sense E_0 is the "best" functional satisfying the conditions of I.2.

I.7. DEFINITION. A *string* is a nontrivial product of $\sim$'s and existential quantifiers. The *order* of a string is the number of existential quantifiers used to form it[6]. A string U is a *universal quantifier* if $\sim U \sim$ is an existential quantifier. A string is a *Kleene string* if it can be represented as an alternating product of existential and universal quantifiers. Two strings are *similar* if their restrictions to the class of total relations are equal. A relation is *k-enumerable* if it can be represented in the form HR, where R is a recursive relation and H is a Kleene string of order k in whose representation as a product an existential quantifier is the left-hand term. A relation P is *anti-k-enumerable* if $\sim P$ is k-enumerable. A relation is *potentially-k-enumerable* or *potentially-anti-k-enumerable* if it has an extension which is k-enumerable or anti-k-enumerable, respectively.

Note that, as defined here, k-enumerable and anti-k-enumerable relations are necessarily total.

At this point some discussion to show our motivation and point of view will be appropriate.

The modern theory of recursive unsolvability had its birth with Church's identification of recursiveness with effective calculability[7]. In the present context we are concerned with a wider class of decision problems than was contemplated by Church. Namely, each of the decision problems with which Church concerned himself could be made to depend on the decision problem for some total relation so that the solvability of the problem could be equated with the recursiveness of the decision problem.

In the present situation we are concerned with decision problems for a partial relation. Our first thought might well be to identify solvability in this case with partial recursiveness. However, it is not difficult to see that this would be unsatisfactory, for the domain of definition of a partial recursive relation is always recursively enumerable. Thus we could not hope to deal with decision problems in connection with a set of objects N, unless the set of representations of these objects were recursively enumerable. Thus N could not even be the set of recursive functions. This situation is clearly intolerable.

Intuitively when we say that such a decision problem is solvable, we mean that there exists an effective procedure which when applied to an object *which happens to belong to N* will terminate and furnish us with the desired information. The behavior of the procedure for objects which do not belong to N is of no concern to us in this connection. Thus we are led to call a decision problem solvable if by suitably extending its domain of definition

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(6) This is independent of the manner in which the string is represented as a product of existential quantifiers and $\sim$'s.

(7) [2].

and suitably defining it for the objects newly introduced it can be rendered partial recursive, i.e., if it is *potentially partial recursive*([8]).

Just as the concept of solvability must be extended to accommodate partial relations, so must the meaning of the scale of unsolvability provided by the Kleene hierarchy be extended in order to include partial relations. Here, however, the situation does not at first glance appear to be nearly so straightforward. It is not at all clear, for example, what the generalization of a recursively enumerable relation is to be.

Several alternatives appear to be equally possible. Thus, for the generalization to partial relations of the property of being recursively enumerable, we might choose the property of expressibility in the form $(Ey)R(x, y)$ where R is potentially recursive, partial recursive, or perhaps potentially partial recursive. Again there is the possibility of possessing an extension expressible in one of the above forms, since any relation having an extension with the property should also have the property.

Furthermore, it is not clear what is meant by $(Ey)R(x, y)$ where R is a partial relation.

These difficulties can, however, be resolved. Whatever one means by an existential quantifier, it is fairly clear that such a quantifier must possess the properties required by I.2. The strongest possible formulation would therefore be that the relation shall have an extension obtainable by applying an existential quantifier to a potentially partial recursive relation. The weakest possible formulation would be that the relation shall have an extension expressible in the form $(Ey)R(x, y)$ where R is recursive. The latter formulation is independent of the meaning of existential quantification over nontotal relations and we shall prove (I. 9) that the strongest formulation is equivalent to this weakest formulation.

Similar questions can be asked, and similar answers given, for higher levels in the Kleene hierarchy. Thus what we have called potentially-k-enumerable and potentially-anti-k-enumerable relations are the constituents of the Kleene hierarchy when modified to allow for partial relations.

The rather involved terminology we use has been introduced primarily to simplify the statement of results. Note that the 0-enumerable relations coincide with the anti-0-enumerable relations and are, in fact, simply the recursive relations. The 1-enumerable relations are the recursively enumerable relations. In general, the k-enumerable relations are what Kleene in [5] has called "predicates expressible in the k-quantifier form with existential quantifier first" and the anti-k-enumerable relations are what he has called "predicates expressible in the k-quantifier form with universal quantifier first."

([8]) This generalization of Church's thesis is by no means new here. It is, e.g., implicit in the early work of Kleene.

I.8. THEOREM. *If H is a string and P_1, P_2 are relations such that $P_1 \subset P_2$, then $HP_1 \subset HP_2$.*

If H_1 and H_2 are similar strings and if P is a relation, then H_1P and H_2P agree on the intersection of their domains of definition.

Proof. By induction from (ii) of I.2 and I.3.

I.9. THEOREM. *If P has an extension of the form HR, where H is a string and R is potentially partial recursive, then P has an extension of the form HR' where R' is recursive.*

Proof. CASE 1. The right-hand factor in a representation of H as a product is an existential quantifier.

Let R be an n-ary relation, having an extension R' such that

$$R' = (\lambda x_1, \cdots, x_n)(U(\mu y(T_n(e, x_1, \cdots, x_n, y)))) = 0).$$

Define:

$$S = (\lambda x_1, \cdots, x_n)(Ey)(T_n(e, x_1, \cdots, x_n, y) \text{ \& } U(y) = 0).$$

Clearly $R \subset R' \subset S$. By I.8, $HR \subset HS$ and hence $P \subset HS$. But S is obtained by existential quantification over a total relation, and hence by (iii) of I.2 we may contract existential quantifiers and obtain the conclusion.

CASE 2. Write $HR = H \sim (\sim R)$. By Case 1, P has an extension $H \sim R'$, where R' is recursive. Take $R'' = \sim R'$. Q.E.D.

In what follows we shall frequently make use of I.9. Its essential content is that one can deal freely with quantifiers over partial relations, using the usual manipulations such as bringing quantifiers forward, just as though the relations concerned were total, without fear of pitfalls.

An example will best clarify the situation. In this example, as well as hereafter, we shall use E_0 as our existential quantifier, and $\sim E_0 \sim$ as our universal quantifier, where E_0 is the functional of the remarks following I.6.

We shall also use the standard notation for quantifiers rather than the functional notation hitherto used. This will enable us to quantify any variable rather than only the variable used to represent the last component of an n-tuple, in the case of n-ary relations.

Let P be a 1-ary relation whose domain is the set of Gödel numbers of total 1-ary recursive functions. For x in the domain of P, $P(x)$ is to be true if and only if the 1-ary function with Gödel number x either vanishes identically or takes on the value 3 infinitely many times.

Let Q_1 be defined by:

$$Q_1 = (\lambda x)(u)(Ev)(v > u \text{ \& } U(\mu z T_1(x, v, z)) = 3)$$

and let Q_2 be defined by:

$$Q_2 = (\lambda x)(u)(U(\mu z T_1(x, u, z)) = 0).$$

It is clear that $Q_1 \vee Q_2$ is an extension of P. By the theorem there are recursive relations R_1 and R_2 such that

$$Q_1 \subset (\lambda x)(u)(Ev)R_1(x,\, u,\, v)$$

and

$$Q_2 \subset (\lambda x)(u)R_2(x,\, u).$$

Hence $Q_1 \vee Q_2 \subset (\lambda x)((u)(Ev)R_1(x,\, u,\, v) \vee (u)R_2(x,\, u))$.

The right-hand side of the above consists now of total relations and may be expressed by well-known techniques, in the form $(u)(Ev)R_3(x,\, u,\, v)$, R_3 recursive. We have

$$P \subset (Q_1 \vee Q_2) \subset (\lambda x)(u)(Ev)(R_3(x,\, u,\, v)).$$

Hence P is potentially-anti-2-enumerable.

In the future we shall give proofs based on I.9 in less detail. Thus in this case we would merely write

$$P \subset (\lambda x)((u)(Ev)(v > u \,\&\, U(\mu z T(x,\, v,\, z)) = 3) \vee (u)(U(\mu z T(x,\, u,\, z)) = 0))$$

and conclude the result.

We shall find the relationship of strong reducibility, defined below, a valuable technical tool.

I.10(^{9}). DEFINITION. If P is an n-ary relation and Q is an m-ary relation, we say that P is *strongly reducible* to Q and write $P \ll Q$, when there are partial recursive n-ary functions $f_1, \cdots, f_m$ such that:

$$P \subset (\lambda x_1, \cdots, x_n)(Q(f_1(x_1, \cdots, x_n), f_2(x_1, \cdots, x_n), \cdots, f_m(x_1, \cdots, x_n))).$$

Note that relations which are strongly reducible to potentially-k-enumerable or potentially-anti-k-enumerable relations also have these respective properties. We shall generally prove that a relation is, say, not k-enumerable, by proving that every anti-k-enumerable relation is strongly reducible to it. The following definition and three theorems are given to facilitate this process.

I.11. THEOREM. *For all relations, P, Q, R:*

(i) $(P \subset Q) \rightarrow (P \ll Q)$,

(ii) $(P \ll Q) \,\&\, (Q \ll R) \rightarrow (P \ll R)$,

(iii) $(P \ll P)$.

Proof. Immediate from I.10.

I.12. DEFINITION. A relation, P, is strictly in a given class of relations

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(^{9}) This definition is, in a special case, Post's. See [7] where the concept was called many-one-reducibility.

if it is in that class and every element of that class is strongly reducible to P.

I.13. THEOREM. *For $k > 0$, a strictly-(potentially)-k-enumerable relation cannot be (potentially)-anti-k-enumerable nor can a strictly-(potentially)-anti-k-enumerable relation be (potentially)-k-enumerable([10]).*

Proof. The theorem is merely a restatement of the hierarchy theorem. See [5, pp. 283–284].

I.14. THEOREM. *Every potentially-(anti)-k-enumerable relation is strongly reducible to a given relation, P, if every 1-ary-(anti)-k-enumerable relation is strongly reducible to it.*

Proof. Let Q be an n-ary potentially-k-enumerable relation, and Q' any k-enumerable extension of Q. For all x, let p_x be the xth prime and $\mathrm{Term}_i(x)$ be the exponent to which the ith prime occurs in x. The functions

$$(\lambda x)\ (\mathrm{Term}_i\ x)$$

are recursive. Define:

$$Q'' = (\lambda x)Q'(\mathrm{Term}_1(x), \cdots, \mathrm{Term}_n(x))$$

then $Q'' \ll Q'$, and hence by the remark following I.10, Q'' is k-enumerable. Also:

$$Q' = (\lambda x_1, \cdots, x_n)Q''(2^{x_1}, \cdots, p_n^{x_n}),$$

so that $Q' \ll Q''$, but $Q'' \ll P$ by hypothesis. Since $Q \subset Q'$ we have, by I.11, $Q \ll P$.

The "anti" case is proved similarly. Q.E.D.

I.15. THEOREM. *A relation is potentially partial recursive in a finite number of k-enumerable and anti-k-enumerable relations if and only if it is both potentially-$(k+1)$-enumerable and potentially-anti-$(k+1)$-enumerable.*

Proof. First let P be both potentially-$(k+1)$-enumerable and potentially-anti-$(k+1)$-enumerable. Then P has completions Q_1 and Q_2, such that for k-enumerable relations R_1 and R_2, we have, where P is assumed to be an n-ary relation:

$$Q_1 = (\lambda x_1, \cdots, x_n)(Ey)(\sim R_1(x_1, \cdots, x_n, y)),$$

$$Q_2 = (\lambda x_1, \cdots, x_n)(y)(R_2(x_1, \cdots, x_n, y)).$$

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[10] This statement as well as similar statements is to be read both with and without the words in parentheses.

Then for all $(x_1, \cdots, x_n)$ in the domain of P we must have $Q_1(x_1, \cdots, x_n) \vee$ $\sim Q_2(x_1, \cdots, x_n)$ and hence P has an extension[11]:

$$(\lambda x_1, \cdots, x_n)(R_1(x_1, \cdots, x_n, \mu y(R_1(x_1, \cdots, x_n, y) \vee R_2(x_1, \cdots, x_n, y))))$$

so that P is potentially partial recursive in R_1, R_2, concluding the proof of the "if" part of the theorem.

In order to establish the "only if" part of the theorem, we must first prove several lemmas.

I.16. LEMMA[12]. *If an n-ary relation P is partial recursive in total relations,* $P_1, \cdots, P_m$, *then P has completions* Q_1 *and* Q_2, *where for relations* R_1 *and* R_2 *recursive in* $P_1, \cdots, P_n$,

$$Q_1 = (\lambda x_1, \cdots, x_n)(Ey)R_1(x_1, \cdots, x_n, y),$$
$$Q_2 = (\lambda x_1, \cdots, x_n)(y)R_1(x_1, \cdots, x_n, y).$$

Proof. By the normal form theorem[13]:

$$P = (\lambda x_1, \cdots, x_n)(U(\mu y T_n^{P_1, \cdots, P_m}(e, x_1, \cdots, x_n, y)) = 0).$$

Hence

$$P \subset (\lambda x_1, \cdots, x_n)(Ey)(T_n^{P_1, \cdots, P_m}(e, x_1, \cdots, x_n, y) \& U(y) = 0)$$

and

$$P \subset (\lambda x_1, \cdots, x_n)(y)(T_n^{P_1, \cdots, P_m}(e, x_1, \cdots, x_n, y) \to U(y) = 0). \quad \text{Q.E.D.}$$

I.17. LEMMA. *Let* $P_1, \cdots, P_m$ *be total relations and let Q be recursive in relations which are recursively enumerable in* $P_1, \cdots, P_m$. *Then Q can be expressed in the form* $(\lambda x_1, \cdots, x_n)(Ey)(z)R_1(x_1, \cdots, x_n, y, z)$ *and in the form* $(\lambda x_1, \cdots, x_n)(y)(Ez)R_2(x_1, \cdots, x_n, y, z)$ *where* R_1 *and* R_2 *are recursive in* $P_1, \cdots, P_m$.

Proof. Let S be the class of relations expressible in both of the above forms. Since S contains all relations which are recursively enumerable in $P_1, \cdots, P_m$, it will be sufficient to prove that S contains with any finite set of relations all relations recursive in that set. To do so it will be sufficient to show that the set, S^*, of total functions whose representing relations are elements of S, is closed under substitution, minimalization, and primitive recursion. This fact may be verified by elementary manipulations.

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[11] If, e.g., $y' = \mu y(\sim R_1(x_1, \cdots, x_n) v \sim R_2(x_1, \cdots, x_n))$ is such that $R_2(x_1, \cdots, x_n, y')$, then we have $\sim Q_2(x_1, \cdots, x_n)$, hence $\sim Q_1(x_1, \cdots, x_n)$, hence $(y)R_1(x_1, \cdots, x_n, y)$, and the displayed formula is false for $(x_1, \cdots, x_n)$, agreeing with P.

[12] This lemma is an immediate generalization of a result of Kleene [5].

[13] See [5].

The proof of the "only if" part of I.15 will be by induction on k.

The case $k=0$ is a consequence of I.16.

Assuming the theorem for $k=p\geq 0$, let Q be an n-ary relation which is potentially partial recursive in relations $P_1, \cdots, P_m$, which we may assume are $(p+1)$-enumerable. By I.16:

(1) $$Q \subset (\lambda x_1, \cdots, x_n)(Ey)R_1(x_1, \cdots, x_n, y)$$

and

(2) $$Q \subset (\lambda x_1, \cdots, x_n)(y)R_2(x_1, \cdots, x_n, y)$$

where R_1 and R_2 are recursive in $P_1, \cdots, P_m$. There are anti-p-enumerable relations $Q_1, \cdots, Q_m$ such that for $1 \leq i \leq m$

$$P_i = (\lambda x_1, \cdots, x_{n_i})(Ey)Q_i(x_1, \cdots, x_{n_i}, y).$$

R_1 and R_2 are recursive in relations which are recursively enumerable in the Q_i and hence, by I.17,

(3) $$R_1 = (\lambda x_1, \cdots, x_n, y)(Ez)(w)R_1'(x_1, \cdots, x_n, y, z, w),$$

(4) $$R_2 = (\lambda x_1, \cdots, x_n, y)(z)(Ew)R_2'(x_1, \cdots, x_n, y, z, w)$$

where R_1' and R_2' are recursive in the Q_i. Hence by the induction hypothesis R_1' is anti-$(p+1)$-enumerable and R_2' is $(p+1)$-enumerable.

Combining this with (1) and (3), Q is potentially-$(p+2)$-enumerable and combining (2) and (4), Q is potentially anti-$(p+2)$-enumerable, completing the induction. Q.E.D.

I.18. THEOREM (POST). *A relation is recursive in a finite number of k-enumerable and anti-k-enumerable relations if and only if it is both $(k+1)$-enumerable and anti-$(k+1)$-enumerable.*

Proof. Immediate from I.15.

Proofs of I.18 appear in [3] and [5]. The proof given here is new, and we could have obtained I.18 directly, omitting I.16 and the complications arising in connection with it. Note that our proof does not use the strong version of the normal form theorem as [5] does, nor does it use the detailed nature of the Gödel arithmetization as [3] does.

I.15 and I.18 prompt the following definition.

I.19. DEFINITION. A relation is *k-recursive* if and only if it is both k-enumerable and anti-k-enumerable. A relation is *potentially-k-recursive* if it has k-enumerable extensions and anti-k-enumerable extensions[14].

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[14] A potentially k-recursive relation need not have a k-recursive extension. If our terminology were not already cumbersome, we would call what we have called potentially k-recursive, potentially-k-partial-recursive and call relations with k-recursive extensions, potentially k-recursive.

PART II

II.1. DEFINITION. By a *dyadic* sequence is meant a recursive sequence of zeros and ones, infinitely many terms of which are zeros. If R is any set of real numbers, by P_R we shall mean the relation whose domain of definition is the set of Gödel numbers of dyadic sequences, where for an x in the domain of P_R, $P_R(x)$ is true if and only if:

$$\sum_{i=0}^{\infty} 2^{-(i+1)} U(\mu y T_1(x,\, i,\, y))$$

is an element of R. That is, $P_R(x)$ is true if and only if the real number whose binary expansion has the Gödel number x belongs to R. A recursive[15] real number, u, in $0 \leq u < 1$, is an *indicator* of R, if R contains all real numbers of the form $(u+v)$ where v has a finite binary expansion. By $\overline{R}$ is meant the complement of R with respect to the set of all real numbers. R is *nontrivial* if both R and $\overline{R}$ have recursive[15] elements in the interval $0 \leq u < 1$.

Part II will constitute a study of the classification of the P_R, for various sets of real numbers, R. Note that $P_{\overline{R}} = \sim P_{\overline{R}}$, $P_{R \cup R'} = P_R \lor P_{R'}$ and $P_{R \cap R'} = P_R \& P_{R'}$.

The reader may have noticed that we have essentially restricted consideration to sets of recursive real numbers in the range $0 \leq u < 1$. We have chosen the more general language merely to facilitate the statement of results and not to give an unjustified appearance of generality. The restriction that $0 \leq u < 1$ is, of course, merely a technical device and is not significant. The restriction to recursive real numbers is however basic[16].

II.2. THEOREM. *If R is nontrivial and if $\overline{R}$ contains an indicator, then every anti-1-enumerable relation is strongly reducible to P_R.*

Proof. Let $Q(x, y)$ be recursive. Let u be an indicator of $\overline{R}$ and v an element of R such that $0 \leq u, v < 1$. Let $u_0, u_1, \cdots$, and $v_0, v_1, \cdots$ be dyadic sequences representing u and v respectively. Define a recursive function $f(x, i)$ as follows:

$$f(x,\, i) = v_i \text{ if } \underset{y \leq i}{(y)} Q(x,\, y),$$

$$f(x,\, i) = u_i \text{ if not.}$$

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[15] A real number is recursive if its binary expansion is a recursive sequence. See [9] for other equivalent definitions. The concept also appears in Specker Robinson (JSL, vol. 16, p. 282) and Myhill (JSL, vol. 18, pp. 7–10), and first appeared in [12].

[16] The notation that we have chosen to represent real numbers is another apparent restriction. It can, however, be shown that any unsolvability result which holds for this notation also holds for any notation into which it is effectively translatable. In particular, all the unsolvability results of Part II hold if we represent real numbers by recursively convergent, recursive sequences of rationals.

For fixed x, the sequence $f(x, 0), f(x, 1), \cdots$ will coincide with $v_0, v_1, \cdots$ and hence represent an element of R, if $(y)Q(x, y)$, but if $\sim(y)Q(x, y)$ the sequence $f(x, 0), f(x, 1), \cdots$ will coincide with $u_0, u_1, \cdots$ at all but a finite number of terms and hence represent an element of $\overline{R}$. Letting e be a Gödel number of $f(x, y)$, we have, for all x:

$$(y)Q(x, y) \leftrightarrow P_R(S_1^1(e, x)).$$

From this we obtain $(\lambda x)(y)Q(x, y) \ll P_R$. The theorem then follows from I.14.

II.3. THEOREM. *If R contains an indicator and is nontrivial, then every 1-enumerable relation is strongly reducible to P_R.*

Proof. II.3 is merely II.2 with R and $\overline{R}$ interchanged.

II.4. THEOREM. *If both R and $\overline{R}$ contain indicators, every 1-enumerable relation and every anti-1-enumerable relation is strongly reducible to P_R.*

Proof. II.4 is a combination of II.2 and II.3.

II.5. THEOREM. *If each element of R is algebraic and if R contains an indicator, then every 2-enumerable relation is strongly reducible to P_R.*

We shall establish a lemma, to be used in the proof of II.5, as well as in Part III.

II.6. LEMMA([17]). *Let P be the relation whose domain of definition is the set of Gödel numbers of recursive sequences of zeros and ones (not necessarily dyadic), where for all x in the domain of P, $P(x)$ is true if and only if the sequence with Gödel number x contains only finitely many ones. Then P is strictly potentially-2-enumerable.*

Proof. The relation $(Ew)(y)(w < y \rightarrow U(\mu z T_1(x, y, z)) = 0)$ is an extension of P so that P is potentially-2-enumerable.

Let $Q(x, y, z)$ be recursive. Define recursive functions $h(x, t)$ and $g(x, t)$ as follows:

$$h(x, 0) = 0,$$
$$h(x, t + 1) = h(x, t) \qquad \text{if } Q(x, h(x, t), t \mathbin{\dot-} (\mu s)(h(x, s) = h(x, t))),$$
$$\qquad\qquad = h(x, t) + 1 \quad \text{if not.}$$
$$g(x, t) = 0 \qquad \text{if } Q(x, h(x, t), t \mathbin{\dot-} (\mu s)(h(x, s) = h(x, t))),$$
$$g(x, t) = 1 \qquad \text{if not.}$$

For fixed x, if $(Ey)(z)Q(x, y, z)$, then $h(x, t)$ ultimately becomes constant

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([17]) Results closely related to this are in [6] and in [11].

and $g(x, t)$ ultimately vanishes, but if $\sim(Ey)(z)Q(x, y, z)$, then $h(x, t)$ increases infinitely many times and $g(x, t)$ does not ultimately vanish. Hence, letting e be a Gödel number of $g(x, t)$, we have for all x:

$$(Ey)(z)Q(x, y, z) \leftrightarrow P(S_1^1(e, x)).$$

Hence $(Ey)(z)Q(x, y, z) \ll P$ and the theorem follows from I.14. Q.E.D.

We shall carry out the proof of II.5 only in the case that 0 is an indicator of R. The general case is proved similarly with added trivial complications. By the lemma, it will be sufficient to prove that $P \ll P_R$. Define a partial recursive function $f(x, t)$, as follows:

$$f(x, t) = 0 \text{ if } \underset{s \leq t}{(s)} (t \neq s!),$$

$$f(x, t) = U(\mu z)T(x, (\mu w)(w! = t), z) \text{ if not.}$$

The sequence $f(x, 0), f(x, 1), \cdots$ is the dyadic representation of the real number, $u = \sum_{i=0}^{\infty} x_i 2^{-(i!+1)}$, where $x_0, x_1, \cdots$ is the sequence of which x is a Gödel number. If this sequence ultimately vanishes, u is an element of R, but if this sequence does not ultimately vanish, it is known[18] that u is not algebraic and hence u is not an element of R. Let f have a Gödel number e. We have for all x in the domain of P:

$$P(x) \leftrightarrow P_R(S_1^1(e, x)).$$

Hence $P \ll P_R$. Q.E.D.

We shall assume an effective correspondence between nonnegative integers and polynomials with integral coefficients, such as can be obtained by a scheme of Gödel numbering. This can be done in such a way that to each nonnegative integer there corresponds exactly one polynomial. The usual operations on polynomials will correspond to recursive functions. Thus the function $(\lambda P)(\deg P)$, where $\deg P$ is the degree of the polynomial with Gödel number P, if the polynomial is irreducible and is zero otherwise, is recursive.

II.7. THEOREM. *There exists a partial recursive relation $V(P, x, n)$ such that if x is the Gödel number of a dyadic sequence representing a real number u, and P is the Gödel number of a polynomial P, then for all n, $V(P, x, n)$ is defined and $P(u) = 0 \leftrightarrow (n) V(P, x, n)$.*

Proof. To give a complete proof would be long and laborious. All that is needed is a formalization of the processes of high school algebra, such as Sturm's method. A formalization of [10] would yield the theorem more directly.

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[18] See [4].

II.8. DEFINITION. If N is any set of positive integers, $R(N)$ is the set of all algebraic real numbers whose algebraic degree over the rationals is an element of N.

II.9. THEOREM. *For any set*([19]), *N, of positive integers $N \ll P_{R(N)}$.*

Proof. Let $f(n, t)$ be the tth digit in the binary expansion of $(1/2)^{1/n}$. The function $f(n, t)$ is recursive. Let it have Gödel number e. Since the algebraic degree of $(1/2)^{1/n}$ is equal to that of $(2)^{1/n}$, which by Eisenstein's criterion is n, we have, for all n:

$$n \in N \leftrightarrow P_{R(N)}(S_1^1(e, n)). \qquad\qquad \text{Q.E.D.}$$

II.10. THEOREM. *If N is a nonempty recursive set of positive integers, then $P_{R(N)}$ is strictly potentially-2-enumerable.*

Proof. By II.5 and I.11 we need only find a 2-enumerable extension of $P_{R(N)}$. Such an extension can be found by the usual use of I.9 and the relation:

$$(EP)((\deg P) \in N \ \& \ (n)(V(P, x, n))) \qquad\qquad \text{Q.E.D.}$$

II.11. THEOREM. *If R is the set of all rational numbers or the set of all algebraic real numbers, then P_R is potentially-2-enumerable and every potentially-2-enumerable relation is strongly reducible to P_R.*

Proof, In II.10 take N to be the unit set of 1 and the set of all positive integers, respectively.

II.12. THEOREM. *The decision problems for the rational numbers and the algebraic numbers are recursively unsolvable, i.e., there exists no effective procedure for: given a real number (via an algorithm for its binary expansion) to tell if that number is rational, or to tell if that number is algebraic.*

This is a weakening of II.11 and even of II.3, and is stated only for emphasis.

I.10 yields other somewhat surprising consequences, in connection with which the following definition is stated.

II.13. DEFINITION. Relations P and Q are *strongly equivalent* if $P \ll Q$ and $Q \ll P$.

II.14. THEOREM. *All the relations of II.10 and hence the decision problems of the related sets of real numbers are strongly equivalent.*

Proof. Immediate.

Thus, for example, if I were supplied with an oracle to tell me if a given

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([19]) In all contexts in which we apply to sets the notions that we have developed for relations, each set is to be identified with the total 1-ary relation which is true for precisely those arguments that are elements of the set.

real number is rational, I could effectively tell if a given real number is algebraic, and conversely.

II.15. THEOREM. *For any set N of nonnegative integers, there exists a set, R, of real numbers, such that N and P_R are strongly equivalent.*

Proof. Let $k(u)$, where u is a recursive real number, be the number of 1's preceding the first zero in the dyadic expansion of u.

Let R be the set of all recursive u, $0 \leq u < 1$, such that $k(u) \in N$ and let:

$$f(x) = (\mu y)(U(\mu z T(x, y, z)) = 0).$$

It is clear that $f(x)$ is partial recursive, and that if x is the Gödel number of a dyadic sequence representing u, $k(u) = f(x)$. We have, for all x in the domain of P_R:

$$P_R(x) \leftrightarrow f(x) \in N.$$

Hence $P_R \ll N$.

Define a recursive function $g(x, t)$:

$$g(x, t) = 0 \quad \text{if } t = x,$$
$$g(x, t) = 1 \quad \text{if not.}$$

We then have, letting g have Gödel number e:

$$x \in N \leftrightarrow P_R(S_1^1(e, x)),$$

so that $N \ll P_R$. Q.E.D.

The content of II.15 is simply that decision problems for sets of real numbers saturate the scale of recursive solvability.

II.16. THEOREM. *If R is a nonempty finite set of recursive real numbers in $0 \leq u < 1$, then P_R is strictly potentially anti-1-enumerable.*

Proof. To show that P_R is potentially-anti-1-enumerable, it will be sufficient to do so in the case that R contains exactly one element u, with dyadic expansion, $u_0, u_1, \cdot \cdot \cdot$. In this case, note that:

$$P_R \subset (\lambda x)(i)(U((\mu y)T(x, i, y)) = u_i)$$

and apply I.9.

The "strictly" follows from II.2 when it is noticed that the additive group of recursive real numbers, modulo the group of real numbers having finite dyadic expansion, has infinite order. Q.E.D.

PART III

III.1. DEFINITION. Let C be a class of recursive 1-ary functions. Q_C is the total relation such that $Q_C(e)$ is true if and only if the 1-ary partial recursive

function with Gödel number e is an element of C. Q_C' is the restriction of Q_C to the set of Gödel numbers of total 1-ary functions.

Both Q_C and Q_C' may be looked upon as decision problems for C. In Part III we will concern ourselves with the classification of the Q_C and of the Q_C'.

III.2. THEOREM. *If C is nonempty, every anti-2-enumerable relation is strongly reducible to Q_C.*

Proof. Let $R(x, y, z)$ be recursive and $f(y)$ an element of C. Define a partial recursive function $g(x, y)$:

$$g(x, y) = f(y) + 0 \cdot (\mu z) R(x, y, z).$$

For fixed x, if $(y)(Ez)R(x, y, z)$, then $(\lambda y)g(x, y)$ coincides with $f(y)$ and is therefore an element of C. But if $\sim(y)(Ez)R(x, y, z)$, $(\lambda y)g(x, y)$ is not total, and hence is not an element of C. We have, letting g have Gödel number e, for all x:

$$(y)(Ez)R(x, y, z) \leftrightarrow Q_C(S_1^1(e, x)).$$

Hence, by I.14, every anti-2-enumerable relation is strongly reducible to P_C.

III.3. THEOREM. *For $k > 2$, Q_C is (anti)-k-enumerable if and only if Q_C' is potentially-(anti)-k-enumerable. If Q_C' is potentially-anti-2-enumerable, then Q_C is anti-2-enumerable and conversely. If Q_C' is potentially-2-enumerable, then Q_C is 3-recursive.*

Proof. The only if part of the first statement follows from I.11.

If Q_C'' is any completion of Q_C', we have, for all x:

$$Q_C(x) \leftrightarrow (y)(Ez)T_1(x, y, z) \,\&\, Q_C''(x),$$

from which the rest of the theorem follows.

III.3 essentially states that Q_C is just as solvable as Q_C' except when prohibited by III.2. The following theorem is a special case of III.2.

III.4. THEOREM[20]. *Let C be the class of all recursive 1-ary functions. Then Q_C is strictly anti-2-enumerable.*

Proof.

$$Q_C(x) \leftrightarrow (y)(Ez)T_1(x, y, z).$$

III.5. DEFINITION. *C is generative if there exists a recursive function $f(x, y)$ such that C consists precisely of the functions $(\lambda y)f(x, y)$. A recursive function $g(x)$ is an indicator of C, if C contains all functions agreeing with C*

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[20] A different proof is given in [3].

at all but a finite number of arguments. By $\overline{C}$ is meant the complement of C in the set of 1-ary recursive functions. C is *nontrivial* if both C and $\overline{C}$ contain elements.

III.6. THEOREM. *If C is nontrivial and if $\overline{C}$ contains an indicator, every anti-1-enumerable relation is strongly reducible to Q'_C .*

III.7. THEOREM. *If C is nontrivial and contains an indicator, every 1-enumerable relation is strongly reducible to Q'_C .*

III.8. THEOREM. *If C and $\overline{C}$ both contain indicators, every 1-enumerable and every anti-1-enumerable relation is strongly reducible to Q'_C .*

Proofs. Similar to those of II.2, II.3, and II.4.

An example of a class satisfying the conditions of III.8 is the class of 1-ary recursive functions with finite range of values. Two other examples are the class, C_1, of 1-ary recursive functions with recursive ranges of values, and the class, C_2, of 1-ary recursive functions having ranges of values to which not all recursively enumerable sets are reducible. It can be shown that every 3-enumerable relation is strongly reducible to both Q'_{C_1} and Q'_{C_2}.

III.9. THEOREM. *If C contains an indicator and if for some 1-ary recursive function, g, no element of C agrees with g infinitely often, then every 2-enumerable relation is strongly reducible to Q'_C .*

Proof. We shall prove that for every such C, the relation P of II.6 is strongly reducible to Q'_C . Let f be an indicator of C and define a partial recursive function:

$$h(x, y) = (1 \doteq U(\mu z T_1(x, y, z)))f(y) + U(\mu z T_1(x, y, z))g(y).$$

Then if the sequence with Gödel number x ultimately vanishes, $(\lambda x)h(x, y)$ differs from f at only a finite number of points, and hence is an element of C. If, on the other hand, this sequence does not ultimately vanish, $(\lambda y)h(x, y)$ agrees with g infinitely often, and hence could not be an element of C. Letting e be a Gödel number of h, we have, for all x in the domain of P:

$$P(x) \leftrightarrow Q'_C(S^1_1(e, x)).$$

Hence $P \ll Q'_C$. Q.E.D.

III.10. THEOREM. *If C is generative, there exists a recursive 1-ary function, g, such that g ultimately exceeds each element f of C (i.e.*

$$(Ex)(y)(x < y \rightarrow f(y) < g(y)).$$

Proof. Let:

$$g(y) = 1 + \underset{0 \le x \le y}{\text{Maximum}} (g(x, y))$$

where $g(x, y)$ is given by III.5.

III.11. THEOREM. *If C is a generative class containing an indicator, Q_C' is strictly potentially-2-enumerable, and hence Q_C is 3-recursive but neither 2-enumerable nor anti-2-enumerable.*

Proof. Let $g(x, y)$ be as given by III.5. Then:

$$Q_C' \subset (\lambda w)(Ex)(y)(g(x, y) = U((\mu z)T_1(w, y, z)))$$

and by I.9 Q_C' is potentially-2-enumerable.

By III.9, III.10, and I.11, Q_C' is strictly potentially-2-enumerable. The rest of the theorem follows from III.2 and III.3. Q.E.D.

Examples of classes, C, to which the above theorem applies are the Kalmár elementary functions, the primitive recursive functions, and the n-fold recursive functions, for each n or for all finite n.

III.12. THEOREM. *For any set, N, of nonnegative integers, there exists a class, C, such that Q_C' and N are strongly equivalent.*

Proof. Let C be the class of 1-ary recursive functions, f, such that $f(0) \in N$. Then:

$$Q_C'(\lambda x)(U(\mu y T_1(x, 0, y)) \in N)$$

and hence $Q_C' \ll N$.

Let $f(x, y) = x$, and let f have Gödel number e, then, for all x:

$$x \in N \leftrightarrow Q_C'(S_1^1(e, x))$$

and hence $N \ll Q_C'$. Q.E.D.

III.13. THEOREM. *If C is a nonempty finite class of 1-ary recursive functions, then Q_C' is strictly potentially anti-1-enumerable and Q_C is strictly anti-2-enumerable.*

Proof. The proof for Q_C' is similar to the proof of II.16. The classification of Q_C follows from that of Q_C' by III.2 and III.3.

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