A GENERAL FUNCTIONAL EQUATION[1]

BY

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1. **Introduction.** In the following, E_m denotes the m-dimensional Euclidean space of points $x = (x_1, \cdots, x_m)$ with the usual norm $|x|$. For $B_1 \subset E_m$, $B_2 \subset E_m$ and λ real, $B_1 + B_2$ or λB_1 denotes the set of all the points $b_1 + b_2$ or λb_1, respectively, $(b_1 \in B_1, b_2 \in B_2)$. Further, the "measure" of a subset of E_m is always meant to be its m-dimensional Lebesgue measure. Unless otherwise stated, all functions considered are complex-valued. A function on E_m is said to be additive when it satisfies the functional equation $f(x+y) = f(x) + f(y)$. A real-valued additive function on E_m is linear, as soon as it is bounded above on a set of positive measure, Ostrowski [15], or on a set B, such that some n-fold sum $B + \cdots + B$ contains an interior point, cf. §3. Thus, the following generalization, (cf. §2), of a theorem due to Steinhaus [20] yields a new proof of Ostrowski's result. Let B_1 and B_2 be two subsets of E_m, each of positive measure; then it is possible to delete sets of measure 0 from B_1 and B_2, in such a way that $B_1 + B_2$ becomes an open set.

Let Ω be the class of all functions $f(x)$ on E_m equal to a polynomial, (in the components of x). Let $b_0, b_1, \cdots, b_k$ be $k+1$ different real numbers and let $f_0(x), \cdots, f_k(x)$ be a set of $k+1$ functions on E_m. The central part of this paper consists of results of the following type. (i) If f_0 is bounded on a (measurable) set of positive measure and, moreover,

$$(1) \qquad \sum_{i=0}^{k} f_i(x + b_i y) \in \Omega \qquad \text{for each } y \in C,$$

where C is a subset of E_m of positive measure, then $f_0 \in \Omega$. (ii) Let θ_1, θ_2 be a fixed pair of rationally independent real numbers, and let p, q denote arbitrary non-negative integers. If $m = 1$, f_0 is measurable and $C = \{c \mid c = p\theta_1 + q\theta_2;\ p+q \leq k\}$ then (1) implies the existence of a $g \in \Omega$, such that $f(x) = g(x)$ for almost all x.

The above result (i) may be generalized by introducing the notion of a "local condition." Whether or not a function $f(x)$ satisfies a given local condition P at a point x_0, depends only on the behavior of $f(x)$ in an arbitrarily small neighborhood of x_0. We only consider local conditions P, such that: (a) If $f(x)$ satisfies P at x_0 then $f(x+h)$ satisfies P at $x_0 - h$, for each $h \in E_m$; (b) If f_1 and f_2 satisfy P at x_0 then also $f_1 - f_2$. For A as an open subset of E_m,

$\Omega_P(A)$ will denote the class of all the functions on A, which satisfy P at each point in A. Thus, $f \in \Omega_P(A)$ implies (*) $\Delta_h f = f(x+h) - f(x) \in \Omega_P(A_h)$, for each $h \in E_m$; here, $A_h = A \cap (A-h)$ is the domain of $\Delta_h f$. A local condition P is said to have the weak difference property if, vice versa, for f as a function on the open set A, (*) implies $f \in \Omega_P(A)$, whenever f is bounded on some subset of A of positive measure. Now, from Theorem 6.1, we have the following generalization of (i). Let A and C be subsets of E_m, where A is open and connected and C is of positive measure. Let P be a given local condition with the weak difference property. Finally, let $b_0 = 0, b_1, \cdots, b_k$ be $k+1$ different real numbers, and let $f_i(x)$ $(i = 0, \cdots, k)$ be a function defined on $A + b_i C$. Assertion: if (1) holds for $\Omega = \Omega_P(A)$ then $f_0 \in \Omega_P(A)$, whenever f_0 is bounded on some subset of A of positive measure.

The results of de Bruijn [2; 3] imply the weak difference property of the local conditions in E_1, that $f(x)$ be Riemann integrable, (or continuous, or absolutely continuous, or of bounded variation, or differentiable, respectively), throughout an interval about x_0. In §5, we establish the weak difference property for certain local conditions in E_m, for instance, the local conditions that $f(x) = f(x_1, \cdots, x_m)$ be continuous, (or analytic, or equal to a polynomial, or bounded, respectively), throughout an entire neighborhood of the point x_0.

In §7, we concern ourselves with results of the type of the above assertion (ii), some of which are generalizations of results due to Boas [1]. Finally, in §8, we give a new proof of the following theorem due to Skitovic [19]. Let a_j and b_j, $(j = 1, \cdots, n)$, be real constants and let $X_1, \cdots, X_n$ be independent random variables, such that $\sum a_j X_j$ and $\sum b_j X_j$ are independent. Then each X_j, for which $a_j b_j \neq 0$, is normally distributed. We note that each of the §§2, 3, 7 and 8 is a self-contained unit with an interest of its own.

Before proceeding, we want to give some counterexamples, showing that the assumptions on f_0, in the above assertions (i) and (ii), cannot be weakened very much without strengthening the assumptions on C, and vice versa[2]. The following construction is similar to one given by Erdös and Golomb [6, 16].

Let $U \subset E_m$ be a Hamel basis of the linear space E_m over the enumerable field R, obtained by adjoining $b_0, \cdots, b_k$ to the field of rational numbers, i.e. each $x \in E_m$ has a unique representation as a finite sum $x = \sum H_\alpha(x) u_\alpha$ with $H_\alpha(x) \in R$ and $u_\alpha \in U$. Let $u_0 \in U$ be fixed; then $H(x) = H_0(x)$ is additive on E_m, such that $H(b_i x) = b_i H(x)$, $i = 0, \cdots, k$. Further, the additive group $G = \{x \mid H(x) = 0\}$ is not of positive measure; for, otherwise, $G = G + G$ would contain an interior point, thus, $G = E_m$, contradicting $u_0 \notin G$. But E_m is the union of the denumerably many cosets $\{x \mid H(x) = r\}$ of G, $(r \in R)$, hence,

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[2] These counterexamples also show, that the same is true for the above stated consequence of Theorem 6.1, provided that each function in $\Omega_P(A)$ is measurable on A.

G is a nonmeasurable set of the power c. Therefore, H is nonmeasurable, thus, unbounded on each set of positive measure.

Suppose that $k+1 \geq 3$, and let $\gamma_0, \cdots, \gamma_k$ be real constants with $\sum \gamma_i = \sum b_i \gamma_i = 0$ and $\gamma_0 \neq 0$. Then the set of functions $f_i(x) = \gamma_i H(x)$, $(i = 0, \cdots, k)$, satisfies

$$(2) \qquad \sum_{i=0}^{k} f_i(x + b_i y) = 0,$$

identically in x and y. But, H being nonmeasurable, $f_0 = \gamma_0 H \notin \Omega$, thus, in assertion (i), the condition on f_0 cannot be omitted, even if $C = E_m$. Further, the set of bounded functions $f_i(x) = \gamma_i$ arc tan $H(x)$, $(i = 0, \cdots, k)$, satisfies (2) for $x \in E_m$, $y \in G$, though $f_0 \notin \Omega$. From the properties of G, it follows that, in assertion (ii), "measurable" cannot be replaced by "bounded," while, in assertion (i), the condition "of positive measure" on C cannot be replaced by (say) "of positive outer measure."

2. **Sums of sets of positive measure.** Let B be a measurable subset of the Euclidean space E_m. The set B is said to be of unit density at the point $x_0 \in E_m$ when

$$\lim_{r \to 0} \mu(B \cap S_r) \mu(S_r)^{-1} = 1.$$

Here, S_r denotes the sphere $|x - x_0| < r$, while $\mu(U)$ denotes the m-dimensional Lebesgue measure of U. By B^* we shall denote the set of all the points x_0 in B at which B is of unit density. Then B is of unit density at almost all points of B, (cf. Sierpinsky [17]) thus, $\mu(B^*) = \mu(B)$.

THEOREM 2.1. *For $i = 1, \cdots, k$, let B_i be a subset of E_m of positive measure and let a_i be any fixed point in B_i^*. Then there exist positive numbers δ and η, such that*

$$\mu((B_1 - x_1) \cap \cdots \cap (B_k - x_k)) > \eta > 0,$$

whenever $|x_i - a_i| < \delta$, $i = 1, \cdots, k$.

Proof. Replacing B_i by $B_i - a_i$, we may assume that $a_i = 0$, thus, B_i is of unit density at 0, $(i = 1, \cdots, k)$. Let ϵ be a fixed real number, $0 < \epsilon < 1/k$, and let S be a fixed sphere $|x| < \rho$ such that

$$(1) \qquad \mu(C_i) \geq (1 - \epsilon)\mu(S) \qquad (i = 1, \cdots, k),$$

where

$$(2) \qquad C_i = B_i \cap S \qquad (i = 1, \cdots, k).$$

From $C_i \subset B_i$,

$$m_0 = \mu((B_1 - x_1) \cap \cdots \cap (B_k - x_k)) \geq \mu(S \cap (C_1 - x_1) \cap \cdots \cap (C_k - x_k)),$$

hence, (letting $\overline{D}$ denote the complement of D).

(3) $$m_0 \geq \mu(S) - \sum_{1}^{k} \mu(S \cap (\overline{C_i - x_i})).$$

Further, from

$$\mu(S \cup (C_i - x_i)) = \mu(S) + \mu(\overline{S} \cap (\overline{C_i - x_i}))$$
$$= \mu(C_i - x_i) + \mu(S \cap (C_i - x_i)),$$

together with (1) and $C_i \subset S$, we have

$$\mu(S \cap (\overline{C_i - x_i})) = \mu(S) - \mu(C_i) + \mu(\overline{S} \cap (C_i - x_i))$$
$$\leq \epsilon\mu(S) + \mu(\overline{S} \cap (S - x_i)) = \epsilon\mu(S) + O(|x_i|).$$

In fact, one easily proves that, in E_m,

$$\mu(\overline{S} \cap (S - x)) \leq m\rho^{-1}\mu(S)|x|.$$

From (3), it follows that

$$m_0 \geq (1 - k\epsilon)\mu(S) - O(|x_1|) - \cdots - O(|x_k|),$$

showing that to each positive number $\eta < (1-k\epsilon)\mu(S)$ there corresponds a positive number δ such that $m_0 > \eta$ whenever $|x_i| < \delta$, $i = 1, \cdots, k$.

The following result is due to Steinhaus [20, p. 99].

Let B_1 and B_2 be two linear sets, each of positive measure. Then $B_1 - B_2$ contains an interval of positive length. Moreover, if $B_1 = B_2$ then 0 is an interior point of $B_1 - B_2$.

Theorem 2.1 yields the following generalization.

THEOREM 2.2. Let B_1 and B_2 be subsets of E_m, each of positive measure. Then, after deleting suitable subsets of measure 0 from B_1 and B_2, the set $B_1 - B_2$ becomes an open set. In fact, $B_1^* - B_2^*$ (and, also, $B_1^* + B_2^*$) is open.

More precisely, to each point d in $B_1^* - B_2^*$ there correspond positive numbers δ and η, such that

$$|y - d| < \delta \text{ implies } \mu(B_1^* \cap (B_2^* + y)) > \eta.$$

Proof. Note that $B_i^* \subset B_i$ and $\mu(B_i^*) = \mu(B_i)$, $(i = 1, 2)$. Let $d = a_1 - a_2$ with $a_i \in B_i^*$, $(i = 1, 2)$. From Theorem 2.1, there exist positive numbers δ and η, such that

$$\mu(B_1^* \cap (B_2^* + y)) = \mu((B_1 - a_1) \cap (B_2 - a_1 + y)) > \eta,$$

whenever $|(a_1 - y) - a_2| = |y - d| < \delta$.

The following lemma will be needed in the proofs of Theorem 4.1 and Theorem 7.1.

LEMMA 2.1. Let B be a subset of E_m of positive measure. Let $\{\xi_n\}$ be a se-

quence of points in E_m, converging to 0. Let B' be the set of all those points x in B for which $x + \xi_n \in B$, infinitely often. Then B' is of positive measure.

Proof. We may assume that B is a bounded set. We have $0 \in B^* - B^*$, hence, from Theorem 2.2, there exist positive numbers δ and η, such that $\mu(B \cap (B - \xi)) > \eta$ whenever $|\xi| < \delta$. Let $C_n = B \cap (B - \xi_n)$; then $\mu(C_n) > \eta$ whenever $n > N_0$. Further,

$$B' = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} C_n = \bigcap_{N=1}^{\infty} D_N \qquad \text{(say)}. \qquad (1)$$

Here, $\mu(D_N) \geq \mu(C_N) > \eta$, for $N > N_0$, while each set of the decreasing sequence $\{D_N\}$ is a subset of the bounded set B. It follows that $\mu(B') \geq \eta > 0$.

Finally, we mention the following generalization of a result due to Ruziewicz [16].

Let B be a subset of E_m of positive measure. Then to each positive integer k there corresponds a positive number δ_k, such that, for each u in E_m with $|u| < \delta_k$, the set B contains an arithmetical progression $b_1, b_2, \cdots, b_k$ with $b_i - b_{i-1} = u$, $(i = 2, \cdots, k)$.

Proof. We may assume that 0 is a point of unit density of B. From Theorem 2.1, applied for $a_i = 0$, $B_i = B$, $(i = 1, \cdots, k)$, there exists a positive number δ, such that $\mu((B - x_1) \cap \cdots \cap (B - x_k)) > 0$, whenever $|x_i| < \delta$, $(i = 1, \cdots, k)$. Letting $x_i = (i-1)u$, $(i = 1, \cdots, k)$ with $(k-1)|u| < \delta$, it follows that the set of points b_1 with $b_1 + (i-1)u \in B$, $i = 1, \cdots, k$, is of positive measure, hence, nonempty.

Added in proof. Let X be a locally compact additive group, let μ denote a regular right Haar measure on X, and let $B_1, \cdots, B_k$ be Borel sets in X. It can be shown that there exists a sequence $\{U_n\}$ of neighborhoods of 0 with the following property. Let $\{S_n\}$ be a sequence of Borel sets with $\mu(S_n) > 0$, $S_n < U_n$; then there exists a Borel set D with $\mu(\overline{D}) = 0$, such that, for each $x \in D$,

$$\lim_{n \to \infty} \mu((B_i - x) \cap S_n)\mu(S_n)^{-1} = 1 \qquad \text{if } x \in B_i,$$

$$= 0 \qquad \text{if } x \in \overline{B}_i,$$

$i = 1, \cdots, k$. Letting $B_i^* = B_i \cap D$, the results and proofs of this section carry over to X. Somewhat simpler proofs are obtained by observing that the function $\mu((B_1 - x_1) \cap \cdots \cap (B_k - x_k))$ is jointly continuous in the x_i, cf. P. R. Halmos, *Measure theory*, New York, 1950, p. 266.

3. Additive functions.

DEFINITION. A subset B of E_m is said to be of *positive type* when a positive integer n can be found, such that the n-fold sum $B^{(n)} = B + \cdots + B$ contains a nonempty open subset. If n is the smallest such integer, we put $\nu(B) = 1/n$; when B is not of positive type we put $\nu(B) = 0$.

THEOREM 3.1. *Let $H(x)$ be a real-valued additive function on E_m and suppose that $H(x)$ is bounded above on a set B of positive type. Then $H(x)$ is of the form $H(x) = ax$, where a is a constant vector in E_m.*

Proof. Let $H(x) \leq M$ for $x \in B$, where M is a constant. Let $\nu(B) = 1/n$ and let $x \in B^{(n)}$; then x can be written as $x = x_1 + \cdots + x_n$ with $x_i \in B$, thus, $H(x) = H(x_1) + \cdots + H(x_n) \leq nM$. Let S be some open sphere $|x - x_0| < \epsilon$, completely contained in $B^{(n)}$. Then $|x| < \epsilon$ implies $x_0 + x \in B^{(n)}$ and $x_0 - x \in B^{(n)}$, hence,

$$H(x) = H(x_0 + x) - H(x_0) \leq nM - H(x_0)$$

and $-H(x) = H(-x) \leq nM - H(x_0)$. Consequently, $|H(x)| \leq K = nM - H(x_0)$ when $|x| < \epsilon$. The rest is classical. For $|x| < \epsilon/q$, where q is a positive integer, we have

$$|H(y + x) - H(y)| = |H(x)| = 1/q|H(qx)| \leq K/q,$$

thus, H is continuous everywhere. Finally, $H(\lambda x) = \lambda H(x)$ for λ rational, hence, for λ real.

REMARK. A slight modification of the above proof yields the following more general result. Let X be a topological linear space over the field of the reals, and let $H(x)$ be a real additive function on X, which is bounded above on a set of positive type. Then $H(\lambda x) = \lambda H(x)$ for λ real. Thus, if an additive function H on X is bounded on a set B, with $B - B$ of positive type, then both the real and imaginary part of H is bounded on $B' = B \cup (-B)$, which is of positive type, i.e. $H(\lambda x) = \lambda H(x)$ for λ real. This implies a result of Kestelman [8] for normed linear spaces.

From Theorem 2.2, if B is a subset of E_m of positive measure then $B^* + B^*$ is open, (thus, $\nu(B) \geq 1/2$). Consequently, if a real-valued additive function H on E_m is bounded above on a set of positive measure, (e.g. when H is measurable), then $H(x)$ is a linear function. In fact, this result[3] in E_m can be easily obtained from the special case $m = 1$, which is due to Ostrowski [15]. However, Theorem 3.1 applies to much thinner sets. For, the following sets B are all of measure 0 but of positive type.

1. In E_1, the closed set B of all real numbers of the form $\sum_1^\infty \epsilon_n p^{-n}$, where $\epsilon_n = 0, 1$ and where p is a fixed integer ≥ 3. Clearly, $\nu(B) = 1/(p-1)$. In general, let B_1 be any *closed* set to which there corresponds a denumerable set C in E_m such that E_m is the union of all the sets $c + B_1^{(n)}$, ($c \in C$; $n = 1, 2, \cdots$). Then B_1 is of positive type. For, at least one of the closed sets $c + B_1^{(n)}$ is not nowhere dense, hence, contains an interior point.

2. In E_2, the set B of points $(\cos \theta, \sin \theta)$, where θ runs through a set of real numbers of positive linear measure.

3. In E_m, the set $B = B_1 \cup \cdots \cup B_m$, where B_j is a set of positive linear

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(3) It can also easily be deduced from Lemma 2.1, cf. the proof of Theorem 4.1.

type situated on a straight line L_j, $(j=1, \cdots, m)$, provided that no hyperplane is parallel to each L_j; cf. Kestelman [8, 146].

4. **Difference properties.** In this paper, a *local condition* in E_m is defined by a class P^* of functions on E_m such that $g_1 \in P^*$, $g_2 \in P^*$, together imply $g_1 - g_2 \in P^*$. Let $x_0 \in E_m$ and let $f(x)$ be a function defined on an open set containing x_0. We say that $f(x)$ satisfies at x_0 the local condition P associated to P^* when a function $g \in P^*$ and a positive number ϵ can be found, such that $f(x) = g(x - x_0)$ for $|x - x_0| < \epsilon$, (thus, a restriction imposed on the behavior of $f(x)$ in the immediate neighborhood of x_0). It follows that any local condition P has the following properties: (i) If $f(x)$ satisfies P at x_0 then, for each $h \in E_m$, the function $f(x+h)$ satisfies P at $x_0 - h$; (ii) If f_1 and f_2 satisfy P at x_0 then also $f_1 - f_2$ and $f_1 + f_2$.

Let P be a local condition and let A be an open subset of E_m. Then the class of all functions f on A, satisfying P at each point of A, will be denoted as $\Omega_P(A)$. Further, for $h \in E_m$, we put

$$A \cap (A - h) = A_h.$$

Thus, when $f(x)$ is defined on A then $\Delta_h f(x) = f(x+h) - f(x)$ is defined on A_h.

DEFINITION. A local condition P is said to be *bounded*, (or *measurable*, respectively), when a function, which satisfies P at a point x_0, is always bounded, (or measurable, respectively), in some (open) neighborhood of x_0.

DEFINITION. A local condition is said to have the *difference property* when the following is true for each open subset A of E_m. Let f be a function on A such that $\Delta_h f \in \Omega_P(A_h)$ for each $h \in E_m$. Then there exists an additive function H on E_m, such that $f - H \in \Omega_P(A)$.

DEFINITION. A local condition P is said to have the *weak difference property* when the following is true for each open subset A of E_m. Let $f(x)$ be a function on A, which is bounded on a subset B of A of positive measure, and such that $\Delta_h f \in \Omega_P(A_h)$ for each $h \in E_m$. Then $f \in \Omega_P(A)$.

DEFINITION. A class Ω of functions on E_m is said to have the *difference property* when to each function f on E_m, such that $\Delta_h f \in \Omega$ for each $h \in E_m$, there corresponds an additive function H on E_m, such that $f - H \in \Omega$.

De Bruijn [2; 3] proved the difference property for a great number of classes of functions on E_1. We mention:

(B1) The class of all polynomials.

(B2) The class of all trigonometric polynomials of period 1.

(B3) The class of all functions, analytic for $-\infty < x < +\infty$.

(B4) The class of all functions, which can be extended to a function $f(x+iy)$, analytic in the strip $|y| \leq 1$.

(B5) The class of all real functions with $|f(x) - f(0)| \leq 1$.

Further, his results easily imply that each of the following "adjectives" represents a (bounded and measurable) local condition in E_1 having the difference property. (Observe that, for P as a local condition in E_m with the

difference property, the class $\Omega_P(E_m)$ always has the difference property.)

(B6) Continuous. (B7) Riemann integrable, (=bounded and continuous almost everywhere). (B8) Absolutely continuous. (B9) Of bounded variation. (B10) k times differentiable. (B11) k times differentiable, kth derivative bounded. (B12) k times differentiable, kth derivative continuous. Here, k is a fixed integer $\geqq 1$.

A function $f(x)$ satisfies such a local condition at the point x_0 when it is defined and continuous (or Riemann integrable, etc.) throughout an open interval containing x_0. We note that the local conditions "measurable" and "bounded" do not have the difference property, cf. [2, p. 195, p. 217]. However, the latter condition does have the weak difference property, as is shown by the following theorem.

THEOREM 4.1. *In E_m, let P be the local condition of boundedness, i.e. a function f satisfies P at a point x_0 if and only if f is defined and bounded in some neighborhood of x_0. Then P has the weak difference property.*

Proof. Let the function f be defined on an open set A, such that $|f(x)| \leqq M$ = constant throughout a subset B of A of positive measure, while $\Delta_h f \in \Omega_P(A_h)$ for $h \in E_m$. Suppose that $f(x)$ does not satisfy P at some point x_0 in A. Then there exists a sequence $\{\xi_n\}$ of points in E_m, converging to 0, and such that $|f(x_0+\xi_n)| \to \infty$ for $n \to \infty$. From Lemma 2.1, there exists a point y_0, $y_0 \in B' \subset B \subset A$, such that $y_0+\xi_n \in B$ for infinitely many n, hence, $|f(y_0+\xi_n')| \leqq M$ for some infinite subsequence $\{\xi_n'\}$ of $\{\xi_n\}$. But then, for $h = y_0 - x_0$, we have $|\Delta_h f(x_0+\xi_n')| \to \infty$ for $n \to \infty$, contradicting the assumption that $\Delta_h f(x)$ satisfies P at x_0.

LEMMA 4.1. *Let P be a local condition such that: (i) P has the difference property; (ii) P is either bounded or measurable; (iii) If a is a constant vector then the linear function $g(x) = ax$ satisfies P at each point x_0. Then P has the weak difference property.*

Proof. Let the function f be defined as in the preceding proof. P having the difference property, it follows that $f = g + H$ with $g \in \Omega_P(A)$ and H additive. It suffices to prove that H is linear; for then, $H \in \Omega_P(A)$, hence, $f \in \Omega_P(A)$. When P is measurable, $g \in \Omega_P(A)$ is measurable on B, thus, bounded on a subset B_1 of B of positive measure. But then the additive function H is bounded on B_1, implying that H is linear, (cf. §3). When P is bounded, $g(x)$ is bounded on each compact subset of A. Hence, letting B_1 be any compact subset of B of positive measure the same conclusion holds.

REMARK. Condition (iii) is in some sense a necessary condition. For let P be any local condition with the weak difference property and let $g(x)$ be a polynomial (in the components $x_1, \cdots, x_m$ of x). Then $g(x)$ satisfies P at each point x_0, as is easily shown by induction with respect to the degree of g.

5. **Local conditions having the difference property.** In this section, we shall establish the difference property for certain local conditions P in E_m. The reader will observe that each of these local conditions P satisfies the assumptions of Lemma 4.1, thus, P has also the weak difference property. The methods of proof are similar to those of de Bruijn [2]; as to his results, we shall only use the fact that, in E_1, each of the local conditions of continuity and differentiability has the difference property.

THEOREM 5.1. *In E_m, the (bounded and measurable) local condition of continuity has the difference property.*

Proof. Let A be an open subset of E_m and let $f(x) = f(x_1, \cdots, x_m)$ be defined on A, such that $\Delta_h f(x)$ is continuous throughout A_h, for each $h \in E_m$. Let y be a fixed point in A. Because Theorem 5.1 holds for $m = 1$, we have for each integer j $(1 \leq j \leq m)$ that there exists an additive function $H_j(x_j)$ on E_1, such that, at the point y, the function $f(x_1, \cdots, x_m) - H_j(x_j)$ is continuous in x_j. Let $H(x)$ be the additive function on E_m, defined by

(1) $\qquad H(x) = H(x_1, \cdots, x_m) = H_1(x_1) + \cdots + H_m(x_m).$

Then, at y, the function $g(x) = f(x) - H(x)$ is continuous in each x_j. From

(2) $\qquad g(x + h) = g(x) + \Delta_h g(x) = g(x) + \Delta_h f(x) - H(h),$

applied for $h = x' - y$, it follows that $g(x)$ is continuous in each x_j at every point x' in A. Now, by a theorem of Hahn [7], there exists a point in A at which $g(x)$ is jointly continuous in $x_1, \cdots, x_m$. Thus, applying (2) again, $g(x)$ is continuous at each point in A.

THEOREM 5.2. *Let k be a fixed positive integer or $k = \infty$, and let D stand for any differential operator in E_m of the form*

(3) $\qquad\qquad D = \left(\dfrac{\partial}{\partial x_{i_1}}\right) \cdots \left(\dfrac{\partial}{\partial x_{i_p}}\right) \qquad\qquad with\ 0 \leq p \leq k,$

$(p < \infty)$. *To each such operator D we associate a local condition P_D with the weak difference property; (note that the condition P_D may be void). Let $g(x)$ be a function defined in a neighborhood of the point x_0 in E_m. We say that $g(x)$ satisfies at x_0 the local condition Q when for each D: (i) Dg exists throughout some neighborhood of x_0; (ii) Dg satisfies at x_0 the local condition P_D. Assertion: the local condition Q is measurable and has the difference property.*

REMARK 1. We shall say that a function $g(x)$ satisfies at x_0 the local condition $Q_1(k)$ when Dg exists throughout a neighborhood of x_0, for each D of the form (3). Taking each P_D as the empty condition, Theorem 5.2 implies that $Q_1(k)$ has the difference property. Clearly, $Q_1(k)$ is the straightforward generalization to E_m of the local condition (B10) in E_1. Theorem 5.2 also im-

plies the difference property of similar generalizations to E_m of (B11), (using Theorem 4.1), and of (B12), (using Theorem 5.1 and Lemma 4.1).

REMARK 2. Let $\phi(x) = \phi(x_1, \cdots, x_m)$ be defined on an open set A, such that all its first partial derivatives exist in A. Then, throughout A, $\phi(x)$ is continuous in each x_j, implying (cf. Lebesgue [10, 201]) that $\phi(x)$ is of the $(m-1)$th Baire class, consequently, $\phi(x)$ is measurable in A. This implies a.o. that the above local condition Q is measurable.

Proof of Theorem 5.2. We first show that $Q_1(k)$ has the difference property. Let $f(x) = f(x_1, \cdots, x_m)$ be defined on an open set A, such that, for each $h \in E_m$ and each D of the form (3), $D\Delta_h f$ exists throughout A_h. Let y be a fixed point in A. Because, in E_1, the local condition of differentiability has the difference property, there exists an additive function H_j on E_1, such that, at the point y, $f(x_1, \cdots, x_m) - H_j(x_j)$ is differentiable with respect to x_j. Let $H(x)$ be the additive function on E_m, defined by (1), and let $g(x) = f(x) - H(x)$. Then $g(x)$ is differentiable with respect to each x_j, first at the point y, but hence, (applying (2) for $h = x' - y$), at each point x' in A. This already shows that $Q_1(1)$ has the difference property. Further, from Remark 2, $g(x)$ is a measurable function on A.

Let $1 \leq k_0 < k \leq \infty$; it suffices to prove that $D'g$ exists in A for each D' of order $k_0 + 1$, given that Dg exists in A for each D of order k_0. Thus, let D be a fixed operator of order k_0; then Dg exists as a measurable function on A. Moreover,

$$(4) \qquad \Delta_h Dg = D\Delta_h g = D(\Delta_h f - H(h)),$$

thus, for $i = 1, \cdots, m$, $\partial \Delta_h Dg / \partial x_i$ exists throughout A_h, for each $h \in E_m$. Because $Q_1(1)$ has the difference property, we have $Dg = g_1 + H_1$, where H_1 is additive on E_m and g_1 has all its first derivatives throughout A. From Remark 2, g_1 is measurable on A; hence, H_1 is measurable and, thus, linear. Consequently, for each D of order k_0, all the first derivatives of Dg exist in A, i.e. $D'g$ exist in A, for each D' of order $k_0 + 1$.

Now, let f be defined on A, such that, for each $h \in E_m$, $\Delta_h f$ satisfies Q and, thus, $Q_1(k)$ throughout A_h. From the above result, $f = g + H$, where H is additive, while g is a measurable function on A satisfying $Q_1(k)$ at each point in A. Let D be fixed. Then Dg is measurable on A, hence, bounded on a subset of A of positive measure. Further, from (4) and the definition of Q, we have, for each $h \in E_m$, that $\Delta_h Dg$ satisfies the local condition P_D at each point in A_h. Because P_D has the weak difference property, it follows that, for each D, the function Dg satisfies P_D in A, i.e., g satisfies Q in A.

As follows, we introduce a new local condition Λ in E_m, which generalizes the local conditions (B1), (B3) and (B4) in E_1. Let R be the set of all the vectors $r = (r_1, \cdots, r_m)$ in E_m with non-negative integral components r_j. We call $s = (s_1, \cdots, s_m)$ a predecessor of $r \in R$, when for some index j, $1 \leq j \leq m$, we have $s_i = r_i$ for $i \neq j$ and $s_j = r_j - 1$. Let Λ^* be a *finite or denumerable* class of

"sequences" $\{\lambda_r\}$, with index set R, and such that $0 \leqq \lambda_r \leqq \infty$, $\lambda_0 > 0$. We assume that: (i) For each pair of sequences $\{\lambda_r\}$ and $\{\lambda_r'\}$ in Λ^*, one can find a sequence $\{\lambda_r''\}$ in Λ^* and a positive constant c with $\lambda_r + \lambda_r' \leqq c\lambda_r''$, for each $r \in R$; (ii) For each sequence $\{\lambda_r\}$ in Λ^*, one can find a sequence $\{\lambda_r'\}$ in Λ^* and a positive constant c, such that to each $r \in R$ with $|r| \geqq 1$ there corresponds a predecessor $s \in R$ of r with $\lambda_s \leqq c\lambda_r'$.

A function $g(x)$, defined in a neighborhood of the point $x_0 \in E_m$, will be said to satisfy the local condition Λ at x_0, when there exist a neighborhood N of x_0, a sequence $\{\lambda_r\}$ in Λ^*, and a positive constant c, such that: (i) All mixed derivatives of g exist and are continuous throughout N, (and, thus, are independent of the order in which the single differentiations are performed); (ii) $|D_r g(x)| \leqq c\lambda_r$, for $r \in R$ and $x \in N$, where

$$D_r = \left(\frac{\partial}{\partial x_1}\right)^{r_1} \cdots \left(\frac{\partial}{\partial x_m}\right)^{r_m}.$$

THEOREM 5.3. *The local condition Λ has the difference property. Obviously, Λ is bounded and measurable.*

Proof. Let A be an open subset of E_m and let $f(x)$ be defined on A, such that, for each $h \in E_m$, $\Delta_h f$ satisfies Λ at each point of A_h; without loss of generality, we may assume that $f(x)$ is real-valued. Especially, for $r \in R$ and $h \in E_m$, $D_r \Delta_h f$ exists and is continuous throughout A_h. In view of Theorem 5.1 and Lemma 4.1, Theorem 5.2, applied for $k = \infty$, yields the decomposition $f = g + H$, where H is additive on E_m, while $D_r g$ exists and is continuous throughout A, for each $r \in R$.

Let x_0 be a fixed point in A; it suffices to prove that $g(x)$ satisfies Λ at x_0. Let ϵ_1 and ϵ_2 be positive numbers, such that the sphere $|x - x_0| \leqq \epsilon_1 + \epsilon_2$ is entirely contained in A. Let K be the sphere $|x - x_0| \leqq \epsilon_1$ and let S be the sphere $|x| \leqq \epsilon_2$; then $K \subset A_h$ for $h \in S$.

For the moment, let h be fixed, $h \in S$. From the assumptions on $\Delta_h f(x) = \Delta_h g(x) + H(h)$, we have for each point $\xi \in K \subset A_h$ that there exist an open neighborhood $N_\xi \subset A_h$ of ξ, a sequence $\{\lambda_r(\xi)\}$ in Λ^* and a positive constant $c(\xi)$, such that $|D_r \Delta_h g(x)| \leqq c(\xi) \lambda_r(\xi)$ for $r \in R$, $x \in N_\xi$. From the compactness of K and property (i) of Λ^*, it follows that to each $h \in S$ there correspond a sequence $\{\lambda_r\}$ in Λ^* and a positive integer M, such that

(5) $|D_r \Delta_h g(x)| \leqq M\lambda_r$ for $r \in R$, $x \in K$.

For a given sequence $\{\lambda_r\}$ in Λ^* and a given positive integer M, let S_j be the set of all the points $h \in S$ for which (5) holds. Then S is the union of all possible sets S_j. Moreover, Λ^* being enumerable, there are only denumerably many such sets S_j. Hence, it is not true that each S_j is nowhere dense in S, i.e. there exists an S_j which is dense in some sphere S', contained in S, (say) the sphere $|x - x_1| \leqq \delta$ with $\delta > 0$. If $\{\lambda_r\}$ and M, (which are fixed from now

on), correspond to this S_j then (5) holds for a set of values h dense in S'. But $D_r g(x)$ is continuous in A; thus, for $x \in K$, $h \in S'$, (hence, $x \in A$, $x+h \in A$), $D_r \Delta_h g(x)$ is a continuous function of h. It follows that (5) holds for each $h \in S'$.

Let x, y be any pair of points in K with $|x-y| \leq 2\delta$. Applying (5) for $h = h_1 = x_1 + 2^{-1}(y-x) \in S'$ and also for $h = h_2 = x_1 - 2^{-1}(y-x) \in S'$, we have from $x + h_1 = y + h_2$ that

$$\left| D_r g(y) - D_r g(x) \right| = \left| D_r \Delta_{h_1} g(x) - D_r \Delta_{h_2} g(y) \right| \leq 2M\lambda_r,$$

for each $r \in R$. Let M' be the smallest integer $\geq \epsilon_1/\delta$; K being the sphere $|x-x_0| \leq \epsilon_1$, we conclude that

(6) $$\left| D_r g(y) - D_r g(x) \right| \leq 2MM'\lambda_r \qquad \text{if } r \in R, \ x \in K, \ y \in K.$$

Let $r \in R$, $|r| \geq 1$, and let s be any predecessor of r, thus, $D_r g = \partial D_s g/\partial x_j$ for some j, $1 \leq j \leq m$. Let e_j be the unit vector parallel to the x_j-axis. Applying (6) for r replaced by s and $x = x_0 - \epsilon_1 e_j$, $y = x_0 + \epsilon_1 e_j$, it follows from the mean value theorem that (for each predecessor s of r) there exists a point $y \in K$, such that

(7) $$\left| D_r g(y) \right| \leq MM'\lambda_s/\epsilon_1.$$

Combining (6) and (7), we obtain that, for each $r \in R$ with $|r| \geq 1$, and any predecessor s of r,

$$\left| D_r g(x) \right| \leq MM'(2\lambda_r + \lambda_s/\epsilon_1) \qquad \text{if } x \in K.$$

Further, for $r = 0$, the continuous function $D_r g(x) = g(x)$ is certainly bounded on K. We now conclude from the properties of the class Λ^* that there exist a sequence $\{\lambda_r'\}$ in Λ^* and a positive constant c such that $|D_r g(x)| \leq c\lambda_r'$ for $r \in R$, $x \in K$. This shows that $g(x)$ satisfies at x_0 the local condition Λ.

We shall now exhibit some applications of Theorem 5.3. As is easily verified, each of the following classes Λ^* satisfies the required properties (i) and (ii). From Theorem 5.3, for each such Λ^*, the corresponding local condition Λ has the difference property, i.e. when A is an open subset of E_m and the function f is defined on A, such that $\Delta_h f \in \Omega_\Lambda(A_h)$ for $h \in E_m$, then $f - H \in \Omega_\Lambda(A)$ for some additive function H on E_m. Consequently, the class $\Omega_\Lambda(E_m)$, of all the functions on E_m satisfying Λ everywhere, has the difference property.

1. Let Λ^* consist of the sequences $\{\lambda_r^{(n)}\} = \{\lambda_{r_1,\dots,r_m}^{(n)}\}$, $(n = 0, 1, \cdots)$, with $\lambda_r^{(n)} = 1$ if $r_1 + \cdots + r_m \leq n$, $\lambda_r^{(n)} = 0$, otherwise. Then $g \in \Omega_\Lambda(A)$ if and only if $g(x)$ is equal to a polynomial throughout each (connected) component of A. Thus, we have a.o. that the class $\Omega_\Lambda(E_m)$ of all polynomials on E_m has the difference property; this generalizes the corresponding result (B1) in E_1, due to de Bruijn [2].

2. Let Λ^* consist of the sequences $\{\lambda_r^{(N)}\}$, $(N = 1, 2, \cdots)$, with

$$\lambda_r^{(N)} = \lambda_{r_1,\ldots,r_m}^{(N)} = r_1! \cdots r_m! N^{r_1+\cdots+r_m}.$$

Then $g \in \Omega_\Lambda(A)$ if and only if, throughout each (connected) component of A, $g(x) = g(x_1, \cdots, x_m)$ is an analytic function of the real variables $x_1, \cdots, x_m$. Thus we have a.o. that the class $\Omega_\Lambda(E_m)$, of all the analytic functions on E_m, has the difference property; this generalizes the corresponding result (B3) due to de Bruijn [2].

3. Let $R_1, \cdots, R_m$ be fixed positive numbers, and let Λ^* consist of all the sequences $\{\lambda_r^{(N)}\}$, $(N = 1, 2, \cdots)$, of the form

$$\lambda_{r_1,\ldots,r_m}^{(N)} = r_1! \cdots r_m! (R_1 + 1/N)^{-r_1} \cdots (R_m + 1/N)^{-r_m}.$$

Then $g \in \Omega_\Lambda(A)$ if and only if the following is true for each convex subset A' of A. In the space of m complex variables $z_j = x_j + iy_j$, $(x_j, y_j$ real; $j = 1, \cdots, m)$, let A'' be the set of points satisfying

(8) $\left| z_j - x_j' \right| \leq R_j,$ $j = 1, \cdots, m$, for some $(x_1', \cdots, x_m') \in A'$.

Then there exists a single-valued function $\bar{g}(x+iy) = \bar{g}(z_1, \cdots, z_m)$ on A'', analytic in the complex variables $z_1, \cdots, z_m$, such that $\bar{g}(x) = g(x)$ for $x \in A'$. It follows that the class $\Omega_\Lambda(E_m)$, of all analytic functions on E_m, extendable to an analytic function of $z_1, \cdots, z_m$ in the region $\left| y_j \right| \leq R_j$ $(j = 1, \cdots, m)$, has the difference property; this generalizes the corresponding result (B4) in E_1, due to de Bruijn [2]. As an easy consequence, the class of "entire" functions on E_m has the difference property.

4. Let $m = 1$, and let Λ^* consist of the sequences $\{N^r B_r\}$, $(N = 1, 2, 3, \cdots)$, where $\{B_r\}$ is a a fixed sequence of positive numbers with $B_r > \epsilon^r B_{r-1}$, ϵ being a positive constant. Then $g \in \Omega_\Lambda(A)$ if and only if $\left| D_r g(x) \right|^{1/r} = O(B_r^{1/r})$, uniformly in any finite closed subinterval of A. Note that the class $\Omega_\Lambda(A)$ is quasi-analytic if and only if A is an open interval and

$$\sum_{r=0}^{\infty} \left[\sup_{s \geq r} B_s^{-1/s} \right] = \infty,$$

cf. Carleman [4]. The latter condition holds, for instance, when $B_r \leq r!$ for infinitely many positive integers r.

6. **A functional equation.** Let $\phi(x, y)$ be a function on $E_m \times E_m$, admitting the following representation as a sum of functions of one variable:

(1) $$\phi(x, y) = \sum_{i=0}^{k} f_i(a_i x + b_i y),$$

a_i, b_i being given real constants with

$$a_i \neq 0, \qquad a_i b_j - a_j b_i \neq 0, \qquad\qquad (i, j = 0, \cdots, k; i \neq j).$$

For $k+1 \geq 3$, this representation is certainly not unique. For, let $\gamma_0, \cdots, \gamma_k$

be real constants, not all zero, with $\sum a_i \gamma_i = \sum b_i \gamma_i = 0$, and let H be an additive function on E_m with $H(a_i x) = a_i H(x)$, $H(b_i x) = b_i H(x)$, $(i = 0, \cdots, k)$[4]. Then $\sum \gamma_i H(a_i x + b_i y) = 0$, hence, if $f_i(x)$ $(i = 0, \cdots, k)$ is a solution of (1) then also $f_i(x) + \gamma_i H(x)$ $(i = 0, \cdots, k)$. But $H(x)$ is, in general, unbounded on each set of positive measure, hence, the same is true for most solutions of (1).

In order to rule out such pathological solutions, we might consider only those solutions $f_i(x)$ $(i = 0, \cdots, k)$ of (1), for which each function $f_i(x)$ is bounded on some set of positive measure. Assertion (i) of the following Theorem 6.1, (applied for $A = C = E_m$), easily implies that, for each such solution of (1), all the functions $f_i(x)$ are necessarily continuous, (or analytic, or polynomial, etc.), whenever, for each $y \in E_m$, the function $\phi_y(x) = \phi(x, y)$ is continuous, (or analytic, or polynomial, respectively). Clearly, this is a generalization of the following classical result, (cf. §3). Let $f(x)$ be a function on E_m, such that: (i) $f(x)$ is bounded on a set of positive measure; (ii) $f(x)$ is a solution of the functional equation

$$f(y) = f(x + y) - f(x).$$

Then $f(x)$ is continuous on E_m, in fact, $f(x)$ is a (linear) polynomial.

THEOREM 6.1. *In E_m, let A be an open connected set and let C be a set of positive measure. Let $\phi(x, y)$ be a given function on $A \times C$ and consider the relation*

$$(2) \qquad\qquad \phi(x, y) = \sum_{i=0}^{k} f_i(x + b_i y),$$

where $k \geq 0$ is a given integer and the b_i are given real numbers, $b_i \neq b_0 = 0$ $(i = 1, \cdots, k)$. By a solution of (2) we mean a set of functions $f_i(x)$ $(i = 0, 1, \cdots, k)$, $f_i(x)$ being defined on $A + b_i C$, such that (2) holds for all pairs of points x, y with $x \in A$, $y \in C$. Let $f_i(x)$ $(i = 0, \cdots, k)$ and $f_i^*(x)$ $(i = 0, \cdots, k)$ be two solutions of (2), such that each of the two functions $f_0(x)$ and $f_0^*(x)$ is bounded on some subset of A of positive measure.

Assertion: (i) *Let P be a local condition in E_m having the weak difference property. Suppose that, for each $y_0 \in C$, the function $\phi(x, y_0)$ satisfies P at each point of A. Then $f_0(x)$ satisfies P at each point of A.*

(ii) *If, in addition to the assumptions of (i), P is either bounded or measurable, then $f_0(x) - f_0^*(x)$ is a polynomial on A of degree $\leq k - 1$.*

(iii) *Suppose that, for each $y_0 \in C$, the function $\phi(x, y_0)$ is a polynomial on A of degree $\leq n$, (n fixed). Then $f_0(x)$ is a polynomial on A of degree $\leq n + k$.*

Here, the condition, that C be of positive measure, cannot be weakened very much, cf. §1. In assertion (iii) we allow the value $n = -1$, provided that a

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[4] One way of obtaining such a function H is by means of a Hamel basis of E_m over the field R, obtained by adjoining the a_i and b_i to the field of rational numbers, cf. §1.

polynomial of degree -1 is interpreted as the zero-function. Observe that (ii) is a consequence of (i) and (iii). For, suppose that the conditions of (ii) hold. From (i), f_0 and f_0^*, thus, $f_0' = f_0 - f_0^*$ satisfy P throughout A. Because P is bounded or measurable, f_0' is bounded on a subset of A of positive measure. Further, $f_i^* = f_i - f_i'$ $(i = 0, \cdots, k)$ is a solution of (2) with $\phi(x, y)$ replaced by 0, a polynomial of degree -1. Hence, from (iii), f_0' is a polynomial on A of degree $\leqq k - 1$.

We shall say that a function $f(x)$ satisfies Q at a point x_0, when $f(x)$ is defined and equal to a polynomial throughout an entire neighborhood of x_0. Clearly, Q is a bounded local condition. From application 1 of Theorem 5.3, Q has the difference property. Hence, from Lemma 4.1, Q has also the weak difference property. Now, suppose that the conditions of assertion (iii) hold. Then (i), (applied to Q instead of P), already shows that f_0 satisfies Q throughout A, i.e. f_0 is a polynomial on A. Consequently, it suffices to prove (i) and the part of (iii) which asserts that the polynomial f_0 is of degree $\leqq n + k$.

Proof of Theorem 6.1. From $b_0 = 0$, everything is obvious when $k = 0$. Let k be a fixed integer, $k \geqq 1$, and suppose that Theorem 6.1 holds when k is replaced by $k - 1$. Let $f_i(x)$ $(i = 0, \cdots, k)$ be a solution of (2), such that $f_0(x)$ is bounded on a subset B of A of positive measure. Let x_0 be an arbitrary but fixed point in A and let ξ_0 be any point of unit density of B, $\xi_0 \in A$. Select any polygonal path (of finite length) L in A with endpoints ξ_0 and x_0, and let ϵ denote the distance between L and the complement of A, $0 < \epsilon \leqq \infty$. Finally, let A' be the set of all the points in E_m having a distance $< \epsilon/2$ from L. Then: (a) A' is open and connected, $x_0 \in A'$; (b) For each $h \in E_m$ with $|h| \leqq \epsilon/2$, we have $A' + h \subset A$, especially, $A' \subset A$; (c) The function $f_0(x)$ is bounded on the subset $B' = A' \cap B$ of A', where B' is of positive measure.

From Theorem 2.2, applied for $d = 0$, (or from the fact that B' and C possess points of unit density), there exists a positive number δ, $\delta \leqq \epsilon/2$, such that, for each $h \in E_m$ with $|h| \leqq \delta$, both sets $B'(h) = B' \cap (B' - h)$ and $C(h) = C \cap (C + h/b_k)$ are of positive measure; (note that $b_k \neq 0$). For the moment, let $h \in E_m$ be arbitrary but fixed, $|h| \leqq \delta$. Then, $B'(h)$ and $C(h)$ are of positive measure and $A' + h \subset A$. Further, let

$$(3) \qquad h_i = (1 - b_i/b_k)h, \quad g_i(x) = f_i(x + h_i) - f_i(x), \qquad (i = 0, \cdots, k).$$

Here, $h_k = 0$ and, (by $b_0 = 0$), $h_0 = h$, thus, $g_k(x) = 0$ and $g_0(x) = \Delta_h f(x)$. Moreover, $x \in B'(h)$ implies $x \in B'$ and $x + h \in B'$, hence, $g_0(x)$ is bounded on $B'(h)$, where $B'(h) \subset B' \subset A'$. Observe that, for $i = 0, \cdots, k$, the function $g_i(x)$ is well-defined on $A' + b_i C(h)$. For, from $A' \subset A$, $A' + h \subset A$ and the definition of $C(h)$, it follows that $x \in A' + b_i C(h)$ implies $x \in A + b_i C$ and, also, $x + h_i \in A' + b_i(C + h/b_k) + h_i = A' + h + b_i C \subset A + b_i C$.

Let y' be any point in $C(h)$. Then $y' \in C$ and $y' - h/b_k = y'' \in C$, while, from (3),

(4) $$b_i y'' + h = b_i y' + h_i, \qquad\qquad (i = 0, \cdots, k).$$

Under the conditions of assertion (i), (denoted as "case (i)"), both functions $\phi(x, y')$ and $\phi(x, y'')$ satisfy P throughout A, hence, (from $A'+h \subset A$, $A' \subset A$), the function

(5) $$\psi(x, y') = \phi(x + h, y'') - \phi(x, y')$$

satisfies P throughout A'. Similarly, under the conditions of assertion (iii), (denoted as case (iii)), $\psi(x, y')$ is a polynomial on A' of degree $\leq n$. Finally, from (2), (3), (4), (5) and $g_k(x) = 0$,

(6) $$\psi(x, y') = \sum_{i=0}^{k-1} g_i(x + b_i y'), \qquad (x \in A', y' \in C(h)).$$

From $b_i \neq b_0 = 0$, $(i = 0, \cdots, k-1)$, and the fact that $g_0(x)$ is bounded on the subset $B'(h)$ of A' of positive measure, it now follows by induction, (replacing A, C and $f_i(x)$, $i = 0, \cdots, k$, by A', $C(h)$ and $g_i(x)$, $i = 0, \cdots, k-1$, respectively): (a) In case (i), that $g_0(x) = \Delta_h f_0(x)$ satisfies P throughout A'; (b) In case (iii), that $\Delta_h f(x)$ is a polynomial on A' of degree $\leq n+k-1$. In fact, the latter conclusions hold for any $h \in E_m$ with $|h| \leq \delta$.

Now, suppose first that case (i) holds. Let $h \in E_m$ and $x_1 \in A'$ be such that $x_1 + h = y_1 \in A'$. Because A' is an open, connected set, there exists a sequence of points $x_2, \cdots, x_p, x_{p+1} = y$ in A', such that, for $h_j = x_{j+1} - x_j$, we have $|h_j| \leq \delta$, $j = 1, \cdots, p$. Thus, the identity

$$\Delta_h f_0(x) = \Delta_{h_1} f_0(x) + \Delta_{h_2} f_0(x + h_1) + \cdots + \Delta_{h_p} f_0(x + h_1 + \cdots + h_{p-1})$$

implies that $\Delta_h f_0(x)$ satisfies P at x_1, (for each $h \in E_m$ and each $x_1 \in A_h'$). Because P has the weak difference property, and $f_0(x)$ is bounded on the subset B' of A' of positive measure, it follows that $f_0(x)$ satisfies P throughout A', especially at the point $x_0 \in A'$. Here, x_0 was chosen as an arbitrary point in A, hence, $f_0(x)$ satisfies P throughout A. This proves assertion (i).

Now, suppose that case (iii) holds. From assertion (i) and the remark preceding this proof, $f_0(x)$ is a polynomial on A. Furthermore, for each $h \in E_m$ with $|h| \leq \delta$, $\Delta_h f_0(x)$ is a polynomial on A' of degree $\leq n+k-1$. Consequently, $f_0(x)$ is a polynomial of degree $\leq n+k$.

Let P be a given local condition in E_m. Suppose that the conditions of Theorem 6.1, together with $\phi(x, y_0) \in \Omega_P(A)$ for each $y_0 \in C$, always imply $f_0 \in \Omega_P(A)$. Then P has the weak difference property, showing that the condition of assertion (i) is, in some sense, a necessary condition.

For, let A be any open set in E_m, and let f be defined on A, such that: (i) For each $h \in E_m$, and each $x_0 \in A$ with $x_0 + h \in A$, the function $\Delta_h f$ satisfies P at x_0; (ii) f is bounded on a subset B of A of positive measure. Let x_1 be a fixed point of unit density for B, $x_1 \in A$, having a distance ϵ from the complement of A. Let A' be the open sphere $|x - x_1| < \epsilon/2$ and let C be the set of

positive measure $|x| \leqq \epsilon/2$. Then, for each $y \in C$, the function

$$\phi(x, y) = - \Delta_y f(x) = f(x) - f(x + y)$$

satisfies P throughout A'. Moreover, f is bounded on the subset $B' = A' \cap B$ of A' of positive measure. From the above assumption concerning P, we have $f \in \Omega_P(A')$, especially, f satisfies P at the point $x_1 \in A' \subset A$. Applying $f(x+h) = f(x) + \Delta_h f(x)$ for $h = x_2 - x_1$, it follows that f satisfies P at each point $x_2 \in A'$. This shows that P has the weak difference property.

As an interesting consequence of Theorem 6.1, we have

THEOREM 6.2. *Let P be a bounded, or measurable, local condition in E_m having the weak difference property. Let $b_0, b_1, \cdots, b_k$ be given real numbers with $b_i \neq b_0$, $(i=1, \cdots, k)$. Let $f_i(x)$ $(i=0, \cdots, k)$ be functions on the open connected subset A of E_m, such that $f_0(x)$ is bounded on the subset B of A of positive measure and that, for each $y \in C$, the function*

$$\phi(x, y) = \sum_{i=0}^{k} f_i(x + b_i y)$$

satisfies P at each point x_0, where it is defined, (i.e. $x_0 + b_i y \in A$ for all i). Here, C is a subset of E_m, such that, to each point $x_0 \in A$, there corresponds a point $c \in E_m$ at which C is of unit density and with $x_0 + (b_i - b_0)c \in A$ for $i = 1, \cdots, k$; (example: C is of unit density at 0). Then the function $f_0(x)$ satisfies P throughout A.

Proof. Let A^* be the set of all the points $x_0 \in A$, such that, within each neighborhood of x_0, one can find a set of positive measure on which $f_0(x)$ is bounded. Clearly, A^* is closed relative to A. Further, A^* is nonempty, because $x_0 \in A^*$ for each point $x_0 \in A$ at which B is of unit density.

Let x_0 be a fixed point in A^*. Then there exists a point c_0 in C, at which C is of unit density, and such that $x_i = x_0 + (b_i - b_0)c_0 \in A$, $(i = 1, \cdots, k)$. Because A is open, there exist positive numbers δ_1 and δ_2, such that $|x - x_0| < \delta_1$ and $|c - c_0| < \delta_2$ together imply $x + (b_i - b_0)c \in A$, $(i = 1, \cdots, k)$. Let A_0 be the open sphere $|x - x_0| < \delta_1$ and let C_0 be the set of all the points c in C with $|c - c_0| < \delta_2$. Then C_0 is of positive measure. Further, from $x_0 \in A^*$, the function $f_0(x)$ is bounded on some subset of A_0 of positive measure. Finally, the function

$$\psi(x, y) = \phi(x - b_0 y, y) = \sum_{i=0}^{k} f_i(x + (b_i - b_0)y)$$

is defined on $A_0 \times C_0$ and, thus, satisfies property P throughout A_0 for each fixed $y \in C_0$. From assertion (i) of Theorem 6.1, $f_0(x)$ satisfies P throughout A_0, especially, at $x_0 \in A_0$. Moreover, P being bounded or measurable, $f_0(x)$ is bounded or measurable on each compact subset of A_0, consequently, each point of A_0 belongs to A^*. It follows that the set A^* is open, hence, $A^* = A$,

because A is connected. Hence, $f_0(x)$ satisfies P at each point x_0 of $A = A^*$.

7. **Characteristic sets.** In this section, for convenience, we restrict ourselves to the case $m = 1$ and to functions $f(x)$ defined for all $x \in E_1$. Let Ω be a class of functions on E_1, containing all constant functions, and such that $f_1 \in \Omega$, $f_2 \in \Omega$ imply $f_1 - f_2 \in \Omega$ and $f_1(x + h) \in \Omega$, for each $h \in E_1$. Let $(\sim)$ denote a fixed equivalence relation defined on Ω, such that $f_1 \sim g_1$, $f_2 \sim g_2$ imply $f_1 - f_2 \sim g_1 - g_2$ and $f_1(x + h) \sim g_1(x + h)$, for each $h \in E_1$. Further, we suppose that, for a constant function f, $f \sim 0$ implies $f = 0$. As an important example, we shall write $f_1 \approx f_2$ if and only if $f_1(x)$ and $f_2(x)$ differ only on a set of measure 0.

DEFINITION. A set S of real numbers is said to be *characteristic* (with respect to Ω) when the following two implications hold. (i) Let $f \in \Omega$ be such that $\Delta_h f \sim 0$, for each $h \in S$. Then f is equivalent to a constant. (ii) Suppose that to each $h \in S$ there corresponds a function $g_h \in \Omega$, in such a way that $\Delta_h g_{h'} = \Delta_{h'} g_h$ for h, $h' \in S$. Then there exists a $g \in \Omega$ and constants a_h, such that $\Delta_h g \sim g_h - a_h$, for each $h \in S$.

Any set S', containing a characteristic set S, is itself a characteristic set. (i) Let $f \in \Omega$ be such that $\Delta_{h'} f \sim 0$ for $h' \in S'$. Then $\Delta_h f \sim 0$ for $h \in S$, hence, f is equivalent to a constant. (ii) For $h' \in S'$, let $g_{h'} \in \Omega$ be such that $\Delta_h g_{h'} = \Delta_{h'} g_h$ when h, $h' \in S'$. By $S \subset S'$, there exists a $g \in \Omega$ with $\Delta_h g - g_h \sim - a_h$ = constant, when $h \in S$. For $h' \in S'$ and each $h \in S$, $\Delta_h(\Delta_{h'} g - g_{h'}) \sim \Delta_{h'}(g_h - a_h)$ $- \Delta_h g_{h'} = 0$, hence, $\Delta_{h'} g - g_{h'}$ is equivalent to a constant $- a_{h'}$.

If C is a set of real numbers then the additive group of real numbers generated by C will be denoted as $C\dagger$. Here, $C\dagger$ is dense in E_1 when, for instance, C contains a pair of rationally independent real numbers.

THEOREM 7.1. *Let C be a set of real numbers, such that $C\dagger$ is a characteristic set. Let $f(x)$ be any function on E_1, such that $\Delta_h f \in \Omega$ for each $h \in C$.*

ASSERTION. (i) *There exists a $g \in \Omega$ and an additive function H on E_1, such that $\Delta_h(f - g - H) \sim 0$, for each $h \in C\dagger$.*

(ii) *Suppose that, in Ω, $f_1 \sim f_2$ implies $f_1 = f_2$, and that to Ω there corresponds at least one characteristic set. Then the class Ω has the difference property.*

(iii) *Suppose that: (a) Each function in Ω is measurable; (b) In Ω, $f_1 \sim f_2$ implies $f_1 \approx f_2$. (c) $C\dagger$ is dense in E_1. Provided that $f(x)$ is bounded on some set of positive measure, we have $f \approx \beta x + g + g^*$, where β is a constant, $g \in \Omega$, and g^* is a bounded function with $\Delta_h g^* \approx 0$ for $h \in C\dagger$. Provided that $f(x)$ is measurable, we have $f \approx \beta x + g$, where β is a constant and $g \in \Omega$.*

Proof. It follows from

(1) $$\Delta_{h-h'} f(x) = \Delta_h f(x) - \Delta_{h'} f(x + h - h'),$$

that $\Delta_h f \in \Omega$, for each $h \in C\dagger$. Let $g_h = \Delta_h f$, hence, $\Delta_h g_{h'} = \Delta_{h'} g_h$, for each h, h'; moreover, $g_h \in \Omega$ when $h \in C\dagger$. Because $C\dagger$ is characteristic, there exists a $g \in \Omega$ and constants $H(h)$, such that $\Delta_h g \sim g_h - H(h)$ when $h \in C\dagger$. Let $f' = f - g$,

thus, $\Delta_h f' = \Delta_h f - \Delta_h g \sim H(h)$ for $h \in C\dagger$. From (1), applied to f' instead of f, $H(h)$ is an additive function on $C\dagger$; it is easy to extend H to an additive function on the entire group E_1. Now, for $h \in C\dagger$, $\Delta_h(f - g - H) = \Delta_h f' - H(h) \sim 0$, proving assertion (i).

Under the assumptions of assertion (ii), E_1 is characteristic. Hence, applying assertion (i) to $C = C\dagger = E_1$, we have that $\Delta_h f \in \Omega$, for each $h \in E_1$, implies the existence of a $g \in \Omega$ and an additive function H, such that $\Delta_h(f - g - H) = 0$, for each $h \in E_1$. Hence, $f - g - H = a = $ constant, thus, $f - H = g + a \in \Omega$.

Suppose that the assumptions of assertion (iii) hold. Let g, $f' = f - g$ and H be as in the proof of assertion (i). Here, f is either measurable or f is bounded on some set of positive measure. Further, g is measurable, hence, f' is bounded on some bounded set B' of positive measure:

$$(2) \qquad\qquad |f'(x)| \leq M \qquad\qquad \text{for } x \in B',$$

where M is a positive real constant. Moreover, for $h \in C\dagger$, we have $\Delta_h f' \sim H(h)$, hence, $\Delta_h f' \approx H(h)$. We want to prove that

$$(3) \qquad\qquad h \cdot H(h') = h' \cdot H(h),$$

for h, $h' \in C\dagger$. Because H is additive, (3) certainly holds when h and h' are rationally dependent. On the other hand, let h and h' be rationally independent elements of $C\dagger$, such that $hH(h') \neq h'H(h)$. Then there exist two sequences of integers $\{\lambda_j\}$ and $\{\mu_j\}$ $(j = 1, 2, \cdots)$, such that $h_j = \lambda_j h + \mu_j h'$ satisfies $h_j \to 0$ and $|H(h_j)| = |\lambda_j H(h) + \mu_j H(h')| \to \infty$ for $j \to +\infty$. From $h_j \in C\dagger$, we have $\Delta_{h_j} f' \approx H(h)$, i.e.

$$(4) \qquad\qquad f'(x + h_j) - f'(x) = H(h_j),$$

except for those values x which are in a set D_j of measure 0, $(j = 1, 2, \cdots)$. Let B'' be the set of all those points x in B', for which $x \notin D_j$ $(j = 1, 2, \cdots)$. Clearly, B'' is of positive measure. From Lemma 2.1, there exists a point $x \in B''$ such that $x + h_j \in B''$, infinitely often. Hence, from (2) and (4), $|H(h_j)| \leq 2M$, infinitely often, contradicting $|H(h_j)| \to +\infty$. This proves (3) for h, $h' \in C\dagger$, thus, $H(h) = \beta h$ for $h \in C\dagger$, where β is a constant.

Let $g^* = f - \beta x - g = f' - \beta x$, thus,

$$\Delta_h g^* = \Delta_h f' - \beta h = \Delta_h f' - H(h) \approx 0,$$

for each number h in the everywhere dense set $C\dagger$. Further, g^* is bounded on the (bounded) set B' of positive measure, while g^* is measurable whenever f is measurable. Assertion (iii) now follows from the following result due to Boas [1], applied to g^* instead of f.

LEMMA 7.1. *Let f be a function on E_1, such that $\Delta_h f \approx 0$, for each number h in a set dense in E_1. If $f(x)$ is bounded on a set of positive measure then $f \approx f_1$, where f_1 is some bounded function. Further, if $f(x)$ is measurable then $f \approx \alpha$, where α is a constant.*

This lemma is optimal in some sense. For, under the assumption of the continuum hypothesis, Sierpinsky [18] constructed a bounded nonmeasurable function (with values 0 and 1), such that $\Delta_h f \approx 0$ for each number h. This remark is due to P. Erdös, cf. [2, p. 195].

In applying Theorem 7.1, our main concern is to find sets S which are characteristic with respect to Ω. Because each superset of a characteristic set is characteristic, it is important to find (so-called minimal) characteristic sets, for which no proper subset is characteristic. For instance, a characteristic set consisting of a single element h is certainly minimal; in this case, the number h will be said to be characteristic. Any set, containing a characteristic number, is necessarily a characteristic set.

In the remaining part of this section, we shall apply Theorem 7.1 to the following classes Ω_j ($j = 1, \cdots, 4$) of continuous functions, in each of which equivalence is defined as a strict equality, (and also to certain related classes Ω_j^*). The proof, that the following stated numbers are indeed characteristic, will be given at the end of this section. Then, from assertion (ii) of Theorem 7.1, we have that each class Ω_j ($j = 1, \cdots, 4$) has the difference property. For $j = 1$, the latter result is due to de Bruijn [2].

(I) Let Ω_1 be the class of all polynomials on E_1. It is easily shown that each number $h \neq 0$ is characteristic. Hence, a set S is characteristic if and only if it contains a nonzero element.

(II) Let Ω_2 be the class of all functions f on E_1, such that: (i) All derivatives $f^{(n)}$ of f exist; (ii) For $x \to +\infty$,

$$f^{(n)}(x) = O(x^{-\alpha_n}), \qquad\qquad (n = 0, 1, \cdots),$$

where $\{\alpha_n\}$ is a nondecreasing sequence of real numbers, depending on f, and converging to $+\infty$. Again, each number $h \neq 0$ is a characteristic number.

(III) Let ρ and σ be fixed positive numbers. Let Ω_3 be the class of all the analytic functions $f(x)$ on E_1, which can be extended to an entire function $f(z) = f(x+iy)$ with

$$f(x + iy) = O(\exp (\rho |x| + \sigma |y|)).$$

We assert that a number h is characteristic if and only if $0 < \sigma |h| < 2\pi$.

(IV) Let δ be a fixed real number, $\delta \geq 0$. Let Ω_4 be the class of all the analytic functions $f(x)$ on E_1, periodic of period 1, which can be extended to an analytic function $f(z) = f(x+iy)$ in the strip $|y| \leq \delta$.

Let $\overline{\Gamma}$ denote the set of all the real numbers h, to which there corresponds a positive number ϵ, such that the inequality $|nh - m| < e^{-\epsilon n}$ holds for infinitely many pairs of integers m, n with $n \geq 1$. Then $h \in \overline{\Gamma}$ implies $r_1 h + r_2 \in \overline{\Gamma}$ for each pair of rational numbers r_1, r_2; especially, $\overline{\Gamma}$ contains all rational numbers. Further, $\overline{\Gamma}$ is of measure 0, in fact, even of Hausdorff dimension 0. Let Γ be the complement of $\overline{\Gamma}$ (with respect to E_1). Then Γ contains all irrational

algebraic numbers and, further, such numbers as π and e. All these results are well-known, cf. Koksma [9].

We assert that a number h is characteristic with respect to Ω_4 if and only if $h \in \Gamma$. Hence, a set S with $S \cap \Gamma \neq 0$ is always a characteristic set.

LEMMA 7.2. *Suppose that Ω is a class of continuous functions, on which equivalence is defined as a strict equality. Let Ω^* be the class of all functions f^* on E_1 to which there corresponds an $f \in \Omega$ with $f^* \approx f$. Finally, in Ω^*, $f_1 \sim f_2$ if and only if $f_1 \approx f_2$. Assertion: a set of real numbers is characteristic with respect to Ω^* if and only if it is characteristic with respect to Ω.*

Proof. If $f_1 \in \Omega$, $f_2 \in \Omega$, $f_1 \approx f_2$, then $f_1 = f_2$, from the continuity of the function $f_1 - f_2$. This remark enables us to prove Lemma 7.2 in a straightforward manner; we omit the details.

Let Ω_j^* $(j = 1, \cdots, 4)$ be the class associated to Ω_j in the sense of Lemma 7.2. It follows from Lemma 7.2 that the above assertions, concerning the characteristic numbers of Ω_j, apply equally well to the characteristic numbers of Ω_j^*, $(j = 1, \cdots, 4)$. Especially, when C is a pair of rationally independent real numbers then $C\dagger$ is characteristic for Ω_j^*, provided that, for $j = 4$, the set $C\dagger \cap \Gamma$ is nonempty.

THEOREM 7.2. *Let j be fixed, $(j = 1, \cdots, 4)$, and let $\Omega^* = \Omega_j^*$. Let $f_i(x)$, $(i = 0, \cdots, k)$, be a set of $k+1$ functions on E_1, $(k \geq 0)$, such that $f_0(x)$ is measurable, and that the function*

$$\phi_{y,y'}(x) = \sum_{i=0}^{k} f_i(x + \lambda_i y + \lambda_i' y')$$

belongs to Ω^*, for each pair of non-negative integers y, y' with $y + y' \leq k$. Here, λ_i, λ_i' $(i = 0, \cdots, k)$ are real constants, such that, for $i = 1, \cdots, k$, the numbers $\lambda_i - \lambda_0$ and $\lambda_i' - \lambda_0'$ are rationally independent, while, for $j = 4$, either $\lambda_i - \lambda_0$ or $\lambda_i' - \lambda_0'$ belongs to Γ.

Assertion: there exists a polynomial P, such that $f_0 - P \in \Omega^*$. If $j \neq 4$ we may take $P = 0$, hence, $f_0 \in \Omega^*$.

Proof. The assertion is obvious when $k = 0$. Let k be a fixed integer, $k \geq 1$, and suppose that the theorem holds when k is replaced by $k-1$. Let

$$h = \lambda_k - \lambda_0 \quad \text{and} \quad h' = \lambda_k' - \lambda_0'.$$

Then h, h' are rationally independent, while, for $j = 4$, either h or h' is in Γ. Further, let

$$g_i(x) = f_i(x + \lambda_k) - f_i(x + \lambda_i), \qquad (i = 0, \cdots, k);$$

then $g_0(x)$ is measurable and $g_k(x) = 0$. We have, for each pair of non-negative integers y, y' with $y + y' \leq k - 1$, that the function

$$(5) \qquad \sum_{i=0}^{k-1} g_i(x + \lambda_i y + \lambda_i' y') = \phi_{y,y'}(x + \lambda_k) - \phi_{y+1,y'}(x)$$

belongs to Ω^*. Let us first assume that $j \neq 4$. Then, by induction, $g_0(x)$ $= \Delta_h f_0(x + \lambda_0) \in \Omega^*$, hence, $\Delta_h f_0 \in \Omega^*$. In a similar way, $\Delta_{h'} f_0 \in \Omega^*$. Let C be the set, consisting of the two rationally independent numbers h, h'. Then $C\dagger$ is dense in E_1 and, moreover, characteristic with respect to $\Omega^* = \Omega_j^*$, $(j \neq 4)$. Because f_0 is measurable, it follows from assertion (iii) of Theorem 7.1 that $f_0 = \beta x + g$, where β is a constant and $g \in \Omega^*$. But Ω^* contains each polynomial, hence, $f_0 \in \Omega^*$.

Now, let $j = 4$. From (5), by induction, there exists a polynomial P_1, such that $g_0 - P_1 \in \Omega_4^*$. By $g_0(x) = \Delta_h f_0(x + \lambda_0)$, we have $\Delta_h f_0 - P = g \in \Omega_4^*$, for some polynomial P. Similarly, $\Delta_{h'} f_0 - P' = g' \in \Omega_4^*$, for some polynomial P'. Thus,

$$\Delta_h \Delta_{h'} f_0 = \Delta_{h'}(g + P) = \Delta_h(g' + P'),$$

or,

$$\Delta_h g' - \Delta_{h'} g = \Delta_{h'} P - \Delta_h P' = u \text{ (say)}.$$

From $g \in \Omega_4^*$, $g' \in \Omega_4^*$, it follows that u is a polynomial of period 1, such that $\int_0^1 u(x)dx = 0$. Consequently, $u(x) = 0$. Let Q be any polynomial, such that $\Delta_h Q = P$. Then $\Delta_h(\Delta_{h'} Q - P') = \Delta_{h'} P - \Delta_h P' = u = 0$, thus, $\Delta_{h'} Q - P' = a = $ constant. Let $f = f_0 - Q$; then $\Delta_h f = \Delta_h f_0 - P = g \in \Omega_4^*$ and $\Delta_{h'} f = \Delta_{h'} f_0 - P' - a$ $= g' - a \in \Omega_4^*$. Thus, f is a measurable function with $\Delta_h f \in \Omega_4^*$ and $\Delta_{h'} f \in \Omega_4^*$.

Let C be the set, consisting of the two rationally independent numbers h and h', where $h \in \Gamma$ or $h' \in \Gamma$. Hence, $C\dagger$ is dense in E_1 and characteristic with respect to Ω_4^*. From assertion (iii) of Theorem 7.1, we have $f = \beta x + g^*$, where β is a constant and $g^* \in \Omega_4^*$. Finally, $f_0 - (Q + \beta x) = f - \beta x = g^* \in \Omega_4^*$, where $Q + \beta x$ is a polynomial.

COROLLARY 7.2. *Let j be fixed, $(j = 1, \cdots, 4)$, and let $\Omega^* = \Omega_j^*$. Let $f_i(x)$ $(i = 0, \cdots, k)$, be functions on E_1, such that $f_0(x)$ is measurable, $f_0(x+1)$ $- f_0(x) \approx 0$, while*

$$(6) \qquad \phi_y(x) = \sum_{i=0}^{k} f_i(x + \lambda_i y) \in \Omega^*,$$

for $y = 0, 1, \cdots, k$. Here, λ_i $(i = 0, \cdots, k)$ are real constants such that, for $j \neq 4, \lambda_i - \lambda_0$ is irrational and, for $j = 4, \lambda_i - \lambda_0 \in \Gamma, (i = 1, \cdots, k)$. Then $f_0 \in \Omega^*$.

Proof. Let y' be an arbitrary integer. Then $f_0(x + y') \approx f_0(x)$, hence, for $y = 0, \cdots, k,$

$$\phi_y(x + y') = \sum_{i=0}^{k} f_i(x + \lambda_i y + \lambda_i' y') \in \Omega^*,$$

where $\lambda_0' = 0$, $\lambda_i' = 1$, $(i = 1, \cdots, k)$. From Theorem 7.2, $f_0 \in \Omega^*$ when $j \neq 4$.

Further, for $j=4$, $f_0-P=g\in\Omega^*=\Omega_4^*$, where P is some polynomial. But $P=f_0-g$ is equivalent to a function of period 1, hence, P is constant, thus, $f_0=g+P\in\Omega_4^*$.

REMARK 1. Given $k+1\geq 2$ real numbers λ_i $(i=0,\cdots,k)$, one can always find a system of $k+1$ bounded, nonmeasurable functions $f_0(x),\cdots,f_k(x)$, each of period 1, and such that

$$\sum_{i=0}^{k} f_i(x + \lambda_i y) = 0,$$

for each y in some nondenumerable additive group C†. This follows from the construction in §1, replacing R by the field, obtained by adjoining the λ_i to the field of rational numbers and taking $1\in U$, $u_0\neq 1$. Thus, in view of $0\in\Omega_j^*$ and $f_0\notin\Omega_j^*$, the condition of the above corollary, that f_0 be measurable, cannot be weakened very much, unless we require that (6) holds for each number y in (say) a set of positive measure, cf. Theorem 6.1.

REMARK 2. Let $\Omega^*=\Omega_j^*$, $(1\leq j\leq 4)$, and let $f(x)$ be a measurable function on E_1, such that $\Delta_1 f\approx 0$ and

$$f(u + x) + f(u - x) \in \Omega^*,$$

for two values u_1 and u_2 of u with an irrational difference u_1-u_2 when $j\neq 4$, and $u_1-u_2\in\Gamma$ when $j=4$; then $f\in\Omega^*$. This result is an immediate consequence of the above corollary, applied for $k=1$, $f_0(x)=f(u_1+x)$, $f_1(x)=f(u-x)$, $\lambda_0=-\lambda_1=u_2-u_1$. Moreover, if $j=1$ or $j=2$, we have that $f\in\Omega^*$ and $\Delta_1 f\approx 0$ together imply $f\approx a=$constant. The above for $j=1$ and $j=4$ yields a generalization of some results due to Boas [1, Theorems 1, 4].

It remains to verify the assertions (I)–(IV) concerning the characteristic numbers of Ω_j, $(j=1,\cdots,4)$. Note that, for each of these classes, equivalence is defined as a strict equality.

(I) Let h be any real number $\neq 0$. If $f\in\Omega_1$ and $\Delta_h f=0$ then the polynomial f is constant. Further, for each $g_1\in\Omega_1$, there exists a polynomial g with $\Delta_h g=g_1$. Hence, h is characteristic with respect to Ω_1.

(II) Let h be any real number $\neq 0$. Let $f\in\Omega_2$ be such that $\Delta_h f=0$. For n sufficiently large, we have for $x\to+\infty$ that $f^{(n)}(x)=O(x^{-\alpha_n})$ with $\alpha_n>0$. Hence, from $\Delta_h f^{(n)}=0$, $f^{(n)}=0$, showing that f is a periodic polynomial and, thus, a constant. Further, let $g_1\in\Omega_2$; then the so-called principal solution g of the difference equation $\Delta_h g=g_1$ exists and is in Ω_2, cf. Nörlund [14, p. 56], showing that h is characteristic with respect to Ω_2. In fact, let $g_1^{(n)}(x)=O(x^{-\beta_n})$ when $x\to+\infty$, $(n=0, 1, \cdots)$. Then this principal solution satisfies

$$g^{(m)}(x) = a_m - \sum_{j=0}^{\infty} g_1^{(m)}(x + jh),$$

$(a_m=$constant$)$, for each non-negative integer m with $\beta_m>1$, thus, $g^{(m+1)}(x)=O(x^{-\beta_m+1})$.

(III) Let h be any real number with $0 < \sigma |h| < 2\pi$. Let $f \in \Omega_3$ be such that $\Delta_h f = 0$. Then

$$g(\exp (2\pi i z / h)) = f(z)$$

defines a function $g(\zeta)$ which is single valued and analytic for $\zeta \neq 0$. Moreover,

$$g(\zeta) = O(|\zeta|^\beta + |\zeta|^{-\beta}) \quad with \quad \beta = \frac{\sigma |h|}{2\pi} < 1.$$

Consequently, g and, hence, f is a constant function. Further, let $g_1 \in \Omega_3$. Then

$$g(x) = \frac{1}{2\pi i h} \int_{-1/2-i\infty}^{-1/2+i\infty} \left\{ \int_0^{x+hw} g_1(\xi) d\xi \right\} \left(\frac{\pi}{\sin \pi w} \right)^2 dw$$

defines a function $g \in \Omega_3$, such that $\Delta_h g = g_1$, cf. Nörlund [14, p. 78]. This shows that h is characteristic w.r.t. Ω_3. On the other hand, let $\sigma |h| \geq 2\pi$. Then $f = \exp (2\pi i x / h)$ satisfies $f \in \Omega_3$ and $\Delta_h f = 0$, showing that the number h is not characteristic.

(IV) Let h be any number in Γ, thus, h is irrational. Moreover, to each positive number ϵ there corresponds a positive constant $K(\epsilon)$ with $|nh - m| \geq K(\epsilon) e^{-\epsilon |n|}$, for each pair of integers m, n with $n \neq 0$. Let n be any integer $\neq 0$ and let m be the integer closest to nh. Then $|\sin \pi nh| \geq 2|nh - m|$, hence,

$$(7) \qquad |e^{2\pi i nh} - 1|^{-1} = |\sin \pi nh|^{-1}/2 \leq (4K(\epsilon))^{-1} e^{\epsilon |n|},$$

for each integer $n \neq 0$. Now, let $f \in \Omega_4$ be such that $\Delta_h f = 0$. Because h is irrational and $\Delta_1 f = 0$, we have $\Delta_{h'} f = 0$ for an everywhere dense set of values $h' = \lambda + \mu h$, (λ, μ = integers). Further, f is continuous on E_1, hence, f is a constant function.

Let g_1 be any function in Ω_4, in other words,

$$g_1(x) = \sum_{-\infty}^{\infty} c_n e^{2\pi i nx},$$

where $\{c_n\}$ is a sequence of complex numbers, satisfying

$$c_n = O(e^{-(2\pi\delta+\eta)|n|});$$

here η is a positive constant, depending on g_1. From (7), applied for $\epsilon = \eta/2$, it follows that

$$g(x) = \sum_{n \neq 0} c_n (e^{2\pi i nh} - 1)^{-1} e^{2\pi i nx}$$

defines a function $g \in \Omega_4$, such that $\Delta_h g - g_1 = -c_0$. This shows that each number h in Γ is characteristic w.r.t. Ω_4.

Vice versa, let the number h be such, that to each $g_1 \in \Omega_4$ of the form

$$g_1(x) = \sum_{-\infty}^{\infty} e^{-(2\pi\delta+\eta)|n|} e^{2\pi inx}, \qquad\qquad (\eta > 0)$$

there corresponds a $g \in \Omega_4$ with $\Delta_h g - g_1 = $ constant. Let

$$g(x) = \sum_{-\infty}^{\infty} d_n e^{2\pi inx} \quad with \quad |d_n| \leqq K_\eta e^{-2\pi\delta|n|},$$

where K_η is a positive constant. Then, for $n \neq 0$,

$$e^{-(2\pi\delta+\eta)|n|} = |d_n(e^{2\pi inh} - 1)| \leqq K_\eta e^{-2\pi\delta|n|} \cdot 2 |\sin \pi nh|,$$

showing that, for each pair of integers m, n with $n \neq 0$,

$$|nh - m| \geqq \frac{1}{\pi} |\sin \pi nh| \geqq \frac{1}{2\pi K_\eta} e^{-\eta|n|}.$$

Here, η is an arbitrary positive constant, hence, $h \in \Gamma$.

8. A theorem of Skitovic. In this section, we shall give a new proof of the following result due to Skitovic [19].

THEOREM 8.1. *Let $a_{\nu j}$ $(\nu = 1, 2; j = 1, \cdots, n)$ be real constants and let $X_1, \cdots, X_n$ be independent random variables, such that $l_1 = \sum a_{1j} X_j$ and $l_2 = \sum a_{2j} X_j$ are independent. Then each X_j, which occurs in both l_1 and l_2, (i.e. $a_{1j} a_{2j} \neq 0$), is normally distributed.*

We shall need the following lemma, (also used by Skitovic).

LEMMA 8.1. *Let X be a random variable with characteristic function $\phi(t)$. Let $\epsilon > 0$, and suppose that, for $|t| < \epsilon$, $\phi(t) = e^{P(t)}$, where $P(t)$ is a polynomial. Then X is normally distributed.*

This lemma was first stated by Linnik [11, 10]. In fact, it is an easy consequence of the following result due to Marcinkiewicz [13, 87].

Let X be a random variable with cumulative d.f. $F(x)$ and characteristic function $\phi(t)$. Let $0 < \epsilon, 0 < R \leqq \infty$, and suppose that the restriction of $\phi(t)$ to the interval $-\epsilon < t < +\epsilon$ can be extended to an analytic function $\phi(z) = \phi(t + it')$ in the rectangle $|t| < \epsilon$, $|t'| < R$. Then the integral

$$(1) \qquad\qquad \phi(z) = \int_{-\infty}^{\infty} e^{izx} dF(x)$$

converges for all complex $z = t + it'$ with $|t'| < R$.

Consequently, (1) defines an analytic extension of the unrestricted function $\phi(t)$ to the strip $|t'| < R$, such that

$$(2) \qquad\qquad |\phi(t + it')| \leqq \phi(it').$$

Under the conditions of Lemma 8.1, this extension is an entire function of the form $\phi(z) = e^{P(z)}$, where $P(z)$ is a polynomial. But then, as was shown by

Marcinkiewicz [12, 616], (2) readily implies that $P(z)$ is at most of second degree, hence, X is normally distributed.

Proof of Theorem 8.1. Permuting the indices, we may assume that

$$a_{1j} \neq 0, \qquad a_{2j} \neq 0, \qquad\qquad (j = 1, \cdots, k),$$

$a_{2j} = 0$, $(j = k+1, \cdots, q)$ and $a_{1j} = 0$, $(j = q+1, \cdots, n)$, where $0 \leq k \leq q \leq n$. Let

$$L_\nu = \sum_{j=1}^{k} a_{\nu j} X_j, \ (\nu = 1, 2); \qquad Y_1 = \sum_{j=k+1}^{q} a_{1j} X_j; \qquad Y_2 = \sum_{q+1}^{n} a_{2j} X_j.$$

Then $X_1, \cdots, X_k, Y_1, Y_2$ are independent, while $l_\nu = L_\nu + Y_\nu$, $(\nu = 1, 2)$. Let $\phi_j(t), \psi_\nu(t), f_\nu(t)$ denote the characteristic function of X_j, L_ν and Y_ν, respectively, $(j = 1, \cdots, k; \nu = 1, 2)$. The joint characteristic function of l_1 and l_2 is given by

$$E(e^{i(t_1 l_1 + t_2 l_2)}) = \prod_{j=1}^{k} \phi_j(a_{1j} t_1 + a_{2j} t_2) f_1(t_1) f_2(t_2),$$

and also by

$$E(e^{it_1 l_1}) E(e^{it_2 l_2}) = \psi_1(t_1) f_1(t_1) \psi_2(t_2) f_2(t_2),$$

from the independence of l_1 and l_2. Any characteristic function $g(t)$ is continuous at $t = 0$; moreover, $g(0) = 1$. Hence, $f_\nu(t_\nu) \neq 0$ for $|t_\nu| < \epsilon_\nu$, where ϵ_ν is a sufficiently small positive number, $(\nu = 1, 2)$. It follows that, for $|t_1| < \epsilon_1$, $|t_2| < \epsilon_2$,

$$\psi_1(t_1) \psi_2(t_2) = \prod_{j=1}^{k} \phi_j(a_{1j} t_1 + a_{2j} t_2).$$

Replacing (if necessary) ϵ_1 and ϵ_2 by smaller positive numbers, we have, for $|t_1| < \epsilon_1$, $|t_2| < \epsilon_2$, that

$$|\arg \psi_\nu(t_\nu)| < \pi/2; \qquad |\arg \phi_j(a_{1j} t_1 + a_{2j} t_2)| < \pi/k;$$

$(\nu = 1, 2; j = 1, \cdots, k)$. Now, the transformation $\zeta = \log z = \log |z| + i. \arg z$, with $-\pi < \arg z \leq +\pi$, is single valued and continuous in the sector $|\arg z| < \pi$ and such that $\log (z_1 z_2 \cdots z_p) = \log z_1 + \cdots + \log z_p$ whenever $|\arg z_j| < \pi/p$ $(j = 1, \cdots, p)$. Hence, for $|t_1| < \epsilon_1$, $|t_2| < \epsilon_2$,

$$\log \psi_1(t_1) + \log \psi_2(t_2) = \sum_{j=1}^{k} \log \phi_j(a_{1j} t_1 + a_{2j} t_2),$$

where $\log \psi_1(t)$ is continuous at $t = 0$. Putting

$$g_0(t) = -\log \psi_1(t), \qquad g_j(t) = \log \phi_j(a_{1j} t), \qquad\qquad (j = 1, \cdots, k),$$

we have, for $|t_1| < \epsilon_1$, $|t_2| < \epsilon_2$, that

$$(3) \qquad \sum_{j=0}^{k} g_j(t_1 + b_j t_2) = \log \psi_2(t_2),$$

where $b_0=0$, $b_j=a_{2j}/a_{1j}\neq 0$, $(j=1, \cdots, k)$. Moreover, $g_0(t)$ is bounded in a neighborhood of $t=0$. The right hand side of (3) being independent of t_1, it follows from assertion (iii) of Theorem 6.1, applied for $A=\{|t|<\epsilon_1\}$, $C=\{|t|<\epsilon_2\}$, (cf. also the following lemma), that the function $g_0(t) = -\log \psi_1(t)$ is equal to a polynomial throughout the interval $|t|<\epsilon_1$. Hence, from Lemma 8.1, the random variable L_1 is normally distributed. But the normal law is closed under decomposition, Cramér [5, 52], thus, the normality of $L_1 = \sum a_{1j}X_j$ implies the normality of $a_{1j}X_j$ and, hence, of X_j, $(j=1, \cdots, k)$.

REMARK. It is easily seen, Skitovic [19, p. 200], that Lemma 8.1 continues to hold when X and t are replaced by vectors in E_m. Further, in E_m, the normal law is closed under decomposition, Cramér [5, 112]. Consequently, the above proof also yields the generalization of Theorem 8.1 to independent random vectors $X_1, \cdots, X_n$ in E_m. On the other hand, as was shown by Skitovic [19], it is not difficult to obtain this generalization directly from Theorem 8.1.

For the benefit of those readers who are mainly interested in the proof of Theorem 8.1 we shall give a separate proof of the following corollary of Theorem 6.1, which will suffice in proving Theorem 8.1.

LEMMA. *Let $\alpha<\beta$, $\epsilon>0$ be real constants. Suppose that, for $t_1\in I=\{\alpha<t<\beta\}$ and $|t_2|<\epsilon$,*

$$(4) \qquad \sum_{j=0}^{k} g_j(t_1 + b_j t_2) = (polynomial\ in\ t_1) = \sum_{r=0}^{N(t_2)} a_r(t_2)t_1^r.$$

Here, the b_j are real constants, $b_j\neq b_0=0$, $(j=1, \cdots, k)$. Further, $g_j(t)$ is a complex-valued function defined for $\alpha-|b_j|\epsilon<t<\beta+|b_j|\epsilon$, $(j=0, \cdots, k)$. Then, in I, the function $g_0(t)$ is equal to a polynomial if (and only if) it is bounded on some nondegenerate subinterval I_1 of I.

Proof. The assertion is obvious for $k=0$. Let $k\geq 1$, and suppose that the lemma holds true when k is replaced by $k-1$. Let $|I_1|$ denote the length of I_1, $0<|I_1|\leq\beta-\alpha$, let $\delta=\mathrm{Min}\,(|I_1|, 1/2|b_k|\epsilon)$, and let h be any fixed positive number $<\delta$. From $0<h<\delta\leq 1/2|b_k|\epsilon$, we have

$$- |b_j|\epsilon/2 < (1 - b_j/b_k)h < h + |b_j|\epsilon/2.$$

Hence, the function

$$(5) \qquad g_j^*(t) = g_j(t + (1 - b_j/b_k)h) - g_j(t), \qquad (j = 0, \cdots, k)$$

is defined for $\alpha-|b_j|\epsilon/2<t<\beta-h+|b_j|\epsilon/2$, especially, $g_0^*(t)=g_0(t+h)-g_0(t)$ is defined throughout $I^*=\{\alpha<t<\beta-h\}$. Further, from $0<h<\delta\leq|I_1|$, the function $g_0^*(t)$ is bounded on some nondegenerate subinterval I_1^* of I^*.

If $|t_2| < \epsilon/2$ then $|t_2 - h/b_k| < \epsilon/2 + \delta/|b_k| \leqq \epsilon$. Thus, if $\phi(t_1, t_2)$ denotes the left hand side of (4), we have, for $|t_2| < \epsilon/2$, $t_1 \in I^*$, that

$$\psi(t_1, t_2) = \phi(t_1 + h, t_2 - h/b_k) - \phi(t_1, t_2) = (\textit{polynomial in } t_1).$$

Moreover, from (4) and (5), (especially, $g_k^*(t) = 0$),

$$\psi(t_1, t_2) = \sum_{j=0}^{k-1} g_j^*(t_1 + b_j t_2).$$

Hence, by induction, (replacing $g_j(t)$ $(j=0, \cdots, k)$, I, I_1 and ϵ by g_j^* $(j=0, \cdots, k-1)$, I^*, I_1^* and $\epsilon/2$, respectively), it follows that, in $I^* = \{\alpha < t < \beta - h\}$, the function $g_0^*(t) = \Delta_h g_0(t)$ is equal to a polynomial. In fact, this is true for each positive number $h < \delta$.

Let h' be fixed, $0 < h' < \delta/2$. Then $\Delta_{h'} g_0(t)$ $(\alpha < t < \beta - h')$ is equal to a polynomial, hence, there exists a polynomial $P(t)$, such that $\Delta_{h'} g_0(t) = \Delta_{h'} P(t)$, $\alpha < t < \beta - h'$. Let

$$f_0(t) = g_0(t) - P(t), \qquad\qquad (\alpha < t < \beta),$$

thus, $\Delta_{h'} f_0(t) = 0$, $\alpha < t < \beta - h'$. Moreover, for $0 < h < \delta/2$, the function $C(h, t) = \Delta_h f_0(t) = \Delta_h g_0(t) - \Delta_h P(t)$, $(\alpha < t < \beta - h)$ is equal to a polynomial, such that

$$\Delta_{h'} C(h, t) = \Delta_{h'} \Delta_h f_0(t) = \Delta_h \Delta_{h'} f_0(t) = 0,$$

$(\alpha < t < \beta - h - h'$ with $h + h' < \delta \leqq \beta - \alpha)$. It follows that, for $0 < h < \delta/2$, the polynomial $C(h, t) = \Delta_h f_0(t)$ is equal to a constant $C(h)$. From

$$\Delta_{h+h'} f_0(t) = \Delta_h f_0(t) + \Delta_{h'} f_0(t + h),$$

$(h \geqq 0, h' \geqq 0, \alpha < t < \beta - h - h')$, we easily see that $\Delta_h f_0(t)$ $(\alpha < t < \beta - h)$ is equal to a constant $C(h)$ for any number h with $0 \leqq h < \beta - \alpha$, in such a way that

(6) $$C(h + h') = C(h) + C(h'),$$

for $h \geqq 0, h' \geqq 0, h + h' < \beta - \alpha$. Because $f_0(t) = g_0(t) - P(t)$ is bounded on a non-degenerate interval $I_1 \subset I$, we have that $C(h) = \Delta_h f_0(t)$ is bounded for $0 \leqq h \leqq \eta$, when η is a sufficiently small positive number. This, together with (6), easily implies the existence of a constant γ, such that $C(h) = \gamma h$ for $0 \leqq h < \beta - \alpha$. But $C(h') = \Delta_{h'} f_0(t) = 0$, hence, $\gamma = 0$ and

$$f_0(t + h) - f_0(t) = \Delta_h f_0(t) = C(h) = \gamma h = 0,$$

for $0 < h < \beta - \alpha$, $\alpha < t < \beta - h$. It follows that $f_0(t)$ is constant in $\alpha < t < \beta$, thus, $g_0(t) = P(t) + f_0(t)$ is equal to a polynomial.

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