A LIMIT THEOREM FOR A FUNCTION OF THE INCREMENTS OF A DECOMPOSABLE PROCESS

BY

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1. **Introduction and summary.** Let X_T be a separable, continuous (in law), decomposable process on a finite time interval $T=[t_1, t_2]$, that is, the increments $X_{[s,t)}=X_t-X_s$ over disjoint intervals in T are independent random variables and $X_{[t_1,t)}$ is continuous in law, hence almost surely (a.s.): $X_t \to X_s$ *a.s. as* $t\to s$ *for each* $s\in T$. For such process the famous Raikov theorem states that if $EX_{[t_1,t)}=0$ for each t, then $X_{[t_1,t_2)}$ is distributed $\mathfrak{N}(0, \sigma^2)$ if, and only if, $\sum_k X_{nk}^2 \to \sigma^2$ in probability as $n\to\infty$, where $X_{nk}=X_{[t_{n,k-1},t_{nk})}$ and $\sum [t_{n,k-1}, t_{nk})=T$ are partitions, $\mathcal{P}_n(T)$, of T with $\max_k (t_{nk}-t_{n,k-1})\to 0$. Paul Lévy then obtained for the Brownian motion process that, if the partitions $\{\mathcal{P}_n(T)\}$ are a sequence ordered by refinement with $\max(t_{nk}-t_{n,k-1})\to 0$, then in fact $\sum X_{nk}^2\to\sigma^2$ a.s., the variance of $X_{[t_1,t_2)}$. The Brownian motion assumption is easily replaced by the assumption that the process is continuous and the increments are normally distributed and centered at expectations.

Recently M. Loève [2] considered the problem for more general functions of the increments, $g(X_{nk})$, than X_{nk}^2. His results concern the general double sequences of independent, uniformly asymptotically negligible random variables of the modern central limit problem. When applied to processes, these results generalize the Raikov theorem to yield convergence in law of $\sum g(X_{nk})$ when g is a continuous function having a second derivative at the origin with $g(0)=0$. The second derivative assumption can be weakened, but then centering constants have to be introduced to obtain convergence.

Finally Rubin and Tucker [3] generalized the Paul Lévy theorem (for squared increments), showing that a.s. convergence held for separable, decomposable processes with stationary increments provided one took partitions with equal increments and the number of elements, k_n, in the nth partition went to infinity very rapidly, namely $\sum(1/k_n)^{1/3}<\infty$.

We will first drop the stationary increments assumption and show that the sum of squared increments converges a.s. along the sequence $\{\mathcal{P}_n(T)\}$, thus

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improving the result of Rubin and Tucker. Then we will show that the result extends to continuous functions, g, of the increments having second derivatives at the origin, paralleling the extension of Raikov's theorem by Loève.

2. **Some lemmas.** *In the sequel,* X_T, Y_T, *and* Z_T, *with or without primes, denote separable, continuous, decomposable processes on* T. For simplicity we take $T=[0, 1]$ and $\{\mathcal{P}_n(T)\}$ denotes a sequence of partitions of T ordered by refinement with $\max\{t_{n,k}-t_{n,k-1} \mid 1\leqq k\leqq k_n\}\to 0$ as $n\to\infty$. Adding some partitions to the sequence if necessary, we can assume without loss of generality that the nth partition divides T into exactly n increments. Unless otherwise stated, all limits will be along $\{\mathcal{P}_n(T)\}$. The truncation of a random variable at $b>0$ will be denoted by $X^{(b)}=XI_{[|X|\leqq b]}$.

The characteristic function of $X_{[0,t)}$ then has the form $f_t=\exp\{\psi_t\}$ where, in standard notation

$$\psi_t(u) = iu\alpha_t + \int \left(e^{iux} - 1 - \frac{iux}{1+x^2}\right)\frac{1+x^2}{x^2}\, d\Psi_t(x)$$

and will be denoted simply (α_t, Ψ_t). The law of the process over T will be denoted by (α_T, Ψ_T).

For the reader who is unfamiliar with this formalism we may remark that for a decomposable process, neglecting a null set of sample functions, the sample functions are bounded, have right and left limits at every point, and the discontinuities are jumps. Furthermore, in any time interval $[0, t)$, the number of jumps of the sample function whose magnitude lies in $(-\infty, x)$ for $x<0$ or (x, ∞) for $x>0$ is finite and Poisson distributed. Letting $L_t(x)$ denote the expected number of jumps in time interval $[0, t)$ whose magnitude is $<x<0$ and $-L_t(x)$ denote the expected number of jumps $\geqq x>0$, we have

$$\Psi_t(x) = \int_{-\infty}^{x} \frac{y^2}{1+y^2}\, dL_t(y), \qquad x<0,$$

$$\Psi_t(\infty) - \Psi_t(x) = \int_{x}^{\infty} \frac{y^2}{1+y^2}\, dL_t(y), \qquad x>0.$$

Finally, $\Psi(0+)-\Psi(0-)=\sigma^2$ is the variance of the normal part of the process, and α_t corresponds to a sure function or "drift" on T.

Our assumptions imply that α_t is a continuous function of t. In addition *we assume* α_t *is of bounded variation on* T *in what follows. All summations will be for* $k=1, 2, \cdots, n$, *unless otherwise stated.*

LEMMA 1. *The sum* $\Gamma_n=\sum w_{nk}X_{nk}$ *is uniformly bounded in probability for all partitions of* T *and constants* w_{nk} *with* $|w_{nk}|\leqq 1$, *i.e., for arbitrary* $\epsilon>0$, *there is a constant* b_ϵ *depending only on* ϵ *such that* $P[|\Gamma_n|>b_\epsilon]<\epsilon$.

Proof. Let X_T have law (α_T, Ψ_T), and let $\pm b$ be continuity points of $\Psi_1(x)$. Define V_T by

$$Y_t = X_t - J_{b,t}$$

where $J_{b,t}$ is the sum of the (finite number of) jumps of the sample function X_t during $[0, t)$ whose absolute magnitude exceeds b. It is obvious that Y_T is a separable, continuous, decomposable process with law (β_T, Ψ_T^b) where

$$\beta_t = \alpha_t - \int_{|x|>b} \frac{1}{x}\, d\Psi_t(x), \qquad \Psi_t^b(x) = \begin{cases} \Psi_t(-b) & \text{if } x \leqq b, \\ \Psi_t(x) & \text{if } -b < x \leqq b, \\ \Psi_t(b) & \text{if } x > b. \end{cases}$$

Let $\Lambda_n = \sum w_{nk} Y_{nk}$. The process Y_T has finite moments and for all partitions

$$| E\Lambda_n | \leqq \text{Total variation } \{\beta_t; 0 \leqq t \leqq 1\} + b\Psi_1[-b, b]$$

and

$$\text{Var}(\Lambda_n) = \sum w^2_{nk} \text{Var}(Y_{nk}) \leqq \text{Var}(Y_1) < \infty.$$

By Chebishev's inequality, the $\Lambda_n - E\Lambda_n$, hence the Λ_n are uniformly bounded in probability. Now let $A = [\sup\{|X_t|; 0 \leqq t \leqq 1\} \leqq b/2]$, $B = [|\Lambda_n| \leqq b_\epsilon]$ and $C = [|\Gamma_n| \leqq b_\epsilon]$. Since $AB = AC$, we obtain $|PB^c - PC^c| \leqq PA^c$. Letting b, b_ϵ be large enough so that $PA^c < \epsilon/2$ and $PB^c < \epsilon/2$ for all partitions completes the proof of the lemma.

LEMMA 2. *Let Y_T and Z_T be independent processes and $V_n = \sum Y_{nk} Z_{nk}$. Then, letting P^Z denote the conditional probability given Z_T, $P^Z[|V_n| > \epsilon] \to 0$ a.s. for each $\epsilon > 0$.*

Proof. Let z_t be a sure function on T having right and left limits at every point, jump discontinuities and at most a finite number of jumps exceeding any $\delta > 0$ in absolute value. Set

$$\sum z_{nk} Y_{nk} = L_{\delta,n} + M_{\delta,n} \quad \text{where}$$
$$L_{\delta,n} = \sum_{\{k; |z_{nk}| > \delta\}} z_{nk} Y_{nk}.$$

Then the almost sure continuity of Y_T yields $L_{\delta,n} \to 0$ in probability (in fact, $L_{\delta,n} \to 0$ a.s.). On the other hand,

$$M_{\delta,n} = \delta \sum w_{nk} Y_{nk} \quad \text{where} \quad w_{nk} = z_{nk}/\delta \text{ or } 0$$

according as $|z_{nk}| \leqq \delta$ or $|z_{nk}| > \delta$. By Lemma 1, $P[|M_{\delta,n}| > \epsilon/2] = P[|\sum w_{nk} Y_{nk}| > \epsilon/2\delta] \to 0$ uniformly in n as $\delta \to 0$.

The conditional law of Y_T given Z_T can, by independence, be taken to be the law of Y_T. Since outside a null set the sample functions $Z_t(\omega)$ satisfy the assumptions on z_t, we can replace z_t by $Z_t(\omega)$ in what precedes and obtain

$$P^Z[|V_n| > \epsilon] \leqq P^Z[|L_{\delta,n}| > \epsilon/2] + P^Z[|M_{\delta,n}| > \epsilon/2] \text{ a.s.}$$

Letting $n \to \infty$ and then $\delta \to 0$ proves the lemma.

LEMMA 3. *Let X_T, Y_T, and Z_T be three independent processes, $U_n = \sum X_{nk}Z_{nk}$, $V_n = \sum Y_{nk}Z_{nk}$ and $W_n = U_n - V_n$. Then*

$$W_n \xrightarrow{a.s.} 0 \text{ implies } U_n \xrightarrow{a.s.} 0 \text{ and } V_n \xrightarrow{a.s.} 0.$$

Proof. Let $\epsilon > 0$ be arbitrary and observe that

$$[|U_n| > 2\epsilon][|V_n| < \epsilon] \subset [|W_n| > \epsilon].$$

Let $A_n = [|U_n| > 2\epsilon]$ and $B_{mm} = A_m$, $B_{mn} = A_m^c A_{m+1}^c \cdots A_{n-1}^c A_n$ for $n > m$. Then $\bigcup_{n \geq m} A_n - \bigcup_{n \geq m} B_{mn}[|V_n| \geqq \epsilon] \subset \bigcup_{n \geq m} [|W_n| > \epsilon]$. Since U_n and V_n are conditionally independent given Z_T we have

$$P^Z\left\{\bigcup_{n \geq m} B_{mn}[|V_n| \geqq \epsilon]\right\} = \sum_{n \geq m} P^Z B_{mn} P^Z[|V_n| \geqq \epsilon] \text{ a.s.}$$

By Lemma 2, given $\delta > 0$ we can find a set C and an m such that

$$P^Z[|V_n| \geqq \epsilon] < \frac{1}{2} \text{ on } C \text{ for all } n \geqq m \text{ and } PC^c < \delta.$$

Then

$$E\left(\sum_{n \geq m} P^Z B_{mn} P^Z[|V_n| \geqq \epsilon]\right) \leqq \frac{1}{2} \sum_{n \geq m} PB_{mn}C + \sum_{n \geq m} PB_{mn}C^c$$
$$\leqq \frac{1}{2} P \bigcup_{n \geq m} A_n + \delta,$$

hence,

$$\frac{1}{2} P \bigcup_{n \geq m} A_n \leqq P \bigcup_{n \geq m} [|W_n| > \epsilon] + \delta.$$

Again, the lemma follows upon letting $n \to \infty$ and then $\delta \to 0$.

LEMMA 4. *Let $\{U_n\}$ and $\{V_n\}$ be any two sequences of random variables which are independent of each other and have the same joint distributions. If $W_n = U_n + V_n \to W$ a.s., then $U_n \to U$ a.s. and $V_n \to V$ a.s. where U and V are independent and identically distributed and $U + V = W$.*

Proof. Since

$$[U_n - U_m > \epsilon][V_n - V_m > \epsilon] \cup [U_m - U_n > \epsilon][V_m - V_n > \epsilon]$$
$$\subset [|W_n - W_m| > 2\epsilon],$$

we have

$$P^2[V_n - V_m > \epsilon] + P^2[V_m - V_n > \epsilon] \leqq P[|W_n - W_m| > 2\epsilon] \to 0$$

as $n, m \to \infty$, hence $P[|V_n - V_m| > \epsilon] \to 0$. Also,

$$\bigcup_{n \geqq m} [|U_n - U_m| > 2\epsilon][|V_n - V_m| \leqq \epsilon] \subset \bigcup_{n \geqq m} [|W_n - W_m| > \epsilon],$$

and, by the lemma for events (Loève [1, p. 246]),

$$\left(\inf_{n \geqq m} P[|V_n - V_m| \leqq \epsilon]\right) P \bigcup_{n \geqq m} [|U_n - U_m| > 2\epsilon]$$
$$\leqq P \bigcup_{n \geqq m} [|W_n - W_m| > \epsilon].$$

It follows that $P\bigcup_{n \geqq m} [|U_n - U_m| > 2\epsilon] \to 0$ as $m \to \infty$, and the conclusion of the lemma is immediate.

3. **Two limit theorems.** The proofs of the following theorems depend upon repeated application of the methods of symmetrization and truncation, leading to the use of the martingale theorems.

THEOREM 1. *Let X_T be a separable, continuous, decomposable process with law (α_T, Ψ_T), where α_t is a function of bounded variation on T. Then along the sequence of successive refinements, $\mathcal{P}_n(T)$, of T with the length of the largest interval converging to* 0,

$$\int_T (dX_t)^2 = \lim_{n \to \infty} \sum X_{nk}^2 = \sigma^2 + \sum J_t^2 \quad a.s.,$$

where σ^2 is the variance of the normal component of $X_{[t_1, t_2)}$ and $\sum J_t^2$ is the sum of the squares of the jumps of X_T.

Proof. The proof is divided into three parts.

1. Let X_T and X'_T be independent, identically distributed processes with symmetric distributions. Recall the definition of $J_{b,t}$ in Lemma 1, introduce the curtailed process $\overline{X}_t^b = X_t - J_{b,t}$, in the same way define $\overline{X}_t'^b = X'_t - J'_{b,t}$, and set

$$Q_{b,n} = \sum (\overline{X}_{nk}^b)^2, \qquad R_{b,n} = \sum \overline{X}_{nk}^b \overline{X}_{nk}'^b$$

and $Q_n = Q_{\infty,n}$ and $R_n = R_{\infty,n}$. For $b > 0$ fixed, we will show that $\{Q_{b,n}\}$ and $\{R_{b,n}\}$ are reversed martingale sequences. For $\{Q_{b,n}\}$ this may be established by observing that for some k, $Q_{b,n} - Q_{b,n+1} = 2\overline{X}_{n+1,k}^b \overline{X}_{n+1,k+1}^b$, and that the σ-field determined by $Q_{b,n+1}, Q_{b,n+2}, \cdots$ is contained in the σ-field, $\mathcal{B}$, determined by $\overline{X}_t^b$, $t \leqq t_{n+1,k}$, and by $|\overline{X}_t^b - \overline{X}_{t'}^b|$, $t, t' \geqq t_{n+1,k}$. Since this σ-field places a symmetric condition on the symmetrically distributed process on $[t_{n+1,k}, t_{n+1,k+1})$,

$$E(\overline{X}^b_{n+1,k}\overline{X}^b_{n+1,k+1} \mid \mathcal{B}) \overset{\text{a.s.}}{=} \overline{X}^b_{n+1,k}E(\overline{X}^b_{n+1,k+1} \mid \mathcal{B}) \overset{\text{a.s.}}{=} 0,$$

thus

$$E(Q_{b,n} - Q_{b,n+1} \mid Q_{b,n+1}, Q_{b,n+2}, \cdots) = 0 \quad \text{a.s.}$$

The assertion about the $R_{b,n}$ sequence follows by a similar but more involved argument. Now

$$EQ_{b,n} = \sum E(\overline{X}^b_{nk})^2 = E(\overline{X}^{b\,2}_{[t_1,t_2)}) < \infty.$$

Since the $Q_{b,n}$ are nonnegative, the martingale convergence theorem applies and $Q_{b,n} \to Q_b$ a.s. where Q_b is finite. Similarly,

$$E(R_{b,n})^2 = \sum E^2(\overline{X}^b_{nk})^2$$

and since $\max\{E(\overline{X}^b_{nk})^2;\ 1 \leqq k \leqq n\} \to 0$ as $n \to \infty$, $E(R_{b,n})^2 \to 0$. It follows by the martingale convergence theorem that $R_{b,n} \to 0$ a.s. as $n \to \infty$.

Almost all sample functions are bounded, hence have bounded increments, and if $C_b = [|X_t| \leqq b/2$ and $|X_t'| \leqq b/2,\ t \in T]$, then $C_b \uparrow C$ as $b \uparrow \infty$ with $PC = 1$, and on C_b, $R_{b,n} = R_n$ and $Q_{b,n} = Q_n$ for all n. It follows easily that $R_n \to 0$ a.s. and $Q_n \to Q$ finite a.s.

2. If X_T is not symmetrically distributed, we then consider the sum $R_n = \sum (X_{nk} - Y_{nk})(X'_{nk} - Y'_{nk})$, where X_T, Y_T, X'_T and Y'_T are independent and identically distributed. Then by part 1 above, $R_n \to 0$ a.s. and, by Lemma 3, $\sum X_{nk}(X'_{nk} - Y'_{nk}) \to 0$ a.s. and $\sum X_{nk}X'_{nk} \to 0$ a.s. Also, by part 1 above $\sum (X_{nk} - X'_{nk})^2$ converges a.s., and thus we obtain that $\sum X^2_{nk} + \sum X'^2_{nk}$ converges a.s. By Lemma 4, $\sum X^2_{nk}$ converges to (some) S. It remains to determine the random variable S.

3. Let $\pm\epsilon$ be continuity points of Ψ_{t_2}, set $S_{\epsilon,n} = \sum (X^{(\epsilon)}_{nk})^2$, $S_n = S_{\infty,n}$ and observe that $S_n = Q_n$. Since almost all sample functions have only a finite number of jumps $\{J_t\}$ in T in absolute magnitude greater than $\epsilon > 0$ and none equal to $\pm\epsilon$, and since right and left limits exist, it follows that

$$S_n - S_{\epsilon,n} \xrightarrow{\text{a.s.}} \sum_{|J_t| > \epsilon} J_t^2.$$

Thus $S_{\epsilon,n} \to S_\epsilon$ a.s., and as $\epsilon \downarrow 0$, the S_ϵ are nonincreasing. If we set $S_0 = \lim_{\epsilon \downarrow 0} S_\epsilon$, we have $S - S_0 = \sum J_t^2$ a.s. It remains to determine S_0. If the process is symmetrically distributed,

$$\operatorname{Var}(S_{\epsilon,n}) = \sum \operatorname{Var}(X^{(\epsilon)}_{nk})^2 \leqq \sum E(X^{(\epsilon)}_{nk})^4 \leqq \epsilon^2 \sum E(X^{(\epsilon)}_{nk})^2 = \epsilon^2 ES_{\epsilon,n}$$

and

$$\Psi(\epsilon) - \Psi(-\epsilon) \leftarrow \sum E \frac{(X_{nk}^{(\epsilon)})^2}{1 + (X_{nk}^{(\epsilon)})^2} \leqq ES_{\epsilon,n} \leqq (1 + \epsilon^2) \sum E \frac{(X_{nk}^{(\epsilon)})^2}{1 + (X_{nk}^{(\epsilon)})^2}.$$

These inequalities yield uniform integrability of the $S_{\epsilon,n}$, hence $ES_0 = \lim_{\epsilon\downarrow 0} \{\Psi(\epsilon) - \Psi(-\epsilon)\} = \sigma^2$. The inequalities also imply the degeneracy of S_0 at a constant. If the process is not symmetric, then by part 2 above, $\sum(X_{nk}^{(\epsilon)})^2 + \sum(X_{nk}'^{(\epsilon)})^2 \to 2\sigma^2$ a.s. as $n\to\infty$ and then $\epsilon\downarrow 0$. Then Lemma 4 implies $\sum(X_{nk}^{(\epsilon)})^2 \to \sigma^2$ a.s. in the iterated limit. The theorem is proved.

THEOREM 2. *Let g be a continuous function on the real line having a second derivative at* 0 *and with* $g(0) = 0$. *Then if X_T satisfies the conditions of Theorem* 1,

$$\int_T g(dX_t) = \lim_{n\to\infty} \sum g(X_{nk}) = g'(0)X_{[t_1,t_2)} + \frac{1}{2} g''(0)\sigma^2 + \sum \{g(J_t) - g'(0)J_t\} \quad a.s.,$$

where, as before, σ^2 is the variance of the normal part of $X_{[t_1,t_2)}$ and the sum on the right is over the jumps of X_T.

Proof. The hypothesis implies that g is of the form $g(x) = ax + bx^2 + h(x)$, where h is a continuous function and $h(x) = o(x^2)$ at the origin. Because of Theorem 1 we may restrict our attention of the function h. We will show that $\sum h(X_{nk}) \to \sum h(J_t)$ a.s. In fact,

$$\left| \sum h(X_{nk}^{(\epsilon)}) \right| \leqq \sup\{h(x)/x^2;\ |x| \leqq \epsilon\} \sum (X_{nk}^{(\epsilon)})^2,$$

and, letting $n\to\infty$ and then $\epsilon\downarrow 0$, we have $\sum h(X_{nk}^{(\epsilon)}) \to 0$ a.s. Then, taking $\pm\epsilon$ to be continuity points of Ψ_{t_2},

$$\sum h(X_{nk}) - \sum h(X_{nk}^{(\epsilon)}) = \sum_{\{k;\,|X_{nk}|>\epsilon\}} h(X_{nk})$$

which converges a.s. to

$$\sum_{|J_t|>\epsilon} h(J_t).$$

The proof is completed by letting $\epsilon\downarrow 0$.

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