ON SOLUTIONS OF CHANDRASEKHAR'S INTEGRAL EQUATION

BY

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1. **Introduction.** In a recent paper [2] C. Fox has transformed Chandrasekhar's integral equation

$$\frac{1}{H(\mu)} = 1 - \mu \int_0^1 \frac{\Psi(u) H(u)}{\mu + u}\, du \tag{1}$$

by means of the relation

$$H(\mu)H(-\mu) = 1 / T(\mu), \tag{2}$$

where

$$T(\mu) = 1 - 2\mu^2 \int_0^1 \frac{\Psi(u)}{\mu^2 - u^2}\, du, \tag{3}$$

into the form

$$T(\mu) H(\mu) = 1 + \mu \int_0^1 \frac{\Psi(u) H(u)}{u - \mu}\, du. \tag{4}$$

He has then obtained a solution for this by the methods developed by Muskhelishvili in [3]. In the above equations $\Psi(u)$ is known; in most physical problems it is non-negative and satisfies the inequality

$$\psi_0 \equiv \int_0^1 \Psi(u)\, du \leqq 1/2. \tag{5}$$

The equation (2) is deduced from (1), and therefore solutions of (1) satisfy (4). But do all solutions of (4) satisfy (1)? The working can be reversed if (2) is known, but this must now be deduced from (4). I show in this paper that (4) does not always imply the truth of (2) and that, in the most important cases, (4) has a family of solutions, of which at most two members satisfy (1).

2. **Solutions of equation** (4). I shall assume the conditions for $\Psi(u)$ adopted in [1], viz: that $\Psi(u)$ is real and non-negative for $0 \leqq u \leqq 1$; that every point of the closed interval $[-1, 1]$ lies in a domain D in the u-plane in which $\Psi(u)$ is regular; and that $\Psi(u)$ satisfies the condition (5).

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In [1], Chapter 2, it is proved that $T(\mu)$ is regular in the μ-plane cut along $(-1,1)$, that

$$T(\mu) = 1 - 2\psi_0 + O(\mu^{-2}) \qquad (\mu \to \infty), \tag{6}$$

$$T(\mu) = 1 + O(|\mu|) \qquad (|\mu| \to 0), \tag{7}$$

the latter holding uniformly in $\arg \mu$ in the cut plane.

Three cases arise according to the zeros of $T(\mu)$:

(i) $\psi_0 = 1/2$, when $T(\mu)$ has a double zero at infinity;

(ii) $\psi_0 < 1/2$, $\lim_{\mu \to 1+0} T(\mu) < 0$, when $T(\mu)$ has simple zeros at $\mu = \pm 1/k$ $(0 < k < 1)$;

(iii) $\psi_0 < 1/2$, $\lim_{\mu \to 1+0} T(\mu) \geqq 0$, when $T(\mu)$ has no zeros.

I shall denote by $H(\mu)$ the solution of equation (1) which is given for re $\mu > 0$ by

$$H(\mu) = \exp\left\{ \frac{\mu}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\log T(z)}{z^2 - \mu^2} dz \right\} .$$

This function is regular in the μ-plane cut along $(-1,0)$ except for a simple pole at infinity in case (i) and a simple pole at $-1/k$ in case (ii). (See [1, Theorem 11.1].)

Let $H_1(\mu)$ denote any solution of (4) which is regular, apart from a finite number of poles, in the μ-plane cut along $(-1,0)$. I shall assume also that $H_1(\mu)$ is continuous in the set

$$0 \leqq |\mu| \leqq 1, \quad |\arg \mu| \leqq \delta, \tag{8}$$

where $\delta > 0$. This condition is satisfied by $H(\mu)$ for any $\delta < \pi$ (cf. [1, (11.15)]).

Since

$$T(\mu) H_1(\mu) = 1 + \mu \int_0^1 \frac{\Psi(u) H_1(u)}{u - \mu}\, du, \tag{9}$$

$T(\mu)H_1(\mu)$ is regular in the plane cut along $(0,1)$ and is bounded as $\mu \to \infty$. From this, equation (6), and from the known zeros of $T(\mu)$, we have the following information in cases (i) – (iii):

(i) $H_1(\mu)$ may have either a simple or a double pole at infinity;

(ii) $H_1(\mu)$ may have simple poles at one or both of $\mu = \pm 1/k$, and it is bounded as $\mu \to \infty$;

(iii) $H_1(\mu)$ has no poles; it is bounded as $\mu \to \infty$.

3. **The derivation of** (2) **from** (4). Let

$$\Sigma(\mu) = T(\mu) H_1(\mu) H_1(-\mu) - 1. \tag{10}$$

We want to find under what conditions $\Sigma(\mu) \equiv 0$.

Since $T(\mu)$ is an even function of μ, so is $\Sigma(\mu)$. From the assumptions about $H_1(\mu)$ and the analysis in the last section it follows that $\Sigma(\mu)$ is regular, apart from possible poles, in the plane cut along $(0,1)$. Since, however, it is an even function, it must be regular in the uncut plane except, possibly, for a singularity at $\mu = 0$ and, in cases (i) and (ii):

(i) a simple or double pole at infinity;

(ii) simple poles at one or both of $\mu = \pm 1/k$.

In cases (ii) and (iii) $\Sigma(\mu)$ is bounded as $\mu \to \infty$.

Now consider $\Sigma(\mu)$ near $\mu = 0$. Let $\mu = \rho e^{i\phi}$, where $\delta \leqq \phi \leqq 2\pi - \delta$ (δ is defined in (8)). Then, for $0 \leqq u \leqq 1$,

$$|u - \mu| \geqq \rho \sin \delta, \quad |u - \mu| \geqq u \sin \delta,$$

and therefore, from (9),

$$|T(\mu)H_1(\mu) - 1| \leqq \operatorname{cosec} \delta \int_0^\eta \Psi(u)H_1(u)du + \rho \operatorname{cosec} \delta \int_\eta^1 \Psi(u)H_1(u)\frac{du}{u} = o(1)$$

by first choosing η small and then making ρ sufficiently small. Hence

$$T(\mu)H_1(\mu) \to 1 \qquad (|\mu| \to 0) \tag{11}$$

uniformly for $\delta \leqq \arg \mu \leqq 2\pi - \delta$. It follows that

$$T(\mu)\{\Sigma(\mu) + 1\} \to 1 \qquad (|\mu| \to 0)$$

uniformly for $\delta \leqq |\arg \mu| \leqq \pi - \delta$ and hence, by (7), that in these sectors

$$\Sigma(\mu) \to 0 \qquad (|\mu| \to 0). \tag{12}$$

In the sector $\pi - \delta \leqq \arg \mu \leqq \pi + \delta$, $H_1(-\mu)$ is, by hypothesis, bounded as $|\mu| \to 0$, and hence, by (11), $\Sigma(\mu)$ is bounded as $|\mu| \to 0$. Since $\Sigma(-\mu) = \Sigma(\mu)$, it is also bounded as $|\mu| \to 0$ in the sector $|\arg \mu| \leqq \delta$ and therefore it is bounded in the neighborhood of $\mu = 0$. Thus $\Sigma(\mu)$ is regular at $\mu = 0$ and (from (12)) $\Sigma(0) = 0$.

We can now consider our three cases:

(i) $\Sigma(\mu)$ is an integral function with a simple or a double pole at infinity. Since $\Sigma(\mu)$ is even and $\Sigma(0) = 0$, the only possible function is

$$\Sigma(\mu) = A\mu^2,$$

where A is an arbitrary constant. Writing $A = -\alpha^2$, we have

$$T(\mu)H_1(\mu)H_1(-\mu) = 1 - \alpha^2\mu^2 \tag{13}$$

and hence by (2), which is true for $H(\mu)$,

$$\frac{H_1(\mu)}{H(\mu)} \cdot \frac{H_1(-\mu)}{H(-\mu)} = 1 - \alpha^2\mu^2.$$

From this it seems probable that

$$H_1(\mu) = (1 + \alpha\mu)H(\mu) \tag{14}$$

is a solution of (9) but not of (1). ($H(\mu)$ is the unique solution of (1) when $\psi_0 = 1/2$ – loc. cit.) This is easily verified.

With $H_1(\mu)$ given by (14),

$$\begin{aligned}
\mu \int_0^1 \frac{\Psi(u)H_1(u)}{u-\mu}du &= \mu \int_0^1 \frac{\Psi(u)H(u)}{u-\mu}du \\
&\quad + \alpha\mu \int_0^1 \Psi(u)H(u)\left(1 + \frac{\mu}{u-\mu}\right)du \\
&= T(\mu)H(\mu) - 1 + \alpha\mu \int_0^1 \Psi(u)H(u)du \\
&\quad + \alpha\mu\{T(\mu)H(\mu) - 1\},
\end{aligned}$$

since equation (1) implies equation (4). But by [1, Theorem 12.1],

$$h_0 \equiv \int_0^1 \Psi(u)H(u)du = 1 - (1 - 2\psi_0)^{1/2},$$

and hence, since $\psi_0 = \frac{1}{2}$,

$$\begin{aligned}
\mu \int_0^1 \frac{\Psi(u)H_1(u)}{u-\mu}du &= (1 + \alpha\mu)T(\mu)H(\mu) - 1 \\
&= T(\mu)H_1(\mu) - 1.
\end{aligned}$$

Thus (14) is a solution of (9) for all α.

(ii) $\Sigma(\mu)$ is bounded at infinity, but may have simple poles at $\pm 1/k$. Since $\Sigma(\mu)$ is even and $\Sigma(0) = 0$, the only possible function is

$$\Sigma(\mu) = \frac{A\mu^2}{1 - k^2\mu^2}.$$

In this case, on writing $A = k^2 - \alpha^2$,

$$T(\mu)H_1(\mu)H_1(-\mu) = \frac{1 - \alpha^2\mu^2}{1 - k^2\mu^2}, \tag{15}$$

and

$$\frac{H_1(\mu)}{H(\mu)} \cdot \frac{H_1(-\mu)}{H(-\mu)} = \frac{1 - \alpha^2\mu^2}{1 - k^2\mu^2}.$$

From this it seems likely that

(16) $$H_1(\mu) = \frac{1 + \alpha\mu}{1 \pm k\mu} H(\mu).$$

Since $H(\mu)$ has a simple pole at $-1/k$ and $H_1(\mu)$ cannot have a double pole there, we must take the negative sign.

With this value for $H_1(\mu)$,

$$\begin{aligned}
\mu \int_0^1 \frac{\Psi(u)H_1(u)}{u-\mu} du &= \mu \int_0^1 \Psi(u)H(u) \frac{1+\alpha u}{(u-\mu)(1-ku)} du \\
&= \mu \int_0^1 \Psi(u)H(u) \left\{ \frac{1+\alpha\mu}{(1-k\mu)(u-\mu)} + \frac{k+\alpha}{(1-k\mu)(1-ku)} \right\} du \\
&= \frac{1+\alpha\mu}{1-k\mu} \left\{ T(\mu)H(\mu) - 1 \right\} + \frac{(k+\alpha)\mu}{1-k\mu} \int_0^1 \frac{\Psi(u)H(u)}{1-ku} du.
\end{aligned}$$

From (4), since $T(1/k) = 0$ and $H(1/k)$ is finite,

(17) $$\int_0^1 \frac{\Psi(u)H(u)}{1-ku} du = 1.$$

Hence

$$\begin{aligned}
\mu \int_0^1 \frac{\Psi(u)H_1(u)}{u-\mu} du &= \frac{1+\alpha\mu}{1-k\mu} \left\{ T(\mu)H(\mu) - 1 \right\} + \frac{(k+\alpha)\mu}{1-k\mu} \\
&= T(\mu)H_1(\mu) - 1.
\end{aligned}$$

Thus

(18) $$H_1(\mu) = \frac{1+\alpha\mu}{1-k\mu} H(\mu)$$

is a solution of (9) for all α. It is only a solution of (1) if $\alpha = \pm k$ (loc. cit.).

(iii) $\Sigma(\mu)$ is a bounded integral function, and it is therefore a constant which must be zero. Thus in this case only does (4) imply (2) and hence (1).

4. **Conclusions.** In general, it is most unsafe to solve (4) in place of (1). In the most important cases there is a family of solutions all satisfyuing the correct conditions at the origin and at infinity.

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