WEAK EIGENVECTORS AND THE FUNCTIONAL CALCULUS

BY

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In this paper we are concerned with the problem of constructing the functional calculus for an operator with continuous spectrum from a given collection of weak eigenvectors. A functional calculus is a homomorphism $f \to T_f$ of an algebra of functions defined over the 'spectrum' V into a subalgebra of bounded linear operators acting in a banach space E; we may regard an 'eigenfunction expansion' for such a functional calculus as an expression for all T_f over a banach space D dense in E of the form $T_f = \int_V f(\lambda) T_\lambda \, dm(\lambda)$, where m is a measure over V and T_λ is a bounded linear operator from D into the space of 'distributions' D'. In general, it is hoped that the range of T_λ, $\operatorname{Im} T_\lambda$, will consist of rather precisely specified weak eigenvectors (specified, for example, by boundary values). Since a family of weak eigenvectors is often available, a problem that often appears in applications is the following: *Given a family of weak eigenvectors at* λ, *when is it true that this family coincides with* $\operatorname{Im} T_\lambda$?

There are many results in the literature for a closed operator with dense domain in a banach space with continuous spectrum which guarantee a rich functional calculus and a representation of the functional calculus in terms of an eigenfunction expansion where, in fact, $\operatorname{Im} T_\lambda$ consists of weak eigenvectors. F. Browder [3] established such results for a large class of nonsymmetric partial differential operators, and Gel'fand and Šilov [5] were the first to consider the problem of the existence of eigenfunction expansions in general operator theoretic terms. However there are few results that tell us *which* weak eigenvectors to choose for the eigenfunction expansion. Most results of this nature are confined to differential operators with strong regularity theorems, the most classical being Weyl's construction of the Plancherel formula for a singular second order ordinary differential operator [10]. The crucial step is the 'Weyl lemma' which asserts that certain weak eigenvectors are actually infinitely differentiable, and so, classical uniqueness theorems may be invoked to determine precisely which weak eigenvectors contribute to the eigenfunction expansion.

The results given here, in contrast, are well within the context of functional

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analysis and so, in some respects, afford more generality. Other than the traditional machinery of functional analysis and harmonic analysis, the results in this paper depend decisively upon certain results of A. Beurling which are stated in §1. For our purposes, these results essentially establish a usable bridge between the resolvent and its boundary values.

The methods used in this paper require, *ab initio*, that we restrict our consideration to closed operators with dense domain in a banach space which generate a one-parameter strongly continuous group of operators in this banach space. In view of the fact that the Hille-Yoshida theorem (cf. Theorem 12.3.2 [6]) gives necessary and sufficient conditions on the resolvent in order that the operator generate such a group of operators and the various exponential formulas can relate conditions on the resolvent to conditions on the group of operators, we formulate our results directly in terms of a one-parameter strongly continuous group of operators in a banach space. A certain amount of complexity in stating our results is avoided in this way and (hopefully) this will not mitigate their usefulness.

§1 presents the various classes of one-parameter groups of operators for which we have results, and gives the essential definitions and technical apparatus that will be needed. §2 constructs the functional calculus and the generalized eigenfunction expansion; it is also shown that under relatively mild circumstances, the functional calculus is particularly rich in operators which approximate spectral projections over a compact set. §3 is the body of the paper; here, the relation between the functional calculus and weak eigenvectors is explored in some detail. Our principal results are Theorems 2 and 5, which give explicit criteria for the selection of the weak eigenvectors that appear in the functional calculus. In §4, by way of an application, we show how a unitary equivalence may, under suitable circumstances, be given a representation by means of weak eigenvectors.

1. **Definitions and preliminaries.** By M_b, we denote the usual space of bounded complex Radon measures over the real line and by L^1, the closed subspace of M_b consisting of those bounded measures which are absolutely continuous with respect to lebesgue measure. As usual, $m \in L^1$ will be labeled by its derivative, and any function which is equal almost everywhere to such a function will be called *integrable*. We make M_b into a banach algebra by defining the convolution product for $m, n \in M_b$: $m^*n(f) = \int\int f(x+t)dm(t)dn(x)$, f continuous with compact support. L^1 becomes a closed subalgebra of M_b. We will denote the fourier transform of $m \in M_b$, $\int e^{it\lambda}dm(\lambda)$, by $\hat{m}(t)$.

All linear spaces considered in this paper will be linear spaces over the field of complex numbers.

Let D and E be fixed separable banach spaces with conjugate duals D' and E' for which $D \subseteq E \subseteq E' \subseteq D'$, where all maps induced by inclusion are assumed continuous with dense range. In addition, we require that the duality between D'

and D should coincide with the duality between E' and E when restricted to E' and D, with this requirement, we may denote without confusion the result of applying an element $x' \in D'$ (or E') to an element $x \in D$ (or E) by $\langle x, x' \rangle$. D' and E' may be regarded as playing the roles of spaces of distributions.

We denote by $\mathscr{R}$ the collection of all strongly continuous representations of the additive group of the real line into a subgroup of $L(E)$, the usual algebra of bounded operators of E into E, where the group composition in $L(E)$ is the composition of operators. If $r \in \mathscr{R}$, then the value of r at a real number t will be denoted by $r(t)$. We make no assumptions of boundedness for the representations in $\mathscr{R}$.

Associated with each $r \in \mathscr{R}$ is the real-valued function p_r given by $p_r(t) = |{}^*r(t)|_{L(D,D')}$ the norm of the adjoint of $r(t)$ considered as an operator from D into D'; since p_r is lower semi-continuous, p_r is lebesgue measurable.

DEFINITION. *$\mathscr{L}$ will denote those $r \in \mathscr{R}$ which satisfy*

(1) *p_r is integrable.*

(2) *For each $a, b \in D$, the fourier transform of the function $t \to \langle r(t)a, b \rangle$ is integrable.*

If r is generated by a spectral operator of scalar type with condition (1) satisfied, then (2) is automatically satisfied. A substantially large class of operators with continuous spectrum encountered in applications generate $r \in \mathscr{L}$. We will write $\rho_{a,b}$ for the fourier transform of the function $t \to \langle r(t)a, b \rangle$.

DEFINITION. *$\mathscr{L}_1$ will denote those $r \in \mathscr{L}$ for which $t \to |r(t)|_{L(E)}$ is continuous and the integral $\int(\log^+ |r(t)|/(1 + t^2))dt$ is finite.*

The function $w_r(t) = \sup(|r(t)|_{L(E)}, |r(-t)|_{L(E)}, 1)$ for $r \in \mathscr{R}$ is a measurable, even, submultiplicative function for which $w_r(t) \geqq w_r(0) = 1$, for all t. We let $S(r)$ denote the algebra under pointwise multiplication of functions which are fourier transforms of those $m \in M_b$ for which $\int w_r d|m| < +\infty$, and, by $A(r)$, we denote the subalgebra of $S(r)$ consisting of the fourier transforms of functions in L^1. If $f \in S(r)$, m_f will denote the measure such that $\hat{m}_f = f$.

Should $r \in \mathscr{L}_1$, then w_r is continuous and satisfies $\int(\log w_r(t)/(1 + t^2))dt < +\infty$. Under these circumstances, a result of A. Beurling gives us the following theorem for $A(r)$ (essentially Theorem II.B [1, p. 356] and [2]—for the terminology concerning banach algebras, cf. [7]).

THEOREM (BEURLING). *Let $r \in \mathscr{L}_1$; give $A(r)$ the norm $|f|_r = \int w_r d|m_f|$ and complex conjugation as an involution. Then $A(r)$ becomes a commutative semi-simple regular B^*-algebra. Let I be a compact interval on the line and let $\mathscr{I}_I$ be the closed $*$-ideal in $A(r)$ given by the set of $f \in A(r)$ vanishing over I. Form the quotient B^*-algebra $A(r)_I$ by the ideal $\mathscr{I}_I$ and denote the norm in this algebra by $|\ \ |_{r,I}$. Then, for each $\tilde{f}$, $\tilde{f}$ denoting the image in $A(r)_I$ by $f \in A(r)$, the spectral norm of $\tilde{f}$ is given by*

$$\lim_{n\to\infty} |\tilde{f}^n|_{r,I}^{1/n} = \sup_{\lambda \in I} |f(\lambda)|.$$

As a consequence of this result, the *-isomorphism defined by mapping f into the restriction of f to I is the Gel'fand representation of $A(r)_I$. Since $A(r)$ is a regular B^*-algebra, $A(r)_I$ has a unit. Hence, by the well-known analogue of the Wiener lemma for commutative B^*-algebras with unit [7, p. 71], $\tilde{f}^{-1}$ exists in $A(r)_I$ if and only if f does not vanish over I. That is to say, there exists a $g \in A(r)$ such that $g = 1/f$ over I if and only if f does not vanish over I.

The last important type of $r \in \mathscr{R}$ for which we have results is the following generalization of the unitary representation.

Definition. *$r \in \mathscr{R}$ will be called subunitary if*

(1) *For each nonzero $x \in E$, $\langle x, x\rangle$ is real and strictly positive.*

(2) *For all t, the restriction of $^*r(t)$ to E coincides with $r(-t)$.*

As is well known, the fact that $\langle x, x\rangle$ is real for all $x \in E$ implies that the sesquilinear form over E, $x, y \to \langle x, y\rangle$ is hermitian.

Finally, for a given r under consideration and a complex number λ, we let $\mathscr{E}_\lambda$ denote the subspace of E' of $e \in E'$ for which $^*r(t)e = e^{i\lambda t}e$ for all t. Clearly $e \in \mathscr{E}_\lambda$ if and only if, for all t, $x \in E$, $\langle r(t)x, e\rangle = e^{i\lambda t}\langle x, e\rangle$. More generally, for any linear space of vectors N dense in D for which $r(t)$ maps N into N for each t, we denote by $\mathscr{N}_\lambda$ the subspace of D' of $e \in D'$ for which $\langle r(t)a, e\rangle = e^{i\lambda t}\langle a, e\rangle$ for all t and all $a \in N$. If $\mathscr{E}_\lambda \neq \{0\}$ ($\mathscr{N}_\lambda \neq \{0\}$), then the nonzero elements of $\mathscr{E}_\lambda$ ($\mathscr{N}_\lambda$) is the set of weak eigenvectors in D' over E (over N). We will let S denote the set of λ, λ an arbitrary complex number, such that $\mathscr{E}_\lambda \neq \{0\}$.

2. The functional calculus and the eigenfunction expansion. Following a procedure originating with R. S. Phillips for semi-groups [8], we construct a spectral calculus for $r \in \mathscr{R}$.

Proposition 1. *Let $r \in \mathscr{R}$. For each $f \in S(r)$.*

$$T_f x = \int r(t)x\, dm_f(t)$$

exists and defines an operator in $L(E)$ which commutes with $r(t)$ for all t. The map defined by $f \to T_f$ is an algebraic homomorphism onto a subalgebra of $L(E)$ for which $|T_f|_{L(E)} \leqq |f|_r$.

Proof. Since $\int r(t)x\, dm_f(t)$ exists as an ordinary vector-valued Riemann integral, we have

$$r(t')\int r(t)x\, dm_f(t) = \int r(t)r(t')x\, dm_f(t)$$

so that $r(t')T_f x = T_f r(t')x$, for all $x \in E$.

An application of the Fubini theorem gives

$$T_f T_g x = \int r(t)x d(m_g * dm_f)(t) = \int r(t)x\, dm_{fg}(t) = T_{fg}x$$

so that the fact that $f \to T_f$ is an algebraic homomorphism onto a subalgebra of $L(E)$ follows from the fact that $S(r)$ is an algebra. Clearly

$$|T_f|_{L(E)} \leqq \int w_r(t)\, d|m_f|(t) \equiv |f|_r.$$

Proposition 2. *Let $r \in \mathscr{L}$. Then, there exists a unique map from the real line into $L(D,D')$, $\lambda \to T_\lambda$, such that for each pair $a, b \in D$, $\lambda \to \langle a, T_\lambda b\rangle$ is a continuous integrable function with*

$$\int f\langle a, T_\lambda b\rangle\, d\lambda = \langle T_f a, b\rangle, \qquad f \in S(r)$$

and $|T_\lambda|_{L(D,D')} \leqq \int p_r(t)dt$ for all λ.

Proof. Uniqueness is a consequence of the fact that any equivalence class of functions in L^1 contains at most one continuous function. As

$$|\rho_{a,b}(\lambda)| \leqq \int |\langle r(t)a,\ b\rangle|\, dt \leqq |a|_D |b|_D \int p_r(t)dt$$

we have for each $b \in D$, a unique $T_\lambda b \in D'$ for which $\langle a, T_\lambda b\rangle = \rho_{a,b}(\lambda)$. Since the map $a,\ b \to \rho_{a,b}(\lambda)$ is sesquilinear, it is evident that T_λ is a linear map from D into D'; as $|\langle a, T_\lambda b\rangle| \leqq |a|_D |b|_D \int p_r(t)\, dt$, we have $|T_\lambda|_{L(D,D')} \leqq \int p_r(t)\, dt$. Now, an application of the Fubini theorem yields:

$$\begin{aligned}\langle T_f a, b\rangle &= \int \langle r(t)a, b\rangle\, dm_f(t) = \int dm_f(t) \int e^{i\lambda t}\langle a, T_\lambda b\rangle d\lambda \\ &= \int \langle a, T_\lambda b\rangle\, d\lambda \int e^{i\lambda t} dm_f(t) = \int f\langle a, T_\lambda b\rangle d\lambda.\end{aligned}$$

For $r \in \mathscr{L}$, we let V_D denote the set of all λ such that $T_\lambda \neq 0$; V_D is an open subset of the real line, since $V_D = \bigcup_{a,b\in D}\{\lambda : \rho_{a,b}(\lambda) \neq 0\}$. Proposition 2 is a rarefied version, for $r \in \mathscr{L}$, of the classical idea of an eigenfunction expansion and V_D plays the role of that part of the spectrum of r which contributes to the eigenfunction expansion.

Proposition 3. *Let $r \in \mathscr{R}$. Then, for all $f \in S(r)$, $e \in \mathscr{E}_\lambda$, ${}^*T_f e = f(\lambda)e$.*

Proof. Let $x \in E$; then

$$\langle x, {}^*T_f e\rangle = \langle T_f x, e\rangle = \int \langle r(t)x, e\rangle\, dm_f(t) = \int e^{i\lambda t}\langle x, e\rangle\, dm_f(t) = \langle x, e\rangle f(\lambda).$$

Since this is true for all $x \in E$, we have ${}^*T_f e = f(\lambda)e$.

Denote by $S(r, V_D)$ the functions defined over V_D which are the restriction of functions in $S(r)$. For $r \in \mathscr{L}_1$, we have the following improvement of Proposition 1.

Proposition 4. *If $r \in \mathscr{L}_1$, then the map $f \to T_f$ considered as a map from $S(r, V_D)$ into $L(E)$ is an algebraic isomorphism for which*

$$|T_f|_{L(D,D')} \leqq |f|_{L^1} \int p_r(t)\,dt.$$

Proof. Let f be such that $T_f = 0$. Then, for all $g \in S(r)$, $0 = T_f T_g = T_{fg}$, so that for all a, b in D,

$$\int fg\rho_{a,b}d\lambda = 0.$$

Assuming that f is not 0 over V_D implies that there is a real number λ_0 and some $a, b \in D$ such that $\mathrm{Re}\{f(\lambda_0)\rho_{a,b}(\lambda_0)\} > 0$. Since f and $\rho_{a,b}$ are continuous, there exist an open nonvoid interval $I \subseteq V_D$ such that for all $\lambda \in I$, $\mathrm{Re}\{f(\lambda)\rho_{a,b}(\lambda)\} > 0$. Since $A(r)$ is a regular B^*-algebra, there exists a real-valued strictly positive function $g_I \in A(r)$ whose support lies entirely within I. Hence,

$$\left|\int fg_I\rho_{a,b}d\lambda\right| \geqq \mathrm{Re}\int fg_I\rho_{a,b}d\lambda = \int \mathrm{Re}(fg_I\rho_{a,b})d\lambda$$

and since the integrand is strictly greater than zero over a set of positive measure, this last integral must be strictly greater than zero. This is a contradiction, so that $f(\lambda) = 0$ for all $\lambda \in V_D$. Finally,

$$|T_f|_{L(D,D')} \leqq \int |f|\,|T_\lambda|_{L(D,D')}d\lambda \leqq |f|_{L^1}\int p_r(t)\,dt,$$

by Proposition 2.

As a consequence of the above, we have the following result which expresses a striking similarity between the functional calculus of $r \in \mathscr{L}_1$ and the functional calculus for a spectral operator of scalar type. Let $\mathscr{V}$ denote the set of pairs of subsets of V_D, (A,B), $\bar{A} \subseteq B$, where in V_D, A and B are nonvoid and open with $\bar{B}$ compact.

THEOREM 1. *If $r \in \mathscr{L}_1$, then for each $(A,B) \in \mathscr{V}$, there exists a nonzero $p(A,B) \in L(E)$ which commutes with $r(t)$ for all t and satisfies the following properties*:

(a) *For (A,B), $(A',B') \in \mathscr{V}$, $p(A,B)p(A',B') = p(A',B')p(A,B)$ and if $B' \cap B = \emptyset$, then $p(A,B)p(A',B') = 0$.*

(b) *If F_A denotes the filter of open sets containing A, then*

$$\lim_{B \in F_A} |p(A,B)^2 - p(A,B)|_{L(D,D')} = 0.$$

(c) *If $\{(A_n,B_n)\}$ is a sequence in $\mathscr{V}$, such that $\{A_n\}$ is a rising sequence whose union is V_D, then for $a, b \in D$, $\lim_n \langle p(A_n,B_n)a, b\rangle = \langle a, b\rangle$.*

Proof. Using, once more, the fact that the B^*-algebra $A(r)$ is regular, for any $(A,B) \in \mathscr{V}$, there is a real positive $f_{A,B} \in A(r)$ which vanishes outside of B and is identically 1 over A; we may also assume that the range of $f_{A,B}$ lies in the interval $[0,1]$. Since $f_{A,B} \neq 0$ over V_D, Proposition 4 implies that $T_{f_{A,B}} \neq 0$. We set $T_{f_{A,B}} = p(A,B)$.

Proposition 1 immediately implies (a) and the fact that $p(A,B)$ commutes with $r(t)$ for all t. Using the estimate in Proposition 4, we have

$$|p(A,B)^2 - p(A,B)|_{L(D,D')} \leqq \left\{\int |(f_{A,B}^2 - f_{A,B})|\,d\lambda\right\}\int p_r(t)\,dt$$

$$\leqq \left\{\int_{B\cap\mathscr{C}A} d\lambda\right\}\int p_r(t)\,dt$$

and since $\inf_{B\in F_A}\int_{B\cap\mathscr{C}A}d\lambda = 0$, this proves (b). Finally, $\lim_n \langle p(A_n,B_n)a,b\rangle = \lim_n \int f_{A_n,B_n}\rho_{a,b}d\lambda = \int \rho_{a,b}\,d\lambda = \langle a, b\rangle$, using the Lebesgue dominated convergence theorem.

Proposition 5. *For each a in the unit ball of D, there exists a decomposition $a = x_1 + x_2$ such that x_1, x_2 are in the images in E of the unit ball of D by the operators $p(A,B)$ and $1-p(A,B)$, respectively. If x_1', x_2' are another pair of vectors satisfying the above conditions, then*

$$|x_1 - x_1'|_{D'} = |x_2 - x_2'|_{D'} \leqq 4\,|p(A,B)^2 - p(A,B)|_{L(D,D')}.$$

Proof. The existence of such a decomposition is clear. Since $x = x_1 - x_1' = x_2' - x_2$ is a vector in the image of the ball of radius 2 in D of the operation $p(A,B)$ and $1-p(A,B)$, there exists b, c in the ball of radius 2 in D such that $x = p(A,B)b = (1 - p(A,B))c$. Hence, $x = (p(A,B) - p(A,B)^2)(b+c)$, or $|x|_{D'} \leqq 4\,|p(A,B)^2 - p(A,B)|_{L(D,D')}$.

For $(A,B)\in\mathscr{V}$, we will denote by $p(A,B)$ any operator in $L(E)$ constructed in the manner of Theorem 1, i.e., from a real positive function in $S(r)$ equal to 1 over A and vanishing outside of B.

3. The main theorems.

Proposition 6. *If $r\in\mathscr{L}_1$ and N is a dense linear space in D invariant under $r(t)$ for all t, then* $\operatorname{Im}T_\lambda \subseteq \mathscr{N}_\lambda'$ *for all $\lambda\in V_D$.*

Proof. For each $f\in A(r)$, $a\in N$, $b\in D$, $\int f e^{i\lambda t}\langle a, T_\lambda b\rangle d\lambda = \langle T_f r(t)a, b\rangle = \int f\langle r(t)a, T_\lambda b\rangle d\lambda$. Since $A(r)$ is regular, $A(r)$ is dense in the space of continuous functions vanishing at infinity, in virtue of the Stone-Weierstrass theorem; hence, $e^{i\lambda t}\langle a, T_\lambda b\rangle = \langle r(t)a, T_\lambda b\rangle$ for almost all λ; as the right and left sides of this equality are continuous functions of λ, this equality holds for all λ. Hence, $\operatorname{Im}T_\lambda \subseteq \mathscr{N}_\lambda'$ for all $\lambda\in V_D$.

For $r\in\mathscr{L}_1$, we have the inclusions $\mathscr{E}_\lambda \subseteq \mathscr{N}_\lambda'$ and $\operatorname{Im}T_\lambda \subseteq \mathscr{N}_\lambda'$; these inclusions—without 'regularity'—will, in general, be proper. The question we now consider is that of finding reasonable conditions under which we can assert that $\mathscr{E}_\lambda = \operatorname{Im}T_\lambda$.

Let $V_D(c)$ denote the open subset of V_D consisting of those λ such that there exists $a\in D$ for which $\rho_{a,c}(\lambda)\neq 0$.

Proposition 7. *If $r\in\mathscr{L}_1$ and there exists $c\in D$ such that the linear span of $\{{}^*r(t)c\}_t$ is dense in E', then $V_D(c)$ is dense in V_D.*

Proof. If $V_D(c)$ is not dense, then there exists $(A,B) \in \mathscr{V}$ such that $B \cap \overline{V_D(c)} = \emptyset$. So, $\langle r(t)p(A,B)a,c\rangle = \int e^{i\lambda t} f_{A,B}\,\rho_{a,c}\,d\lambda$, and as the support of $\rho_{a,c}$ is contained in $\overline{V_D(c)}$, it follows that $\langle p(A,B)a, {}^*r(t)c\rangle = 0$, for all t, and so $p(A,B) = 0$, which contradicts Theorem 1.

THEOREM 2. *Let $r \in \mathscr{L}_1$ and satisfy*:

(a) $\int p_r(t)w_r(t)dt < +\infty$.

(b) *There exists a $c \in D$ such that the linear span of the set of vectors $\{{}^*r(t)c\}_t$ is dense in E'. Then, for $\lambda \in S \cap V_D(c)$, $\mathscr{E}_\lambda$ is the one-dimensional space spanned by $T_\lambda c$.*

Proof. In virtue of the fact that $\mathscr{E}_\lambda \neq 0$, to show that $\mathscr{E}_\lambda$ is the linear span of $T_\lambda c$ is equivalent to showing that the hyperplane X in $D\{a: \langle a, T_\lambda c\rangle = 0\} = X \subseteq \{a: \langle a,e\rangle = 0$ for all $e \in \mathscr{E}_\lambda\}$. Let $a \in X$. Since $\lambda \in V_D(c)$, there exists $a_0 \in D$ such that $\rho_{a_0,c}(\lambda) \neq 0$, so that there is a compact nonvoid interval I containing λ in its interior such that $\rho_{a_0,c} \neq 0$ over I.

Now, (a) implies that $\rho_{a,b}$ is in $A(r)$ for all $a, b \in D$, so that the aforementioned result of Beurling asserts the existence of an $f \in A(r)$ such that

$$(*) \qquad f = \frac{\rho_{a,c}}{\rho_{a_0,c}} \quad \text{over } I.$$

Let $g \in A(r)$ where $g(\lambda) = 1$ and g vanishes outside of I. Rewriting (*), we have $fg\,\rho_{a_0,c} = g\,\rho_{a,c}$ everywhere and it follows that $\langle r(t)T_{fg}a_0,c\rangle = \langle r(t)T_g a,c\rangle$, or $\langle T_{fg}a_0, {}^*r(t)c\rangle = \langle T_g a, {}^*r(t)c\rangle$. From this expression and condition (b), we have

$$(**) \qquad T_{fg}a_0 = T_g a.$$

Let e be an arbitrary element in $\mathscr{E}_\lambda$. Using Proposition 3 and (**), we have

$$\langle a,e\rangle = g(\lambda)\langle a,e\rangle = \langle T_g a,e\rangle = \langle T_{fg}a_0,e\rangle = f(\lambda)g(\lambda)\langle a_0,e\rangle;$$

from (*), $f(\lambda) = 0$, so that $\langle a,e\rangle = 0$. That is to say $X \subseteq \{a: \langle a,e\rangle = 0$, for all $e \in \mathscr{E}_\lambda\}$, which is what was to be proved.

Theorem 2 is essentially a uniqueness theorem for weak eigenvectors in E'. The following two corollaries emphasize this aspect of the theorem.

COROLLARY 1. If *r and c satisfy the conditions of Theorem 2, then for any $\lambda \in V_D(c)$ such that* $\operatorname{Im} T_\lambda \subseteq \mathscr{E}_\lambda$, $\operatorname{Im} T_\lambda = \mathscr{E}_\lambda$ *and both of these spaces are one-dimensional.*

Proof. The fact that $\lambda \in V_D(c)$ implies that $T_\lambda \neq 0$, so that $\mathscr{E}_\lambda \neq \{0\}$; hence, Theorem 2 is applicable.

COROLLARY 2. *Let r and c satisfy the conditions of Theorem 2 and assume that N is a linear space dense in D and left invariant by $r(t)$, for all t. Then, given any $\lambda \in V_D(c)$ for which $\mathscr{N}_\lambda \subseteq E'$, we must have: $\mathscr{N}_\lambda = \operatorname{Im} T_\lambda = \mathscr{E}_\lambda$, and each of these spaces is one-dimensional.*

Proof. Since $\mathscr{N}_\lambda \subseteq E', \mathscr{E}_\lambda = \mathscr{N}_\lambda^\perp$, so that $\mathscr{E}_\lambda = \mathscr{N}_\lambda^\perp \supseteq \operatorname{Im} T_\lambda$; from this, Corollary 1 gives the result.

We now give a prescription for the construction of the functional calculus from weak eigenvectors.

COROLLARY 3. *Assume that r and c satisfy the conditions of Theorem 2 and in addition, $S \supseteq V_D(c)$ and* Im T_λ *is one-dimensional for all $\lambda \in V_D(c)$. Then there exists a pair of continuous functions $\lambda \to e_\lambda$ and $\lambda \to f_\lambda$ from $V_D(c)$ into D', D' given the weak* topology, such that e_λ is a nonzero element of $\mathscr{E}_\lambda$ and for all $f \in S(r)$ whose support lies in V_D(c),*

$$(*) \qquad \langle T_f a, b\rangle = \int f \langle a, e_\lambda\rangle \overline{\langle b, f_\lambda\rangle}\, d\lambda.$$

Proof. Let $e_\lambda = T_\lambda c$. Since $\operatorname{Im} T_\lambda$ is one-dimensional for $\lambda \in V_D(c)$, there exists a unique $f_\lambda \in D'$ such that $\langle a, T_\lambda b\rangle = \langle a, e_\lambda\rangle \overline{\langle b, f_\lambda\rangle}$ for all $a, b \in D$ and $\lambda \in V_D(c)$. By construction, the functions $\lambda \to \langle a, e_\lambda\rangle$ and $\lambda \to \overline{\langle b, f_\lambda\rangle}$ are continuous; Corollary 2 gives us the fact that $e_\lambda \in \mathscr{E}_\lambda$ and Proposition 2 gives us (*).

Without restrictions on $^*r$ more precise information, (e.g., concerning the role of f_λ) does not seem obtainable in the present context. However, with the additional restriction that r be subunitary, we can obtain a fairly classical picture of the relation between the functional calculus and the weak eigenvectors. We now restrict our attention to $r \in \mathscr{R}$ which are subunitary.

The requirement that r be subunitary implies that the sesquilinear form $x, y \to \langle x, y\rangle$ is a positive definitehermitian form over E; that is to say, E is a dense subspace of a hilbert space H that one obtains by completing the pre-hilbert space E with inner product $x, y \to \langle x, y\rangle$. Now, $\langle r(t)x, r(t)y\rangle = \langle x, {}^*r(t)r(t)y\rangle = \langle x, r(-t)r(t)y\rangle = \langle x, y\rangle$, so that $r(t)$ in E preserves the inner product of H. Hence, there exists a unique unitary operator $R(t) \in L(H)$ whose restriction to $L(E)$ is $r(t)$. It is clear that $t \to R(t)$ is a unitary representation of the real line into $L(H)$, and the Banach-Steinhaus theorem assures us that $t \to R(t)$ is strongly continuous. From this, we have the following characterization of subunitary $r \in \mathscr{R}$:

PROPOSITION 8. *r is subunitary if and only if there exists a hilbert space H and a weakly continuous unitary representation R of the real line satisfying:*

(i) *E is a dense subspace of H and the inner product of H coincides with $\langle , \rangle$ over $E \times E$.*

(ii) *$R(t)$, the value of R at t, maps E continuously into E, and the restriction of $R(t)$ to $L(E)$ is a strongly continuous representation of the line in $L(E)$.*

The natural map of E into H is continuous, for if k is the norm of the natural map of E into E', then $|x|_H^2 = \langle x,x\rangle \leqq |x|_E |x|_{E'} \leqq k|x|_E^2$.

The verification of whether or not r is in $\mathscr{L}$ is comparatively easy.

PROPOSITION 9. *Let $r \in \mathscr{R}$ be subunitary. Then $r \in \mathscr{L}$ if and only if p_r is integrable.*

Proof. Since r is subunitary, the function $t \to \langle r(t)a, a\rangle$ for each $a \in D$ is positive definite. If p_r is integrable, the Bochner theorem implies that its fourier transform $\rho_{a,a}$ is integrable and polarization then demonstrates that $\rho_{a,b}$ is also integrable for all $a, b \in D$; that is, $r \in \mathscr{L}$. Conversely, by definition of $\mathscr{L}$, p_r is integrable.

As is well known, R has an infinitesimal generator iA, where A is a closed self-adjoint operator with dense domain in H. Let V denote the closed subset of the real line which is the spectrum of A. Now, the spectral theorem of M. H. Stone (Theorem 22.4.3 [6]) yields the existence of a family of measures $m_{x,y} \in M_b$ for all $x, y \in H$ such that $\langle R(t)x, y\rangle = \int e^{i\lambda t} dm_{x,y}(\lambda)$; we recall that this family of measures satisfies the following:

(a) The map $x, y \to m_{x,y}$ from $H \times H$ into M_b is sesquilinear and for all $x \in H$, $m_{x,x}$ is positive.

(b) $|m_{x,y}|_{M_b} \leqq |x|_H |y|_H$.

(c) The closure of the union of the supports of $m_{x\,x}$ for all $x \in H$ is precisely V.

PROPOSITION 10. *Let $r \in \mathscr{L}$ be subunitary and R its associated unitary representation in $L(H)$. Then,*

(i) *$\rho_{a,b}$ is the derivative of $m_{a,b}$ for all $a, b \in D$.*

(ii) *$m_{x,y} \in L^1$ and $m_{x,y}(V \cap \mathscr{C} V_D) = 0$ for all $x, y \in H$.*

(iii) *V_D is an open dense subset of V.*

(iv) *For each $f \in S(r)$, the restriction of ${}^*T_f$ to E is $T_{\bar{f}}$.*

(v) *For $f, g \in S(r)$ and arbitrary $x, y \in E$,*

$$^m T_f x, T_g y = f \bar{g}\, m_{x,y}.$$

Proof. (i) follows from the uniqueness of the fourier transform. From (b) and the fact that D is dense in H we have that $m_{x,y}$ is the limit in M_b of a sequence $\{m_{a_n,b_n}\}_n$ and since L^1 is a closed subspace of M_b, we have $m_{x,y} \in L^1$. From (i) it is evident that $V_D \subseteq V$; if K is any compact set contained in $V \cap \mathscr{C} V_D$, we have $m_{a,b}(K) = 0$ so that on taking the sup over all $K \subseteq V \cap \mathscr{C} V_D$, $m_{a,b}(V \cap \mathscr{C} V_D) = 0$ for all a, $b \in D$. Taking limits in L^1 gives us $m_{x,y}(V \cap \mathscr{C} V_D) = 0$ for all $x, y \in H$. From the fact that $m_{x,x}(V \cap \mathscr{C} V_D) = 0$ for all $x \in H$, it follows that $V \cap \mathscr{C} V_D$ has a void interior since any nonvoid interior must contain an interior point of the support of $m_{x,x}$ for some $x \in H$. For each $x, y \in E$, $\langle x, {}^*T_f y\rangle = \langle T_f x, y\rangle = \int \langle r(t)x, y\rangle dm_f(t) = \int \langle x, {}^*r(t)y\rangle\, dm_f(t) = \int \langle x, r(-t)y\rangle dm_f(t)$ from which it follows that ${}^*T_f y = \int r(-t)y\, dm_f(t) = \int r(t)y\, dm_{\bar{f}}(t) = T_{\bar{f}} y$, which proves (iv). Again, using (b) and the fact that D is dense in H, it is evident that it is sufficient to prove the equality in (v) over D; now, for arbitrary $a, b \in D$,

$$\int e^{i\lambda t}\, dm_{T_f a, T_{\bar{g}} b}(\lambda) = \langle r(t)T_f a, T_g b\rangle = \langle T_{\bar{g}} r(t) T_f a, b\rangle, \text{by (iv)}.$$

Propositions 1 and 2 imply that $\langle T_{\bar{g}} r(t) T_f a, b\rangle = \int e^{i\lambda t} f\bar{g}\rho_{a,b}\, d\lambda$. Hence $\int e^{i\lambda t} f\bar{g}\rho_{a,b} d\lambda = \int e^{i\lambda t} dm_{T_f a, T_g b}(\lambda)$ and with uniqueness of the fourier transform we have $f\bar{g}\, m_{a,b} = m_{T_f a, T_g b}$.

We recall that a basic measure for R [4, p. 112] is a positive measure m over V such that for every subset N of V, $m(N) = 0$ if and only if $m_{x,x}(N) = 0$ for all $x \in H$. Hence, from (ii), we have $m(V \cap \mathscr{C} V_D) = 0$.

We are now able to prove an approximation theorem for the weak eigenvectors of subunitary $r \in \mathscr{L}_1$.

THEOREM 3. *If $r \in \mathscr{L}_1$ is subunitary, then for arbitrary $(A,B) \in \mathscr{V}$ and $\lambda \in A, \mathscr{E}_\lambda$ is in the uniform closure in E' of* $\operatorname{Im} p(A,B)$.

Proof. Let e be an arbitrary element of $\mathscr{E}_\lambda$; since E is dense in E', there exists a sequence $\{x_n\}_n \subseteq E$ such that $\lim_n x_n = e$ in E'. Hence, using (iv) of Proposition 10, $\lim_n T_{\bar{f}} x_n = \lim_n {}^*T_f x_n = {}^*T_f e = f(\lambda)e$, so that for $p(A,B)$, which comes from a real function in $S(r)$, we have $\lim_n p(A,B)x_n = e$ in E'.

The following is a somewhat more dramatic version of the same idea.

THEOREM 4. *If $r \in \mathscr{L}_1$ is subunitary, then for arbitrary $\lambda \in V_D$, there exists a sequence E_n of closed subspaces of E invariant under the action of r such that $\bigcap_n E_n = \{0\}$ and for each n, $\mathscr{E}_\lambda$ is contained in the uniform closure of E_n in E'.*

Proof. Since $\lambda \in V_D$, there exists a sequence $(A_n, B_n) \in \mathscr{V}$ such that $\bigcap B_n = \{\lambda\}$. Define E_n as the closure in E of $\operatorname{Im} p(A_n, B_n)$; E_n is evidently invariant under the action of r and Theorem 3 asserts that $\mathscr{E}_\lambda$ is contained in the uniform closure of E_n in E'.

Let $z \in \bigcap_n E_n$. $m_{z,z}$, by Proposition 10, is in L^1 and so is a nonatomic measure. We will now show that the support of $m_{z,z}$ reduces to $\{\lambda\}$ and so it will follow that $m_{z,z} = 0$ and consequently $z = 0$.

As $z \in E_n$, for each $n, z = \lim_i x_i$ in E, $x_i \in \operatorname{Im} p(A_n, B_n)$; hence, using (v) of Proposition 10, we conclude that the support of m_{x_i,x_i}, for each i is contained in B_n, and as $\lim_i m_{x_i,x_i} = m_{z,z}$ in M_b, we may conclude that the support of $m_{z,z}$ lies in B_n for all n. From this, it follows that the support of $m_{z,z}$ reduces to $\{\lambda\}$.

As a preliminary to the statement and proof of our main result for subunitary $r \in \mathscr{L}_1$, we need a statement of the connection between the multiplicity of R and the eigenfunction expansion for $r \in \mathscr{L}$. We recall that the multiplicity of R may be defined in terms of a direct integral decomposition of R $\oplus\int H_\lambda\, dm(\lambda), m$ a basic measure for R, namely, the multiplicity of R is the class of m-measurable functions $[\sigma]$ determined by the function σ defined by $\sigma(\lambda) =$ the dimension of H_λ and the relation $\sigma_1 \in [\sigma]$ if and only if $\sigma_1 = \sigma$ almost everywhere with respect to m. The class $[\sigma]$ does not depend upon the particular direct integral diagonalizing R. For an exposition of the theory of direct integrals, we refer the reader to Chapter II of [4].

LEMMA. *If $r \in \mathscr{L}$ is subunitary and R is its associated unitary representation in $L(H)$ whose multiplicity $\sigma = 1$ a.e. with respect to m, then for all $\lambda \in V_D$,* $\dim \operatorname{Im} T_\lambda = 1$.

Proof. By virtue of (i) of Proposition 10 and the fact that for any point $\lambda \in V_D$ there exists an $a \in D$ such that $\rho_{a,a}$ is strictly positive in a neighborhood of λ, it follows that the restriction of m to V_D is equivalent to the restriction of the lebesgue measure to V_D. As a consequence of this and (ii) of Proposition 10 m may be chosen to be the measure over V which is the restriction of lebesgue measure to V_D and satisfies $m(V \cap \mathscr{C} V_D) = 0$. We now construct a direct integral for R.

For each $\lambda \in V_D$, define H_λ to be the nonzero hilbert space obtained from the pre-hilbert space D given the positive hermitian form $a, b \to \rho_{a,b}(\lambda)$, and for $\lambda \in V \cap \mathscr{C} V_D$, define $H_\lambda = 0$. Hence, by construction, the natural map of D into $H_\lambda, a \to a(\lambda)$ satisfies $[a(\lambda), b(\lambda)]_{H_\lambda} = \rho_{a,b}(\lambda)$. Since $\rho_{a,b}$ is integrable, we may form the direct integral ${}^{\oplus}\int_V H_\lambda \, d\lambda$, and it is easy to verify that the map of D into ${}^{\oplus}\int_V H_\lambda \, d\lambda$ defined by $a \to \{a(\lambda)\}_{\lambda \in V}$ extends to an isometry of H onto ${}^{\oplus}\int_V H_\lambda \, d\lambda$ which diagonalizes R. Hence, ${}^{\oplus}\int_V H_\lambda \, d\lambda$ is a direct integral for R, and so $\dim H_\lambda = 1$ a.e. with respect to the restriction of the lebesgue measure to V_D; or equivalently, we have $\dim \operatorname{Im} T_\lambda = 1$ a.e. over V_D.

We know that $\dim \operatorname{Im} T_\lambda \geqq 1$ for all $\lambda \in V_D$. Let us assume there exists a $\lambda_0 \in V_D$ such that $\dim T_{\lambda_0} > 1$. Then, there exists $b_1, b_2 \in D$ such that $\{T_{\lambda_0} b_1, T_{\lambda_0} b_2\}$ is linearly independent. Since $\dim \operatorname{Im} T_\lambda \geqq 1$ a.e. over V_D, there does not exist a neighborhood about λ_0 such that $\{T_\lambda b_1, T_\lambda b_2\}$ is linearly independent for all λ in this neighborhood. Hence, there exists a compact neighborhood V_0 of λ_0 such that $T_\lambda b_1 \neq 0$, $T_\lambda b_2 \neq 0$ for all $\lambda \in V_0$, a sequence $\{\lambda_n\}_n \subseteq V_0$ such that $\lim_n \lambda_n = \lambda_0$, and a sequence of complex numbers $\{k_n\}_n$ whose moduli are bounded above and below, such that

$$(*) \qquad T_{\lambda_n} b_1 = k_n T_{\lambda_n} b_2.$$

Now, there exists a subsequence k_{n_i} such that $\lim_i k_{n_i} = k \neq 0$. Therefore, applying $a \in D$ to each side of (*) and using continuity we have $\langle a, T_{\lambda_0} b_1 \rangle = \lim_i \langle a, T_{\lambda_{n_i}} b_1 \rangle = \lim_i \langle a, k_{n_i} T_{\lambda_{n_i}} b_2 \rangle = \langle a, k T_{\lambda_0} b_2 \rangle$ which implies that $T_{\lambda_0} b_1 = k T_{\lambda_0} b_2$, which is a contradiction. Hence, $\dim \operatorname{Im} T_\lambda = 1$ for all $\lambda \in V_D$.

We may now prove our principal result concerning subunitary $r \in \mathscr{L}_1$.

THEOREM 5. *Let $r \in \mathscr{L}_1$ be subunitary and R its associated unitary representation in $L(H)$ with multiplicity function σ. If*

(a) $\int p_r(t) w_r(t) dt < +\infty$,

(b) $\sigma = 1$ *a.e. with respect to a basic measure for R,*

then, for $\lambda \in S \cap V_D$, $\mathscr{E}_\lambda$ is one-dimensional and $\mathscr{E}_\lambda = \operatorname{Im} T_\lambda$.

Proof. The preceding lemma tells us that the kernel of T_λ, since $\lambda \in V_D$, is a closed hyperplane in D, which we denote by X; since $\mathscr{E}_\lambda \neq 0$, to show that $\mathscr{E}_\lambda = \operatorname{Im} T_\lambda$, it is sufficient to show that $X \subseteq \{a\colon \langle a, e\rangle = 0$, for all $e \in \mathscr{E}_\lambda\}$. Let $a \in X$; since $a, b \to \rho_{a,b}(\lambda)$ is a positive hermitian form over D so that the Cauchy-Schwartz inequality holds, we have $\rho_{a,b}(\lambda) = 0$ for all $b \in D$. Since $\lambda \in V_D$, there exists $c \in D$ such that $\rho_{c,c} > 0$ over some closed interval I containing λ in its interior. Condition (a) again gives us the fact that $\rho_{a,a}$ and $\rho_{a,c}$ are functions in $A(r)$, so that Beurling's theorem asserts that there exists $f_1 \in A(r)$ such that $f_1 = 1/\rho_{c,c}$ over I. Set $f_0 = \rho_{a,c} f_1$. Since $A(r)$ is regular, there exists $g \in A(r)$ such that $g(\lambda) = 1$ and g vanishes outside of I. Therefore, $f_0 g \rho_{c,c} = g\rho_{a,c}$ everywhere, or, in terms of the vector fields in a direct integral diagonalizing R, $\{x(\xi)\}_{\xi\in V}$, $\{y(\xi)\}_{\xi\in V}$, and $\{c(\xi)\}_{\xi \in V}$, where $x = T_{f_0 g} c$ and $y = T_g a$, we have

$$(*) \qquad [x(\xi),\ c(\xi)]_H = [y(\xi),\ c(\xi)]_H \text{ for all } \xi \in V.$$

Since $\sigma = 1$ a.e. over V_D and the vector field $c(\xi)$ does not vanish a.e. over I (*) implies that $x(\xi) = y(\xi)$ a.e. over I, and, as both vector fields vanish outside of I, we have $x(\xi) = y(\xi)$ a.e. over V with respect to a basic measure for R; from this it follows that $x = y$, or

$$(**) \qquad T_{f_0 g} c = T_g a.$$

Now, exactly as in the proof of Theorem 2, (**) and Proposition 3 yield, for arbitrary $e \in \mathscr{E}_\lambda$, the relation $f_0(\lambda) g(\lambda) \langle c, e\rangle = \langle T_{f_0 g} c, e\rangle = \langle T_g a, e\rangle = g(\lambda)\langle a, e\rangle$, and the fact that $\rho_{a,c}(\lambda) = 0$ implies that $f_0(\lambda) = 0$, so that $\langle a, e\rangle = 0$.

We have the following two corollaries analogous to the first two corollaries from Theorem 2 and whose proofs are exactly the same.

Corollary 1. *If r satisfies the conditions of Theorem* 5, *then for any $\lambda \in V_D$ such that* $\operatorname{Im} T_\lambda \subseteq \mathscr{E}_\lambda$, $\operatorname{Im} T_\lambda = \mathscr{E}_\lambda$ *and both of these spaces are one-dimensional.*

Corollary 2. *Let r satisfy the conditions of Theorem* 5 *and assume that N is a linear space dense in D and left invariant by $r(t)$, for all t. Then, given any $\lambda \in V_D$ for which $\mathscr{N}_\lambda \subseteq E'$, we must have $\mathscr{N}_\lambda =$* $\operatorname{Im} T_\lambda$ *$= \mathscr{E}_\lambda$, and each of these spaces is one-dimensional.*

We conclude this section with the following 'concrete' representation theorem for the functional calculus of r.

Theorem 6. *If r satisfies the conditions of Theorem* 5 *and the complement of S in V_D is of lebesgue measure zero, then there exist nonzero $e_\lambda, f_\lambda \in \mathscr{E}_\lambda$ for all $\lambda \in V_D \cap S$ such that for all $f \in S(r)$, $\langle T_f x, y\rangle = \int_{V_D \cap S} f \langle x, e_\lambda\rangle \overline{\langle y, f_\lambda\rangle}\, d\lambda$, for all $x, y \in E$.*

Proof. Since $\operatorname{Im} T_\lambda$ is one-dimensional for all $\lambda \in V_D$ there exists e_λ, $f_\lambda \in D'$ such that $\langle a, T_\lambda b\rangle = \langle a, e_\lambda\rangle \overline{\langle b, f_\lambda\rangle}$ for all $a, b \in D$; as $T_\lambda \neq 0$ over V_D, e_λ and f_λ also cannot vanish over V_D. Now if $a \in \ker e_\lambda$, it follows that

$$0 = \langle a, e_\lambda \rangle \overline{\langle b, f_\lambda \rangle} = \overline{\langle b, e_\lambda \rangle} \langle a, f_\lambda \rangle,$$

for all $b \in D$, since $(a,b) \to \langle a, T_\lambda b \rangle$ is hermitian; so, choosing b such that $\langle b, e_\lambda \rangle$ does not vanish, we have $a \in \ker f_\lambda$. Since $\ker f_\lambda$ and $\ker e_\lambda$ are both closed hyperplanes in D, we have $\ker e_\lambda = \ker f_\lambda$, so that e_λ and f_λ must be in the same one-dimensional subspace of D'. If, in addition, $\lambda \in S$, Theorem 5 tells us e_λ and f_λ are in $\mathscr{E}_\lambda$.

Given arbitrary $x, y \in E$, we write $\rho_{x,y}$ for the function having the property $dm_{x,y}(\lambda) = \rho_{x,y}(\lambda) d\lambda$. Proposition 10, (ii) implies that $\rho_{x,y} = 0$ a.e. over $V \cap \mathscr{C} V_D$ and this fact together with the hypothesis that $\mathscr{C} S \cap V_D$ is of lebesgue measure zero means that for all $f \in S(r)$, we may write

$$(*) \qquad \langle T_f x, y \rangle = \int f \, dm_{x,y} = \int f \rho_{x,y} d\lambda = \int_{S \cap V_D} f \rho_{x,y} d\lambda.$$

Now, we construct sequences $\{a_n\}_n$, $\{b_n\}_n$ in D converging to x and y, respectively, in E such that $\lim_n \rho_{a_n, b_n} = \rho_{x,y}$ exists in L^1 and, in addition, $\lim_n \rho_{a_n,b_n}(\lambda) = \rho_{x,y}(\lambda)$ a.e. with respect to lebesgue measure. If $\lambda \in S \cap V_D$, we have $\lim_n \rho_{a_n,b_n} = \lim_n \langle a_n, e_\lambda \rangle \overline{\langle b_n, f_\lambda \rangle} = \langle x, e_\lambda \rangle \overline{\langle y, f_\lambda \rangle}$ so that $\rho_{x,y}(\lambda) = \langle x, e_\lambda \rangle \overline{\langle y, f_\lambda \rangle}$ a.e. (lebesgue measure) in $S \cap V_D$ (**). The result now follows from (*) and (**).

4. **An application: the representation of a unitary equivalence.** Let $r, r' \in \mathscr{R}$ be subunitary, where we will distinguish the various structures associated with r and r' by a prime ($'$). Let R and R' denote the associated unitary representations of r and r' in $L(H)$ (H is independent of r and r'). We will assume there exists a unitary operator $U \in L(H)$ with adjoint $*U$ such that (i) $UR(t)^*U = R'(t)$ for all t and (ii) U maps E into E. Condition (ii) and the Closed Graph theorem imply that the restriction of U to E is a continuous operator from E into E; we can denote, without ambiguity, the adjoint of the restriction of U to E by $*U \in L(E')$. For these circumstances, we have the following representation theorem for U.

THEOREM 7. *If $r, r' \in \mathscr{R}$ and a unitary operator $U \in L(H)$ satisfy the conditions of the preceding paragraph, and, in addition, r satisfies the conditions of Theorem 6, then, there exists $e'_\lambda \in \mathscr{E}'_\lambda$, $f_\lambda \in \mathscr{E}_\lambda$ for all $\lambda \in S \cap V_D$ such that*

$$(*) \qquad \langle Ux, y \rangle = \int_{S \cap V_D} \langle x, e'_\lambda \rangle \overline{\langle y, f_\lambda \rangle} d\lambda \qquad \textit{for all } x, y \in E.$$

Proof. Since $Ux, y \in E$, for $x, y \in E$, and r satisfies the hypothesis of Theorem 6, we have

$$\langle Ux, y \rangle = \int_{S \cap V_D} \langle Ux, e_\lambda \rangle \overline{\langle y, f_\lambda \rangle} d\lambda,$$

where $e_\lambda, f_\lambda \in \mathscr{E}_\lambda$, for all $\lambda \in S \cap V_D$. Let $e'_\lambda = *Ue_\lambda$. In order to prove (*), we have only to show that $e'_\lambda \in \mathscr{E}'_\lambda$. For each $x \in E$, we have $\langle R'(t)x, e'_\lambda \rangle = \langle UR'(t)^* UUx, e'_\lambda \rangle = \langle R(t)Ux, e_\lambda \rangle = e^{i\lambda t} \langle Ux, e_\lambda \rangle = e^{i\lambda t} \langle x, e'_\lambda \rangle$, which proves $e'_\lambda \in \mathscr{E}'_\lambda$.

COROLLARY. *If in addition to the hypotheses of Theorem 7, U maps E onto E and $p'_r \leqq Mp_r$ for some constant M, then r' also satisfies the conditions o Theorem 5 so that $\mathscr{E}_\lambda$ and $\mathscr{E}'_\lambda$ are one-dimensional for all $\lambda \in S \cap V_D \cap V'_D$. Furthermore,*

$$(**) \qquad \langle Ux, y\rangle = \int_{S\cap V_D \cap V'_D} \langle x, e_\lambda{}'\rangle \overline{\langle y, f_\lambda\rangle}\, d\lambda, \qquad x, y \in E.$$

Proof. U, by the Closed Graph theorem, is a topological isomorphism of E onto E, so that $U^{-1} = {}^*U$ over E and, $S = S'$. Also,

$$|r'(t)|_{L(E)} \leqq |U|_{L(E)} |{}^*U|_{L(E)} |r(t)|_{L(E)}$$

so that $r' \in \mathscr{L}_1$ and $\int p'_r w'_r dt < +\infty$, which proves (a) of Theorem 5. Since unitary equivalence preserves in an obvious way the spectrum, the class of equivalent basic measures, and the multiplicity function, we evidently have condition (b) of Theorem 5 also satisfied for r'. Hence, $\mathscr{E}_\lambda$ and $\mathscr{E}'_\lambda$ are one-dimensional for all $\lambda \in S \cap V_D \cap V'_D$.

Let m denote a basic measure for R and R'; since the complement of $V_D \cap S$ and $V'_D \cap S$ in V_D and V'_D, respectively, is of lebesgue measure zero and, over V_D and V'_D, m is equivalent to lebesgue measure, we have, using (ii) of Proposition 10, $m(V \cap \mathscr{C}(V_D \cap S)) = m(V \cap \mathscr{C}(V'_D \cap S)) = 0$. Therefore, $m(V \cap \mathscr{C}(V_D \cap V'_D \cap S)) = 0$, so that we have $\langle Ux, y\rangle = \int dm_{Ux,y} = \int_{S\cap V_D\cap V'_D} dm_{Ux,y} = \int_{S\cap V_D \cap V'_D} \langle Ux, e_\lambda\rangle \overline{\langle y, f_\lambda\rangle} d\lambda$, using Theorem 6 for r. Now, in the proof of Theorem 7, we showed that $\langle Ux, e_\lambda\rangle = \langle x, e'_\lambda\rangle$, where $e'_\lambda \in \mathscr{E}'_\lambda$ for all $\lambda \in S$, and this establishes (**).

5. **An example** [2]. We will give an example of an invariant subspace where our results show that the functional calculus may be explicitly constructed from the weak eigenvectors over this space. For the group of operators in $L^2(R_3)$ whose infinitesimal generator is $-i\,\Delta$ where Δ is the laplacian, we will show that every cyclic subspace generated by a function $x_0 \in L^2(R_3)$ with compact support has such an invariant subspace.

We consider a strongly continuous one-parameter group of unitary operators R in $L(H)$ where H is a cyclic space for R generated by $x_0 \in H$. Denote by A the usual banach algebra of fourier transforms of L^1-functions and, by W, the subalgebra of A consisting of all twice differentiable functions such that f, f', and f'' are all in A. Let G denote the interior of the spectrum V of the infinitesimal generator of R (G is also the interior of the support of the measure m_{x_0,x_0}). We impose the following condition on x_0:

$$(*) \qquad m_{x_0,x_0} \text{ is locally in } W \text{ over } G \text{ and } m_{x_0,x_0}(V \cap \mathscr{C}G) = 0.$$

We construct E. Define Φ to be the subspace of H consisting of all $T_f x_0$, where $f \in W$ and f has compact support in G.

(2) Added March, 23 1963.

Φ is invariant under the action of R and we have $\langle R(\ )x, x_0\rangle \in L^1$ for all $x \in \Phi$. Introduce over Φ the norm $\sup\{\int |\langle R(t)x, x_0\rangle|\,dt, |x|_H\}$ and define E to be the completion of Φ with respect to this norm. It may be shown that E may be regarded as a subset of H and the natural map of E into H is continuous. Furthermore, E is invariant under R, and if r denotes the representation obtained by restricting $R(t)$ to E, we have that $r(t)$ is an isometry for each t; using the fact that translation is continuous in L^1, it follows that r is a strongly continuous one-parameter group of operators in $L(E)$.

PROPOSITION 11. *If (*) holds, then there exists a separable banach space $D \subseteq E$, continuously and densely imbedded in E for which*

$$\sup_{|a|_D, |b|_D \leq 1} |\langle r(t)a, b\rangle|$$

is in L^1 and the union of the interiors of the supports of $m_{a,a}$ for $a \in D$ is precisely G.

This construction of D has been relegated to the Appendix.

Regarding D and E imbedded in D' via the transpose of the natural map of D and E in H, we note that our general requirements are satisfied and we have:

PROPOSITION 12. *r satisfies the hypotheses of Theorem 6 with $V_D = G$ and $S \subseteq G$ (i.e., for each $\lambda \in G$, $\mathscr{E}_\lambda \neq 0$).*

Proof. All except the fact that $S \subseteq G$ is immediate from our construction. To prove $S \subseteq G$, we note that the map

$$x \to \rho_{x,x_0}(\lambda) \quad (\rho_{x,x_0} = m_{x,x_0} = [\langle R(\ )x, x_0\rangle]\hat{\ })$$

for $\lambda \in G$ is *a* continuous nonzero linear functional over E and so is represented by a nonzero $e_\lambda \in E'$; as $\rho_{r(t)x,x_0}(\lambda) = e^{i\lambda t}\rho_{x,x_0}(\lambda)$, we have $e_\lambda \in \mathscr{E}_\lambda$.

Theorem 6 now informs us that for all $\lambda \in G$, $\mathscr{E}_\lambda$ is one-dimensional and there exists nonzero $e_\lambda, f_\lambda \in \mathscr{E}_\lambda$ such that

$$\langle R(t)x, y\rangle = \langle r(t)x, y\rangle = \int_G e^{i\lambda t}\langle x, e_\lambda\rangle \overline{\langle y, f_\lambda\rangle}\, d\lambda, \qquad x, y \in E.$$

Furthermore, E is invariant under an operator T_f in the spectral calculus for R corresponding to a function $f \in A$. We emphasize here the fact that this result and Proposition 12 are independent of D.

For the group of operators $R(t) \in L^2(R_3)$ given by the solution of the Schroedinger equation

$$\frac{1}{i}\frac{\partial}{\partial t} = -\Delta,$$

we know that a basic measure for R is equivalent to lebesgue measure over the interior of the spectrum of R, the interval $[0, \infty)$. Also,

$$\rho_{x_0, x_0}(\lambda) = \frac{1}{4\pi^2} \int_{R_3 \times R_3} \frac{\sin|\xi - \eta|\sqrt{\lambda}}{|\xi - \eta|} x_0(\xi)\overline{x_0(\eta)} d\xi d\eta \text{ a.e. } \lambda > 0$$

[9, p.132]. If x_0 has its support in a sphere of radius k, then

$$\rho_{x_0, x_0}(\lambda) = \sum_{n=0}^{\infty} \frac{(-1)^n (\lambda^{1/2})^{2n+1}}{(2n+1)!} \alpha_n \quad \text{a.e.}, \lambda > 0,$$

where $\alpha_n = (1/4\pi^2) \int_{R_3 \times R_3} |\xi - \eta|^{2n} x_0(\xi) \overline{x_0(\eta)} \, d\xi d\eta$ and $|\alpha_n| \leqq (2k)^{2n} |x_0|^2_{L'} / 4\pi^2$.

It follows that $\rho_{x_0, x_0}(\lambda)$ for $\lambda > 0$ is the restriction of a function analytic in a disk about $\lambda' > \lambda$ not containing 0; hence, ρ_{x_0, x_0} is locally in W over $G \subseteq (0, \infty)$. As the zeros of ρ_{x_0, x_0} are at most countable in $[0, \infty)$ and m_{x_0, x_0} is absolutely continuous, we have that $m_{x_0, x_0}(V \cap \mathscr{C}G) = 0$ $(V = [0, \infty))$. In short, we have shown that (*) is satisfied for R generated by $-i\nabla$ in $L^2(R_3)$ (with its maximal domain) and x_0 with compact support. A similar result holds for the laplacian in $L^2(R_n)$ for arbitrary $n \geqq 2$.

Appendix to §5.

Proof of Proposition 11. First, we observe that every $x \in \Phi$ may be written uniquely in the form $T_f x_0$, where $f \in W$ with support in G; furthermore, denoting $\rho^{1/2}_{x_0, x_0}$ by θ, we have $\theta f \in W$, which implies $|[\theta f]\hat{\ }(t)| = O(1 + t^2)^{-1}$.

Now, define Q as the set of $x \in \Phi$, such that

(i) $|x|_E \leqq 1$.

(ii) $|[\theta f]\hat{\ }(t)| \leqq 1/(1 + t^2)$, $T_f x_0 = x$.

Q is convex, so that the closure $\bar{Q}$ of Q in E is bounded, convex, and circled; consequently, there is a unique banach space D_1 such that $\bar{Q}$ is precisely the unit ball of D_1. D_1 is not, *a priori*, separable, but we can construct a smaller subspace which is separable. In order to do this, we need the following:

LEMMA. *The topology induced by E over Φ is separable.*

Proof. Since Φ is given the least upper bound of two topologies one of which is known to be separable, we need only show Φ is separable with respect to the topology induced by the norm $\int |\langle R(t)x, x_0 \rangle| \, dt = |x|_1$. A is separable, so that the subspace of all $f\,\theta^2$ where f is in W and f has support in G, is also separable; call this subspace of A, I. The map of $f\theta^2$ onto $T_f x_0$ is an isometry of Φ with the norm $|\quad|_1$ onto I; hence Φ with topology induced by the norm $|\quad|_1$ is separable. This completes the proof of the lemma.

Φ is dense in E so that by the lemma, there exists a countable set Φ_c total in E where $\Phi_c \subseteq \Phi$. We may also require that the union of the interiors of the supports of $m_{x,x}$ for $x \in \Phi_c$ be precisely G. (by adding elements of the form $T_{f_i} x_0 = x_i$, where the union of the supports of f_i is G, $x_i \in \Phi$).

Now, define D to be the closure *in* D_1 of the linear span of Φ_c; in consequence of this, D is separable, and $V_D = G$.

We now estimate $\langle R(t)a, b\rangle$ for a, b in the unit ball of D; as $Q \cap D$ is dense in the unit ball of D, we need only estimate $\langle R(t)a, b\rangle$ for a, $b \in Q$. As $a = T_f x_0$, $b = T_g x_0$, we have $|\langle R(t)a, b\rangle| = |\int e^{i\lambda t} f\bar{g}\theta^2 d\lambda| = |[f\theta]\hat{} * [\bar{g}\theta]\hat{}(t)|$ and the latter convolution is dominated by the convolution of $1/(1 + t^2)$ with itself, which is in L^1. This completes the proof.

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