THE DIMENSION THEORY OF CERTAIN CARDINAL ALGEBRAS(1)

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Introduction. The von Neumann-Murray dimension theory of operator algebras [6] has been given an abstract setting by several authors, notably by Loomis [3] and Maeda [4] in the domain of lattice theory, and by Kaplansky in his theories of AW^*-algebras [1] and Baer$*$-rings [2]. The equivalence classes of a dimension lattice possess many of the properties of a generalised cardinal algebra (GCA) [9]; the purpose of this paper is to investigate this similarity. After a preliminary result on the axiomatization of GCA's we are able to show that the equivalence classes form a GCA. This is followed by an investigation of the structure of the resulting GCA's, leading to the representation of certain of them as algebras of continuous functions. Unfortunately, some of the dimension theory arguments needed here require completeness, whereas cardinal algebras have only σ-completeness. To overcome this difficulty we make the natural assumption, namely, the countable chain condition, with satisfactory results.

We recall now some pertinent notions and results. A *cardinal algebra* (CA) is an algebraic system consisting of a set A, a binary operation $+$, and an operation $\sum$ of countable rank satisfying conditions 1.1 I–1.1 VII of [9]. The first five of these assert that A is closed under the operations, that $\sum$ is a generalization of $+$, that there is a zero element, and that the operations are unrestrictedly commutative and associative.

VI REFINEMENT POSTULATE. *If $a + b = \sum c_i$, then there are elements a_0, $a_1, \cdots$ and $b_0, b_1, \cdots$ such that $a = \sum a_i$, $b = \sum b_i$, and $a_i + b_i = c_i$ for all i.*

VII REMAINDER POSTULATE. *If $a_n = a_{n+1} + b_n$ for all n, then there is an element c such that $a_n = c + \sum_i b_{n+i}$ for all n.*

(We make the agreement that, unless explicitly stated to the contrary, any index which appears will range over the non-negative integers.) In a *generalized cardinal algebra* (GCA) the operations need not always be defined, but conditions I–VII, when provided with suitable existential assumptions, are satisfied. Although we shall usually work with CA's , GCA's are perhaps more suitable for our purposes. This is because any w^*-algebra of operators gives rise to a

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GCA which is a CA precisely when the identity projection is purely infinite. However the following fact makes possible the extension of our results to GCA's. Any GCA A has a *closure* $\bar{A}$ in the following sense: $\bar{A}$ is a CA containing A as a subalgebra, and any isomorphism of A into a CA B can be extended to an isomorphism of $\bar{A}$ into B. In a GCA there is a natural partial order: $a \leqq b$ means $a + c = b$ for some element c. The element $\sum a_i$ is the supremum in this order of the elements $a_0 + a_1 \cdots + a_n$, $n = 0, 1, \cdots$; thus the operation $\sum$ can be defined in terms of the binary operation. We say that x *absorbs* y if $x + y = x$, that x is *finite* if it absorbs only the zero element, and that x is *idem-multiple* if it absorbs itself.

A partially ordered system $(L, \leqq)$ is a *lattice* if any two elements x and y have a least upper bound $x + y$ and a greatest lower bound xy. Our lattices will usually be *σ-complete*; i.e., any sequence $x_0, x_1, \cdots$ will have a least upper bound and a greatest lower bound $\sum x_i$ and $\prod x_i$. In a lattice with a smallest element 0, a *relative orthocomplementation* is a mapping which associates with each pair of elements x and y, $x \leqq y$, an element $y - x$ such that

(1) $x + (y - x) = y$, $x(y - x) = 0$,

(2) $y - (y - x) = x$, and

(3) $x \leqq y \leqq z$ implies $(z - y) + (y - x) \leqq z - x$.

Elements x and y are *orthogonal* with respect to a given relative orthocomplementation if $x = (x \dot{+} y) - y$, or equivalently if $y = (x + y) - x$. We shall use the symbols $\dot{+}$ and $\dot{\sum}$ in place of $+$ and $\sum$ in case the summands are pairwise orthogonal.

A *Boolean space* is a Hausdorff topological space in which the compact open subsets form a base for the open sets. In such a space the compact open subsets form a GBA (generalized Boolean algebra; i.e., a distributive relatively complemented lattice with a smallest element 0), and conversely, Stone's representation theorem [7] asserts that every GBA may be so obtained. A Boolean space X will be called *σ-complete* if the associated algebra is σ-complete. An equivalent condition is the following: if $U_1, U_2, \cdots$ are compact open subsets of X, then so is the closure of their union.

1. New axioms for cardinal algebras. We shall need, and we give here, a result on the axiomatization of CA's. As immediate consequences of VI we have [9, 2.2 and 2.3]:

VIa. If $a + b = c + d$, then there are elements a_1, a_2, b_1 and b_2 such that $a = a_1 + a_2, b = b_1 + b_2, a_1 + b_1 = c$ and $a_2 + b_2 = d$.

VIb. If $a \leqq \sum b_i$, then there are elements $a_0, a_1, \cdots$ such that $a = \sum a_i$ and $a_i \leqq b_i$ for all i.

We shall show that, conversely, any algebra A satisfying 1.1I–1.1V and 1.1VII of [9], together with VIa and VIb, is a CA. The corresponding result for GCA's may be proved similarly.

For the remainder of the section, we consider an algebra A satisfying the above conditions. Observe first that the proofs in [9] apply verbatim to show that, with 2.1, 2.18, and 2.20 as possible exceptions, the theorems there up to and including 2.34 are valid for the algebra A. Three of these [9, 2.8, 2.19, and 2.34] are of particular importance to us.

THEOREM 1.1. *If $a + c = b + c$, then there is an element z such that $a + z = b + z$ and $c = c + z$.*

THEOREM 1.2. *If n is finite and $a, b_0, b_1, \cdots, b_n$ are elements of A such that*

$$a + b_0 + b_1 + \cdots + b_n = b_0 + b_1 \cdots + b_n,$$

then there are elements $a_0, a_1, \cdots, a_n$ of A with $a = a_0 + a_1 + \cdots + a_n$ and $a_i + b_i = b_i$ for $i = 0, 1, \cdots, n$.

THEOREM 1.3. *If m is finite and $ma = mb$, then $a = b$.*

The lemmas which follow are special cases of VI, from which we shall prove the general statement.

LEMMA 1.4. *If $a = \sum c_i$ is idem-multiple, then there are elements $x_0, x_1, \cdots, y_0, y_1, \cdots$ of A such that*

$$a = \sum x_i = \sum y_i \text{ and } c_i = x_i + y_i \text{ for all } i.$$

Proof. We shall use a recursive procedure to obtain decompositions
(1) $c_j = c_{j,0} + c_{j,1} + \cdots + c_{j,j}$, and
(2) $c_{j,j} = e_{j,0} + \cdots + e_{j,j} + u_j + v_j$
for all j with the following properties:
(3) $v_j = \sum_i c_{i+j+1,j}$,
(4) $\sum_i (c_{i+j+1,0} + \cdots + c_{i+j+1,j})$ absorbs u_j, and
(5) c_j absorbs $e_{j+i,j}$
for all i and j.

Set $c_{0,0} = c_0$. Since a is idem-multiple it absorbs $c_{0,0}$, and therefore by 1.2 we can write $c_{0,0} = e_{0,0} + v_0$ (let $u_0 = 0$) such that c_0 absorbs $e_{0,0}$ and $\sum c_{i+1}$ absorbs v_0. Applying VIb, there exist decompositions

$$c_{i+1} = c_{i+1,0} + d_{i+1,0}$$

such that $\sum c_{i+1,0} = v_0$. These elements satisfy (1) – (4) for $j = 0$.

Letting $c_{1,1} = d_{1,0}$, condition (1) is satisfied for $j = 1$, and we can repeat the above procedure. Observing that

$$a = c_0 + c_1 + \sum c_{i+2,0} + \sum d_{i+2,0}$$

absorbs $c_{1,1}$, we can write (according to 1.2)

$$c_{1,1} = e_{1,0} + e_{1,1} + u_1 + v_1$$

such that c_i absorbs $e_{1,i}$ for $i = 0, 1$, $\sum c_{i+2,0}$ absorbs u_1, and $\sum d_{i+2,0}$ absorbs v_1. In view of VIb we can write

$$d_{i+2,0} = c_{i+2,1} + d_{i+2,1}$$

such that $\sum c_{i+2,1} = v_1$. Conditions (1) – (4) are now satisfied for $j = 1$.

Putting $c_{2,2} = d_{2,1}$, condition (1) is satisfied for $j = 2$, and our process can again be repeated. After $n-1$ repetitions we arrive at decompositions

$$c_j = c_{j,0} + \cdots + c_{j,j}, \ j < n, \text{ and}$$

$$c_j = c_{j,0} + \cdots + c_{j,n-1} + d_{j,n-1} \text{ for } j \geqq n.$$

By letting $c_{n,n} = d_{n,n-1}$, we satisfy (1) for $j = n$. We have

$$a = c_0 + \cdots + c_n + \textstyle\sum_i (c_{i+n+1,0} + \cdots + c_{i+n+1,n-1}) + \sum_i d_{i+n+1,n-1},$$

and since a absorbs $c_{n,n}$ we can write

$$c_{n,n} = e_{n,0} + \cdots + e_{n,n} + u_n + v_n$$

such that c_i absorbs $e_{n,i}$ for $i \leqq n$, $\sum_i(c_{i+n+1,0} + \cdots + c_{i+n+1,n-1})$ absorbs u_n, and $\sum_i d_{i+n+1,n-1}$ absorbs v_n. Now VIb applies to give decompositions

$$d_{i+n+1,n-1} = c_{i+n+1,n} + d_{i+n+1,n}$$

such that $\sum_i c_{i+n+1,n} = v_n$. It is obvious that (1)–(4) are satisfied for $j = n$. Moreover, (5) is satisfied for those e's which have been defined up to this point, namely, the $e_{i+j,j}$ for which $i + j \leqq n$.

In this manner we arrive at the situation described in (1)–(5). Since c_j absorbs $e_{i+j,j}$ for all i, 1.2 implies the existence of decompositions

$$e_{i+j,j} = f_{i+j,j} + g_{i+j,j}$$

such that $c_{j,0} + \cdots + c_{j,j-1}$ absorbs $f_{i+j,j}$ and $c_{j,j}$ absorbs $g_{i+j,j}$. For $j=0$ it is understood that $c_{j,0} + \cdots + c_{j,j-1} = 0$, and we therefore let $f_{i,0} = 0$ and $g_{i,0} = e_{i,0}$ for all i. If we now let $x_j = c_{j,j}$ and

$$y_j = \sum_i g_{i+j,j} + c_{j,0} + \cdots + c_{j,j-1}$$

for all j, it follows from (1) and [9, 1.47] that $x_j + y_j = c_j$ for all j. Furthermore, conditions (1)–(5) imply that

$$\begin{aligned}
\textstyle\sum x_i &= \textstyle\sum c_{i,i} \\
&= \textstyle\sum (e_{i,0} + \cdots + e_{i,i} + u_i + v_i) \\
&= \sum\Big(f_{i,0} + \cdots + f_{i,i} + g_{i,0} + \cdots + g_{i,i} + u_i + \sum_j c_{i+j+1,i}\Big) \\
&= \textstyle\sum (g_{i,0} + \cdots + g_{i,i}) + \sum\limits_{i,j} c_{i+j+1,i} \\
&= \sum\limits_{i,j} g_{i+j,i} + \textstyle\sum (c_{i,0} + \cdots + c_{i,i-1}) \\
&= \textstyle\sum y_i .
\end{aligned}$$

Finally, since a is idem-multiple, we have

$$2a = a = \sum x_i + \sum y_i = 2\sum x_i,$$

and so $a = \sum x_i = \sum y_i$ by 1.3. This completes the proof of the lemma.

LEMMA 1.5. *If $a + b = a = \sum c_i$, then there are elements $a_0, a_1, \cdots, b_0, b_1, \cdots$ of A such that $a = \sum a_i$, $b = \sum b_i$, and $a_i + b_i = c_i$ for all i.*

Proof. Let $z = \infty\, b$. Since $a + b = a$ we get $z \leqq a$ by [9, 1.29]. Applying VIb, there are decompositions $c_i = x_i + y_i$ such that $z = \sum y_i$, and so by 1.4 (since $2z = z$) there are further decompositions $y_i = u_i + v_i$ such that $z = \sum u_i = \sum v_i$. We now have

$$a = \sum x_i + \sum y_i = \sum x_i + z = \sum (x_i + u_i).$$

Because $b \leqq \sum v_i$ we can again use VIb to write $v_i = b_i + w_i$ such that $b = \sum b_i$. Letting $a_i = x_i + u_i + w_i$ it is evident that we have $a_i + b_i = c_i$, and moreover

$$\sum a_i = \sum x_i + \sum u_i + \sum w_i = \sum x_i + \sum u_i = a,$$

with which the lemma is proved.

THEOREM 1.6. *If $(A, +, \sum)$ is an algebraic system satisfying postulates* 1.1I–1.1V *and* 1.1VII *of* [9], *as well as conditions* VIa *and* VIb, *then* $(A, +, \sum)$ *is a* CA.

Proof. We have to verify the refinement postulate VI. Let $a, b, c_0, c_1, \cdots$ be elements of A for which $a + b = \sum c_i$. Applying VIb there are decompositions $c_i = x_i + y_i$ such that $a = \sum x_i$. Then

$$a + b = a + \sum y_i,$$

and so by 1.1 there is an element z of A such that

$$b + z = \sum y_i + z \text{ and } a + z = a.$$

According to 1.5 we can write $x_i = u_i + z_i$ such that $a = \sum u_i$ and $z = \sum z_i$. If we now put $w_i = y_i + z_i$ we get $b \leqq \sum w_i$, and so by VIb there are decompositions $w_i = v_i + g_i$ such that $b = \sum v_i$.

Combining the above results, we have

$$c_i = u_i + v_i + g_i, a = \sum u_i, \text{ and } b = \sum v_i.$$

Therefore $a + b + \sum g_i = a + b$, and so

$$a + b + g_i = a + b$$

for all i. Theorem 1.2 now allows us to write $g_i = e_i + f_i$ so that a absorbs e_i and b absorbs f_i for all i, and thus a absorbs $\sum e_i$ and b absorbs $\sum f_i$ by [9, 1.47]. It follows that the elements $a_i = u_i + e_i$ and $b_i = v_i + f_i$ satisfy

$$a = \sum a_i, \quad b = \sum b_i, \text{ and } a_i + b_i = c_i$$

for all i, and therefore the theorem is proved.

2. **Cardinal lattices.** In this section we carry out the first part of our program; it is shown that in a certain class of lattices bearing an equivalence relation, the equivalence classes form in a natural way a GCA.

DEFINITION 2.1. *A relatively orthocomplemented lattice L is weakly modular if, for any elements a, b, and c of L, the condition* $a \leqq b \leqq c$ *implies* $b = a + b(c-a)$.

DEFINITION 2.2. *A lattice L is a cardinal lattice* (CL) *if it is σ-complete, relatively orthocomplemented, and weakly modular and if it bears an equivalence relation* $\sim$ *with the following properties*:

(i) *If* a, b_1, *and* b_2 *are elements of L such that* $a \sim b_1 \dotplus b_2$, *then there are elements* a_1 *and* a_2 *such that* $a = a_1 \dotplus a_2$, $a_1 \sim b_1$, *and* $a_2 \sim b_2$.

(ii) *If* $a, a_0, a_1, \cdots, b, b_0, b_1, \cdots$, *are elements of L such that* $a = \dot{\sum} a_i$, $b = \dot{\sum} b_i$, *and* $a_i \sim b_i$ *for all i, then* $a \sim b$.

(iii) *If a, b, and x are elements of L with* $a \leqq x$, $b \leqq x$, *and* $a(x-b) = 0$, *then there is an element c of L such that* $c \leqq b$ *and* $c \sim a$.

Speaking roughly, these are the countable versions of the axioms given by Loomis and Maeda for a dimension lattice. We admit, however, the possibility that there exist nonzero elements of dimension zero (i.e., nonzero elements x such that $x \sim 0$). Condition (iii) is a weakened form of axiom D′ in [3].

Let L be a CL, and denote by $A(L)$ the totality of equivalence classes of the elements of L. If $a_0, a_1, \cdots$ are elements of $A(L)$ possessing a system of pairwise orthogonal representatives in L, we define $\sum a_i$ to be the equivalence class of the lattice sum of these representatives. In this case 2.2(ii) obviously implies that $\sum a_i$ is independent of the particular system of pairwise orthogonal representatives employed in its definition. In the contrary case we leave $\sum$ undefined. Our aim is to show that, with respect to this operation, $A(L)$ is a GCA.

Among the lemmas which follow, several will be recognized as the counterparts of results given by Loomis and Maeda. Thus 2.4(ii), 2.6, and 2.8 are just Lemmas 4, 13, and 43 of [3]. The next two lemmas are valid in any relatively orthocomplemented weakly modular lattice.

LEMMA 2.3. DE MORGAN'S LAWS. *If* $a_i \leqq x$ *for all i in I, then*

$$x - \sum_{i \in I} a_i = \prod_{i \in I} (x - a_i),$$

in the sense that if either member exists then so does the other, and the equality holds.

LEMMA 2.4. *If* $a, b, c \in L$ *are such that* $a \leqq b \leqq c$, *then*

(i) $b - a = b(c - a)$,
(ii) $c - a = (c - b) + (b - a)$, *and*
(iii) $c - (b - a) = (c - b) + a$.

Proof. (i) Apply the weak modular law to the inequality $b-a \leqq b(c-a) \leqq b$.

(ii) We have $c-b \leqq c-a \leqq c$; apply the weak modular law and simplify with the help of (i).

(iii) Apply (ii) to the inequality $b-a \leqq b \leqq c$.

LEMMA 2.5. (i) *If $a \leqq b$ and b is orthogonal to c, then $a = (a + c)b$.*
(ii) *If $a \leqq b$ and b is orthogonal to c, then a is orthogonal to c.*
(iii) *If each of the elements $a_0, a_1, \cdots$ is orthogonal to c, then so is $\sum a_i$.*
(iv) *If a and b are orthogonal to c, then $(a + c)b = ab$.*

Proof. (i) Applying 2.4 (ii) to $b-a \leqq b \leqq b+c$ we get

$$(b + c) - (b - a) = ((b + c) - b) + (b - (b - a)) = c + a\,.$$

Therefore by 2.4 (i)

$$b(c + a) = b((b + c) - (b - a)) = b - (b - a) = a\,.$$

(ii) Since $c \leqq a + c \leqq b + c$, 2.4 (i) implies

$$(a + c) - c = (a + c)((b + c) - c) = (a + c)b = a\,,$$

the last equality from (i).

(iii) Since $a_i = (a_i + c) - c \leqq (\sum a_i + c) - c$ for all i, we have

$$\sum a_i \leqq (\sum a_i + c) - c\,,$$

and therefore by (i)

$$\sum a_i = (\sum a_i + c)((\sum a_i + c) - c) = (\sum a_i + c) - c.$$

(iv) According to (iii), $a+b$ is orthogonal to c, and therefore $a = (a + c)(a+b)$ by (i). Multiplying by b gives the desired equality.

We now assume that L is a CL and begin the investigation of the properties of its equivalence relation. We shall write $a \precsim b$ in case there is $c \leqq b$ such that $a \sim c$. The first result is Lemma 13 of [3], the proof applying without change.

LEMMA 2.6. SCHROEDER-BERNSTEIN THEOREM. *If $a \precsim b$ and $b \precsim a$, then $a \sim b$.*

LEMMA 2.7. *For any elements a, b, and x such that $a, b \leqq x$ we have $a - a(x - b) \sim b - b(x - a)$.*

Proof. Let $u = a - a(x - b)$ and $v = b - b(x - a)$. Then $u, v \leqq x$, so that if we show $u(x - v) = 0$, 2.2(iii) will imply $u \precsim v$. We have

$$
\begin{aligned}
u(x - v) &= u(x - (b - b(x - a))) \\
&= u((x - b) + b(x - a)) && \text{by 2.4(iii)} \\
&= u(x - b) && \text{by 2.5(iv)} \\
&= (a - a(x - b))(x - b) \\
&= (a - a(x - b))\,a\,(x - b) \\
&= 0\,.
\end{aligned}
$$

The argument is symmetric in u and v, and so an application of 2.6 completes the proof.

THEOREM 2.8. THE FINITE REFINEMENT PROPERTY. *If* $a \dotplus b \sim c \dotplus d$, *then there are decompositions* $a = a_1 \dotplus a_2$ *and* $b = b_1 \dotplus b_2$ *such that* $a_1 + b_1 \sim c$ *and* $a_2 + b_2 \sim d$.

Proof. Apply 2.2 (i) to write $a \dotplus b = u \dotplus v$ such that $u \sim c$ and $v \sim d$. Letting $x = a + b$ we have

$$a = (a - av) \dotplus av = (a - a(x - u)) \dotplus av\,, \text{ and}$$

$$b = bu \dotplus (b - bu) = bu \dotplus (b - b(x - v))\,.$$

Moreover we get from 2.7 that

$$(a - a(x - u)) + bu \sim (u - u(x - a)) + bu = u \sim c$$

and similarly that

$$av + (b - b(x - v)) \sim av + (v - v(x - b)) = v \sim d\,.$$

We note that the interpretation of the above theorem in $A(L)$ is just condition VIa of 1.6. The next theorem corresponds similarly to VIb, and the theorem following to [9, 1.1 VII].

THEOREM 2.9. *If* $a \leqq \dot{\sum} b_i$, *then there is a decomposition* $a = \dot{\sum} a_i$ *such that* $a_i \precsim b_i$ *for all* i.

Proof. Let $c_n = \sum_i b_{i+n}$ and $a_n = ac_n - ac_{n+1}$ for all n, and let $u = \sum a_i$ and $v = a - u$. If $i < j$ we have $a_j \leqq ac_{i+1}$, and because ac_{i+1} and a_i are orthogonal so are a_j and a_i by 2.5 (ii). Thus the a_i are pairwise orthogonal. Furthermore $v = 0$. For again by 2.5 (ii), v is orthogonal to

$$a_0 + a_i + \cdots + a_n = a - ac_{n+1}$$

for all n, so that

$$v = va = v((a - ac_{n+1}) + ac_{n+1}) = vac_{n+1}$$

by 2.5(iv), and therefore $v \leqq c_{n+1}$ for all n. Thus v and b_n are orthogonal for all n, and so are v and a by 2.5(iii) and (ii). Hence $v = 0$ and $a = \sum a_i$.

It remains to show that $a_i \precsim b_i$ for all i. Since b_i and b_j are orthogonal for $i \neq j$, an application of 2.5 (iii) gives

$$(\sum b_j) - b_i = b_0 + \cdots + b_{i-1} + c_{i+1} .$$

Moreover, because $a_i \leqq c_i$ we see that a_i and b_j are orthogonal for $j < i$, and that therefore by 2.5(iv)

$$a_i((\sum b_j) - b_i) = a_i(b_0 + \cdots + b_{i-1} + c_{i+1}) = a_i c_{i+1} = 0 .$$

Hence $a_i \precsim b_i$ by 2.2(iii), and the theorem is proved.

THEOREM 2.10. THE REMAINDER PROPERTY. *Let $a_0, a_1, \cdots, b_0, b_1, \cdots$ be elements of L such that $a_n \sim a_{n+1} \dotplus b_n$ for all n. Then there are pairwise orthogonal elements $x, c_0, c_1, \cdots$, of L such that, for all n, $c_n \sim b_n$ and*

$$a_n \sim \dot{\sum_i} c_{i+n} \dotplus x.$$

Proof. Since $a_0 \sim a_1 \dotplus b_0$ we can use 2.2 (i) to write $a_0 = x_1 \dotplus c_0$ such that $x_1 \sim a_1$ and $c_0 \sim b_0$. Assuming the elements x_{i+1} and c_i have been defined for $i < n$ such that $x_{i+1} \sim a_{i+1}$, $c_i \sim b_i$, and $a_0 = x_n \dotplus c_0 \dotplus \cdots \dotplus c_{n-1}$, we can define c_n and x_{n+1} as follows. Because $x_n \sim a_n = a_{n+1} \dotplus b_n$, we can write $x_n = x_{n+1} \dotplus c_n$ with $x_{n+1} \sim a_{n+1}$ and $c_n \sim b_n$. But now $a_0 = x_{n+1} \dotplus c_0 \dotplus \cdots \dotplus c_n$ and the procedure can be repeated.

Having obtained the othogonal subelements $c_0, c_1, \cdots$ of a_0 in this manner, we let $x = a_0 - \sum c_i$. Then $a_0 = x \dotplus \dot{\sum} c_i$, and it follows by induction on n that $x_n = x \dotplus \dot{\sum} c_{i+n}$ for all n. Since $a_n \sim x_n$ for all n, the proof is complete.

THEOREM 2.11. *Let L be a* CL, *let $A(L)$ be the family of all equivalence classes of L, and let the operations + and $\sum$ be defined in $A(L)$ as discussed above. Then $(A(L), +, \sum)$ is a* GCA; *it is a* CA *if and only if the following condition is satisfied*:

Every element a of L is contained in an element b of L for which there exists a decomposition $b = b_1 \dotplus b_2$ such that $b \sim b_1 \sim b_2$.

Proof. Conditions 1.1 I-1.1V of [9] are obviously satisfied, and VIa, VIb, and VII are just 2.8, 2.9, and 2.10. Thus the first statement of the theorem is a consequence of 1.6. Turning to the second statement, by [9, 5.25] $A(L)$ is a CA if and only if every sequence $a_0, a_1, \cdots$ in $A(L)$ has an idem-multiple upper bound. But this condition on $A(L)$ is easily seen to be equivalent to the above condition on L.

3. **Structure of cardinal algebras.** We shall confine our attention to CA's $(A, +, \sum)$ with the following two properties:

3.1(i) Any elements $a, b \in A$ have a greatest lower bound (glb) $a \cap b \in A$.

(ii) For any $a \in A$ there are finite elements $a_0, a_1, \cdots$ of A such that $a = \sum a_i$.

The role of the first assumption is not easy to assess. It is conceivable that it is redundant, if for example any CA satisfying 3.1 (ii) can be embedded in a CA satisfying both 3.1 (i) and (ii). The second assumption seems to be required, inasmuch as we determine the relative sizes of two elements by comparing their multiples, and the operation $\sum$ is of countable rank. It also excludes type III algebras, which for similar reasons is perhaps desirable.

LEMMA 3.2. *If $z, a_0, a_1, \cdots$ are elements of A and z is idem-multiple, then*

$$(\sum a_i) \cap z = \sum (a_i \cap z).$$

Proof. The forward inclusion is [9, 3.10], and the other is immediate using the fact that z is idem-multiple.

LEMMA 3.3. *Let z be idem-multiple. Then any element a has a decomposition $a = b + c$ such that $b \leqq z$ and $c \cap z = 0$.*

Proof. Consider first a finite element a. Let b be the glb of a and z, and let $b + c = a$. If d is the glb of c and z we have

$$b + d \leqq b + c = a, \quad b + d \leqq 2z = z,$$

and therefore $b + d = b$. Now $d = 0$ because b is finite.

For the general case we let $a = \sum a_i$ such that the a_i are finite. Write $a_i = b_i + c_i$ so that $b_i \leqq z$ and the glb of c_i and z is 0. Letting $b = \sum b_i$ and $c = \sum c_i$, we have $a = b + c$, $b \leqq z$, and $c \cap z = 0$, the last by 3.2.

THEOREM 3.4. *Let Z be the set of all idem-multiple elements of A. Then Z is a σ-complete* GBA *with respect to the natural partial ordering.*

Proof. Relative complements exist by 3.3, and Z is a σ-complete distributive lattice by [9, 3.24, 3.27, 3.28, 3.32, and 3.34].

DEFINITION 3.5. *Let z be an idem-multiple element of A. Then for any elements a and b of A we say that*

(i) *$a = b$ on z if $a \cap z = b \cap z$.*

(ii) *$a \leqq b$ on z if $a \cap z \leqq b \cap z$, and*

(iii) *$a < b$ on z if $a \leqq b$ on z, and if, whenever z_1 is an idem-multiple subelement of z such that $a = b$ on z_1, then $z_1 = 0$.*

We shall say that elements a and b of A are disjoint in case their glb is 0.

THEOREM 3.6. COMPARABILITY. *Let a and b be any elements of A, and let $z = \infty (a + b)$. Then there is a decomposition of z into pairwise disjoint idem-multiple elements z_1, z_2, and z_3 such that $a < b$ on z_1, $a = b$ on z_2, and $b < a$ on z_3.*

Proof. Suppose first that a is finite. Let c be the glb of a and b, let $a = x + c$, and let $b = y + c$. If u is the glb of x and y we have $u + c \leqq a$ and $u + c \leqq b$, hence $u + c = c$, and therefore $u = 0$ since c is finite. If we now let $z_1 = \infty y$ and $z_3 = \infty x$, it follows from 3.2 that z_1 and z_3 are disjoint. Let z_2 be the complement of $z_1 + z_3$ in z. On z_1 we have

$$a = x + c = c \leqq y + c = b,$$

on z_3 we similarly have $b \leqq a$, and on z_2 we have simply $a = c = b$. Suppose that $a = b$ on an idem-multiple subelement z' of z_1. Then on z' we have

$$a = b = y + c = y + a,$$

the last equality since $a = c$ on z_1. Because a is finite this implies that $y = 0$ on z', and since $z' \leqq z_1 = \infty y$ we get $z' = 0$ by 3.2. Therefore $a < b$ on z_1, and in the same way $b < a$ on z_3.

The general case is an immediate consequence of the following lemma.

LEMMA 3.7. *Any element a of A can be written $a = w + b$ where w and b are disjoint, w is idem-multiple, and b is finite.*

Proof. Let $a_0, a_1, \cdots$ be finite elements such that $a = \sum a_i$, and let $z = \infty a$. By the special case of 3.6 which has just been established, there is for each n a decomposition $z = z_n + z_n'$ into disjoint idem-multiple elements such that

$$a < 2(a_0 + \cdots + a_n) \text{ on } z_n, \text{ and } 2(a_0 + \cdots + a_n) \leqq a \text{ on } z_n'.$$

Then $z_0 \leqq z_1 \leqq \cdots$, and we let u be their supremum and v the complement of u in z. Then $v \leqq z_n'$ for all n, hence

$$2(a_0 + \cdots + a_n) \leqq a \text{ on } v$$

for all n, and therefore a is idem-multiple on v. Moreover, a is finite on z_n for all n, and so a is finite on u by [9, 4.18]. Thus the elements $w = a \cap v$ and $b = a \cap u$ meet our requirements.

Before proceeding, we wish to introduce the function algebras in which we shall be interested. Let R be the GCA of non-negative real numbers, let $\bar{R}$ denote the closure of R (formed by adjoining an infinity), let X be a σ-complete Boolean space, and let $C(X, R)$ (resp. $C(X, \bar{R})$) be the totality of continuous functions from X into R (resp. $\bar{R}$) and having compact support (the support of f being the closure of $\{f > 0\}$). It is relatively easy to see that with respect to the binary operation $+$ of coordinate-wise addition $(C(X, R), +)$ is a semigroup which is a conditionally σ-complete lattice in the natural order; as is shown in [8] the lub of a bounded sequence of functions is that uniquely determined continuous function which differs from the pointwise supremum on a set of first category. Therefore $(C(X, R), +)$ is a GCA by [9, 13.28], and one readily verifies that $C(X, \bar{R})$ is its closure. If N is the GCA of non-negative integers, the corresponding assertions hold for $C(X, N)$ and $C(X, \bar{N})$.

DEFINITION 3.8. *An element $a \in A$ is called*

(i) *multiple-free if $a \neq 0$, and, for any $b \in A$, $2b \leqq a$ implies $b = 0$;*

(ii) *type* I *if every nonzero $b \leqq a$ contains a multiple-free element;*

(iii) *type* II *if whenever $0 \neq b \leqq a$ there exists $c \in A$ such that $c \neq 0$ and $2c \leqq b$.*

An algebra will be called type I (II) in case all its elements are type I (II). Note that an element is type II if and only if it contains no multiple-free element.

Now let $Z(A) = Z$ be the GBA of idem-multiple elements of A, let $X(A) = X$ be the Stone space of Z, and for each $x \in X$ let $D(x)$ be the corresponding maximal dual ideal of Z. If $a \in A$ we denote by $E(a)$ the compact open subset of X corresponding to the element ∞a of Z. Since A need not possess an order unit, we select instead a maximal family H of pairwise disjoint finite nonzero elements of A. If U is the union of the sets $E(h)$ for all $h \in H$, then U is an open dense subset of X, and we associate with each $a \in A$ a continuous function $F(a)$ from U to $\bar{R}$ as follows:

DEFINITION 3.9. *If $a \in A$, $h \in H$, and $x \in E(h)$, then*

$$F(a)(x) = \inf \{m/n \mid na \leqq mh \text{ on some } z \in D(x)\}.$$

The comparison theorem 3.6 enables us to prove the following dual of 3.9, of which the continuity of $F(a)$ and the fact that its support is $E(a) \cap U$ are easy consequences.

LEMMA 3.10. *If $h \in H$ and $x \in E(h)$, then*

$$F(a)(x) = \sup \{m/n \mid mh \leqq na \text{ on some } z \in D(x)\}.$$

In order that there exist continuous extensions of the functions $F(a)$ to all of X, it is necessary to restrict further the algebra A. Unfortunately such an assumption must say something about the completeness of A, and is therefore rather foreign to the spirit of cardinal algebra. Various conditions are possible; we could for example assume that Z is complete, from which it follows that the desired extensions exist and that F embeds A isomorphically in $C(X(A), \bar{R})$. We begin by assuming that A has an "order unit"; that is, a finite element h such that $a \leqq \infty h$ for all $a \in A$. Then, taking $H = \{h\}$, $F(a)$ is defined and continuous on all of X, has support $E(a)$, and thus is an element of $C(X, \bar{R})$. We now show that F is an isomorphism.

The first step is an easy verification of the equations $F(a+b) = F(a) + F(b)$ and $F(a \cap z) = F(a)\chi_z$ for all $a, b \in A$ and $z \in Z$, where χ_z is the characteristic function of $E(z)$. We next note that $a \leqq b$ is equivalent *to* $F(a)(x) \leqq F(b)(x)$ for all $x \in X$. If $a \leqq b$ then $a + c = b$ for some $c \in A$, so that $F(a) + F(c) = F(b)$ and $F(a) \leqq F(b)$. For the other implication we need the fact that $F(c) = 0$ implies $c = 0$. The comparison theorem 3.6 implies that $nc \leqq h$ for all positive integers n, hence $\infty c \leqq h$, and so $c = 0$ because h is finite. Suppose now that $a \nleqq b$.

By 3.6 there exists $z \in Z$, $z \neq 0$, on which $b < a$. If c is an element so that $b + c = a$ on z, we have $c \neq 0$ on z and $F(a) = F(b) + F(c)$ on $E(z)$. Moreover $F(c) > 0$ on $E(z)$, and therefore $F(a) \lneqq F(b)$. We note here two consequences of what has just been shown: F is one-to-one, and the ordering induced in the image of A by the binary addition coincides with the pointwise ordering.

Turning now to the infinite operations, we shall show that if $a \in A$ is the supremum of elements $a_0, a_1, \cdots$ of A, then $F(a)$ is the supremum in $C(X,\bar{R})$ of the $F(a_i)$. Before doing this we make one observation. Let $b \in A$, let r be any non-negative rational number, and let $z, z' \in Z$ be complementary in ∞h such that $b \leqq rh$ on z and $b > rh$ on z'. Then

$$E(z) = \operatorname{int} \{x \mid F(b)(x) \leqq r\}.$$

Certainly $E(z) \subseteq \{F(b) \leqq r\}$, and because $E(z)$ is open we even have $E(z) \subseteq \operatorname{int} \{F(b) \leqq r\}$. On the other hand, if $E(u) \subseteq \{F(b) \leqq r\}$, $u \in Z$, then $b \leqq sh$ on u for all rational $s > r$, hence $b \leqq rh$ on u by 3.6, therefore $u \leqq z$, and so $E(u) \subseteq E(z)$. Returning now to the proof, we have $F(a_i) \leqq F(a)$ for all i by what has already been shown. Suppose that $f \in C(X, \bar{R})$ and $F(a_i) \leqq f$ for all i. Then $\{f \leqq s\} \subseteq \{F(a_i) \leqq s\}$ for all $s \in \bar{R}$ and all i, and therefore

$$\operatorname{int} \{f \leqq s\} \subseteq \operatorname{int} \bigcap_i \operatorname{int} \{F(a_i) \leqq s\}.$$

Let r be a non-negative rational, and for each i let $z_i, z_i' \in Z$ be complementary in ∞h such that $a_i \leqq rh$ on z_i and $a_i > rh$ on z_i'. Also let z and z' be complementary on ∞h such that $a \leqq rh$ on z and $a > rh$ on z'. Then $z = \bigcap z_i$. Because $a \leqq rh$ on z we have $a_i \leqq rh$ on z for all i, and therefore $z \leqq z_i$ for all i. On the other hand, $a_i \leqq rh$ on $\bigcap z_i$ for all i implies $a \leqq rh$ on $\bigcap z_i$, and so $\bigcap z_i \leqq z$. We now have

$$\begin{aligned} \operatorname{int} \{F(a) \leqq r\} &= E(z) \\ &= \operatorname{int} \bigcap_i E(z_i) \\ &= \operatorname{int} \bigcap_i \operatorname{int} \{F(a_i) \leqq r\} \\ &\supseteq \operatorname{int} \{f \leqq r\}. \end{aligned}$$

From this it follows that $F(a) \leqq f$, and therefore $F(a)$ is the supremum in $C(X, \bar{R})$ of the $F(a_i)$.

To see that F preserves infinite sums, we need only recall that any infinite sum is by [9, 3.19] the supremum of its partial sums. This completes the proof of

THEOREM 3.11. *Let A be a* CA *satisfying* 3.1 (i), *and suppose there exists a finite element h of A such that $a \leqq \infty h$ for all $a \in A$. Then A is isomorphic to a cardinal subalgebra of* $C(X(A), \bar{R})$.

If A is type I and the element h can be chosen to be multiple-free, it may even be shown that $A \cong C(X(A), \bar{N})$. Again, if h can be chosen so that the equation $h = nx$ has a solution in A for all $n < \infty$, then the methods of 3.12 may be used to show that $A \cong C(X(A), \bar{R})$. Another observation is the following: any CA satisfying 3.1 (i) and (ii) is isomorphic to a cardinal subalgebra of a complete direct sum of CA's of the type $C(X, \bar{R})$, where X is a σ-complete Boolean space.

It should be noted that the hypotheses of 3.11 do not provide enough completeness to split the algebra into its type I and type II parts, or to identify the range of F. Perhaps the most natural assumption in our context is the countable chain condition.

THEOREM 3.12. *Let A be a* CA *satisfying* 3.1 (i) *and* (ii), *and suppose that any family of pairwise disjoint nonzero elements of A is at most countable. Then $X(A)$ is the union of disjoint compact open subsets X_1 and X_2 such that $A \cong C(X_1, \bar{N}) \oplus C(X_2, \bar{R})$.*

Proof. Let H_1 be a maximal family of pairwise disjoint multiple-free elements of A, and H_2 a maximal family of finite nonzero pairwise disjoint type II elements of A. Then the elements $h_1 = \bigcup \{h | h \in H_1\}$ and $h_2 = \bigcup \{h | h \in H_2\}$ exist by the countable chain condition, and are multiple-free and finite type II respectively. Thus $z_1 = \infty h_1$ is type I and $z_2 = \infty h_2$ is type II. Moreover, $X(A)$ is the union of the disjoint subsets $E(z_1)$ and $E(z_2)$, for $z_1 \cap z_2 = 0$ by 3.2, and $a \leqq z_1 + z_2$ for all $a \in A$ by 3.3 and the maximality of H_1 and H_2. Letting $I(z_i) = \{a \mid a \in A$ and $a \leqq z_i\}$ for $i = 1, 2$, we have $A = I(z_1) \oplus I(z_2)$ since $a = a \cap z_1 + a \cap z_2$ for all $a \in A$. Now 3.11 implies that $I(z_1) \cong C(E(z_1), \bar{N})$ and that $I(z_2)$ is isomorphic to a subalgebra of $C(E(z_2), \bar{R})$. Thus it remains to show that F maps $I(z_2)$ onto $C(E(z_2), \bar{R})$.

For each $x \in E(z_2)$ let $R(x) = \{F(a)(x) \mid a \in I(z_2)\}$. It is easy to see that either $R(x) = \bar{R}$ or else $R(x) = \{rn \mid n \in \bar{N}\}$ for some nonzero $r \in \bar{R}$. Consider the function r defined on $E(z_2)$ by letting $r(x) = 0$ if $R(x) = \bar{R}$ and $r(x) = r$ if $R(x) = \{rn \mid n \in \bar{N}\}$. The set $\{r < 1/n\}$ is open for $n = 1, 2, \cdots$, and is even dense because z_2 is type II. Therefore

$$\{r = 0\} = \bigcap_n \{r < 1/n\}$$

is dense in $E(z_2)$. Consider now an element f of $C(E(z_2), \bar{R})$. Let n be a non-negative integer, and let M_n be a maximal family of pairwise disjoint elements $I(z_2)$ such that $a \in M_n$ implies

$$(1 - 1/n) f(x) \leqq F(a)(x) \leqq f(x)$$

for all $x \in E(a)$. Since $\{r = 0\}$ is dense in $E(z_2)$ it follows that $\bigcup \{E(a) | a \in M_n\}$ is also dense in $E(z_2)$. Therefore, letting $a_n = \bigcup \{a \mid a \in M \}$, we have

$$(1 - 1/n) f(x) \leqq F(a_n)(x) \leqq f(x)$$

for all $x \in E(z_2)$. If now $a = \bigcap a_n$ we evidently have $F(a) = f$, and so the proof is complete.

A consequence of this theorem which is of interest from the point of view of w^*-algebras on an inseparable Hilbert space is the following: let A be a CA satisfying 3.1 (i) and (ii), and suppose that each nonzero $a \in A$ has a nonzero subelement b such that the CA $\{x \mid x \in A \text{ and } x \leqq \infty b\}$ satisfies the countable chain condition of 3.12. Then A is isomorphic to a subdirect sum of the complete direct sum of CA's of the types $C(X, \bar{N})$ and $C(Y, \bar{R})$, where X and Y are σ-complete Boolean spaces.

To conclude this section we mention that the properties 3.1 (i) and (ii) persist under the extension of a GCA to a CA, and so the above results may be applied to GCA's.

4. **Remarks.** It is desirable to apply the results of the previous section to the GCA $A(L)$ associated with a cardinal lattice L. It is relatively easy to see that $A(L)$ wil satisfy the hypotheses of 3.12 if we assume that L satisfies the countable chain condition, and that each nonzero element of L has a finite nonzero subelement ($a \in L$ is called finite if $a \sim b \leqq a$ implies $a = b$). Thus under these conditions $A(L)$ may be embedded in $C(X, \bar{N}) \oplus C(Y, \bar{R})$ for suitable σ-complete Boolean spaces X and Y, the embedding being onto precisely when the unit element of L is purely infinite.

In closing we mention an application to the work of Maharam [5] on the representation of abstract measures. She considers a σ-complete Boolean algebra B satisfying the countable chain condition and bearing an equivalence relation $\sim$ with the following properties:

I. If $x = \dot{\sum} x_i$, $y = \dot{\sum} y_i$, and $x_i \sim y_i$ for all i, then $x \sim y$.

II. If $x \sim x'$ and $y \leqq x$, then there exists $y' \leqq x'$ such that $y \sim y'$.

III. If $x \sim y$, then there are decompositions $x = \dot{\sum} x_i$ and $y = \dot{\sum} y_i$ into finite elements such that $x_i \sim y_i$ for all i.

It follows that there is a countably additive measure μ on B assuming values in a space $C(X, \bar{R})$ such that $x \sim y$ if and only if $\mu(x) = \mu(y)$. To arrive at this result, we note that 2.2 (i) is a consequence of II with the help of III and the countable chain condition (see [5, 6.2]). The other postulates for a CL are obvious, and so the algebra $A(B)$ of equivalence classes is a GCA. Moreover B obviously has the properties mentioned in the first paragraph of this section, and so $A(B)$ may be embedded in a space $C(X, \bar{N}) \oplus C(Y, \bar{R})$, the embedding being onto precisely when the unit element of B is purely infinite.

More generally, let B be a σ-complete Boolean algebra bearing an equivalence relation $\sim$ satisfying I and

II′. If $x \sim y_1 + y_2$, then $x = x_1 \dot{+} x_2$ such that $x_1 \sim y_1$ and $x_2 \sim y_2$.

Then the algebra $A(B)$ of equivalence classes is a GCA, and so the equivalence relation is induced by a measure assuming values in a GCA.

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