# SOME RESULTS ON STABILITY IN SEMIGROUPS(1)

BY

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A semigroup $S$ is called stable if (1) $a, b \in S$ and $Sa \subseteq Sab$ imply that $Sa = Sab$ and (2) $a, b \in S$ and $aS \subseteq baS$ imply that $aS = baS$. This notion was introduced by Koch and Wallace [5], who showed that in a stable semigroup the $\mathscr{D}$ and $\mathscr{J}$ equivalences coincide. Previously, Green [4], to obtain this equality, had imposed certain minimum conditions which were of a somewhat strong nature. Our principal concern here is the connection of stability with the $\mathscr{H}$-equivalence. In presenting a number of characterizations and consequences of stability, the bicyclic semigroup plays a rather special rôle. As we shall see, the fact that this semigroup cannot be immersed in a stable semigroup has, by itself, a number of consequences.

Frequently, we shall use fairly well-known results in the theory of semigroups which can be found in [1].

The equivalences of Green [4], defined for any semigroup $S$, are as follows:

$$a\mathscr{R}b \rightleftarrows a \cup aS = b \cup bS,$$

$$a\mathscr{L}b \rightleftarrows a \cup Sa = b \cup Sb,$$

$$\mathscr{H} = \mathscr{L} \cap \mathscr{R}, \quad \mathscr{D} = \mathscr{L} \circ \mathscr{R} = \mathscr{R} \circ \mathscr{L},$$

$$a \mathscr{J} b \rightleftarrows a \cup Sa \cup aS \cup SaS = b \cup Sb \cup bS \cup SbS.$$

Since we shall frequently assume the existence of an identity element it should be mentioned that the adjunction of an identity does not disturb the above equivalences(2).

For subsets $A$ and $B$ of a semigroup $S$, we shall write $A \;.\!\cdot\; B$ for $\{x \mid Bx \subseteq A\}$. Similarly, $A \;\cdot\!.\; B = \{x \mid xB \subseteq A\}$, and $A \;..\; B = \{(x, y) \mid xBy \subseteq A\}$. The complement of $A$ in $B$ is $B \backslash A$, the empty set is $\varnothing$, and the notation $A \between B$ means $A \cap B \neq \varnothing$. In most instances our notation and terminology follow [1].

PROPOSITION 1. *Let S be a semigroup with identity and let H, L, R, D be*

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(2) While adjoining an identity does not disturb the Green equivalence, there is, nevertheless, conceivably some loss of generality since it is at least conceptually possible to stabilize a semigroup through this adjunction.

*$\mathscr{H}$, $\mathscr{L}$, $\mathscr{R}$, and $\mathscr{D}$ classes respectively such that $H = L \cap R \subseteq D$. The following are then equivalent:*

(1) *$S$ is stable.*

(2) *$sHt \between D$ implies $sH$ is an $\mathscr{H}$-class in $L$, $Ht$ is an $\mathscr{H}$-class in $R$ and $sHt$ is an $\mathscr{H}$-class in $D$ for all $s,t \in S$ and all $H$ in $S$.*

(3) *$sHt \between H$ implies $sH = H = Ht = sHt$, for all $s,t \in S$ and all $H$.*

(4) *$H \mathbin{..} H = H \mathbin{^{\cdot}.} H \times H \mathbin{.^{\cdot}} H$ for all $H$.*

**Proof.** (1)$\rightarrow$(2). First of all, $sHt \between D$ implies that $sht \in D$ for some $h \in H$, so that $h\mathscr{J}sht$. Hence, there are elements $p$ and $q$ such that $pshtq = h$. Letting $a = ps$ and $b = tq$ we have $h = ahb$. We show next that $hb\mathscr{R}h$. Note that $hbS \subseteq hS = a(hbS)$. Since $S$ is stable, $hbS = hS = ahbS$ and $hb\mathscr{R}h$. Hence, $Hb = Htq$ is an $\mathscr{H}$-class in $R$. Since $htq\mathscr{L}h$, we have $Htq = H$. It follows that $Ht \between R$ so by [1], $Ht$ is an $\mathscr{H}$-class in $R$. Similarly, $sH$ is an $\mathscr{H}$-class in $L$ and $sHt$ is an $\mathscr{H}$-class contained in $D$.

(2)$\rightarrow$(3). We immediately have $sHt = H$. Since $sH$ is an $\mathscr{H}$-class in $L$ and $\mathscr{L}$ is right regular, we have $(sHt)\mathscr{L}(Ht)$. But $Ht$ is already an $\mathscr{H}$-class in $R$. Since $Ht$ lies in both $L$ and $R$ we conclude that $H = Ht$. Similarly, $sH = H$ establishing (3).

(3)$\rightarrow$(4). The statement $H \mathbin{^{\cdot}.} H \times H \mathbin{.^{\cdot}} H \subseteq H \mathbin{..} H$ is always true. To show that $H \mathbin{..} H \subseteq H \mathbin{^{\cdot}.} H \times H \mathbin{.^{\cdot}} H$, let $(x,y) \in H \mathbin{..} H$ so that $xHy \subseteq H$. By (3), we have $xH = H = xHy = Hy$ and the result follows.

(4)$\rightarrow$(1). Suppose $aS \subseteq saS$. Then $a = sat$, so that $(s,t) \in H_a \mathbin{..} H_a$. Since $H_a \mathbin{..} H_a \subseteq H_a \mathbin{^{\cdot}.} H_a \times H_a \mathbin{.^{\cdot}} H_a$, we have $sH_a \subseteq H_a$, so that $sa \in H_a$. But then, $aS \subseteq saS \subseteq H_aS = aS$ so that $aS = saS$. Similarly, $Sa \subseteq Sat$ implies $Sa = Sat$ and the proof is complete.

COROLLARY 1.1. *With all things as in Proposition 1 we have the following:*

(1) *$SL \cap D = L$ and $RS \cap D = R$.*

(2) *$L \mathbin{.^{\cdot}} L = H \mathbin{.^{\cdot}} H$ and $R \mathbin{^{\cdot}.} R = H \mathbin{^{\cdot}.} H$.*

(3) *$SH \cap R = H$ and $HS \cap L = H$.*

(4) *If $D$ is regular and $x \in H$ then $xSx \cap D = H$.*

(5) *If $x,y$ and $xy$ lie in $D$ then $xy \in R_x \cap L_y$ so that $R_y \cap L_x$ contains an idempotent. Thus, if $D^2 \between D$ then $D$ is regular.*

(6) *If $e$ and $f$ are idempotent elements in $D$ such that $ef = fe$ then $ef = fe \in S \setminus D$ and $SefS \subseteq SDS \setminus D$.*

(7) *If $x \in D$ and there exists an integer $n = n(x)$ such that $x \in Sx^nS$ then $H_x$ is a group.*

**Proof.** (1) Suppose $ta \in D$ where $a \in L$. Then $ta\mathscr{D}a$ so that $ta\mathscr{J}a$. Hence there exist elements $p$ and $q$ such that $a = ptaq$. From the stability of $S$ we have $pta\mathscr{L}a$. Hence, $Sa = Spta \subseteq Sta \subseteq Sa$, and $Sta = Sa$ which is to say, $ta\mathscr{L}a$. Clearly, $L \subseteq SL \cap D$ so that $L = SL \cap D$. Similarly $RS \cap D = R$.

(2) If $t \in L \,{}^{\cdot}.\, L$, then $Lt \subseteq L$ so $at\mathscr{L}a$ for all $a \in L$. Since $at \in R_aS \cap D = R_a$, we see that $at\mathscr{H}a$ so that $t \in H \,{}^{\cdot}.\, H$. The other containment is clear since any $\mathscr{H}$-class contained in $L$ is a left translate of $H$. In the same way, $R \,{}^{\cdot}.\, R = H \,{}^{\cdot}.\, H$.

(3) Suppose $ta\mathscr{R}a$, where $a \in H$. Then $tR \subseteq R$ so by (2), $tH \subseteq H$, so that $ta \in H$. Clearly $H \subseteq SH \cap R$ so that we have $SH \cap R = H$. Similarly, $HS \cap L = H$.

(4) If $D$ is regular then there exist idempotents $e$ and $f$ such that $xS = eS$ and $Sx = Sf$. Clearly then $xS \cap Sx = xSx$ and the result follows from (1).

(5) From (1) it follows that $xy \in R_x \cap L_y$. The existence of an idempotent in $R_y \cap L_x$ follows from [1].

(6) This is immediate from (1) and the fact that $SDS \setminus D$ is an ideal.

(7) If $x \in Sx^nS$ then $x^n$ lies in $D$ since $SDS \setminus D$ is an ideal. In particular $x^2$ lies in $D$ and from (1) must lie in $H$. This implies that $H$ is a group [4].

In his original investigations, Schützenberger, for his definition of the group of an $\mathscr{H}$-class, dealt with a somewhat restricted type of $\mathscr{D}$-class. (This restriction was later removed.) A $\mathscr{D}$-class $D$ is said to be of *elementary type* if for any $a, b \in D$ we have

(i) $Sa = Sb$ and $aS \subseteq bS$ imply $aS = bS$,

(ii) $aS = bS$ and $Sa \subseteq Sb$ imply $Sa = Sb$.

Thus, noting conclusion (3) in Corollary 1.1 we see that *if $S$ is a stable semigroup then each $\mathscr{D}$-class is of elementary type.*

The bicyclic semigroup $\mathscr{C}(p,q)$ is the semigroup with identity $e$ generated by two elements $p$ and $q$ subject to $qp = e$. We write $\mathscr{C} = \mathscr{C}(p,q)$.

We summarize some properties of $\mathscr{C}(p,q)$. First of all, it has the following form:

$$\begin{array}{cccc} e & q & q^2 & \cdots \\ p & pq & pq^2 & \cdots \\ p^2 & p^2q & p^2q^2 & \cdots \\ \vdots & \vdots & \vdots & . \end{array}$$

For $x \in \mathscr{C}(p,q)$, $L_x$ consists of those elements in the same column with $x$, and $R_x$ consists of those elements in the same row with $x$. Further $E = \{e\} \cup \{p^nq^n\colon n = 1,2,3,\cdots\}$, and $e > pq > p^2q^2 > \cdots$. Note that $\mathscr{C}(p,q)$ is a $\mathscr{D}$-simple inverse semigroup.

(1) If $fSf < eSe \subseteq SfS$, then $e$ is the unit of a bicyclic semigroup contained in $J_e$ (see [1, p. 81]).

(2) If $S$ is a simple semigroup (with or without zero) which contains a nonzero idempotent, then $S$ is completely simple if and only if $S$ does not contain a bicyclic semigroup (Olaf Andersen, see [1, p. 81]).

(3) An element $p$ of a semigroup $S$ lies in a bicyclic semigroup if there is an idempotent $e$ and an element $q$ such that $p, q \in eSe$, $pq = e \neq qp$ [1].

The $\mathscr{L}$-, $\mathscr{R}$-, and $\mathscr{H}$-relations in $\mathscr{C}(p,q)$ are compatible with those in $S$. In fact we have the following more general proposition.

PROPOSITION 2. *Let a and b be regular elements in a semigroup S, and suppose S is embedded in a semigroup T. Then a and b are* $\mathscr{R}$- ($\mathscr{L}$-, $\mathscr{H}$-) *related in T if and only if a and b are* $\mathscr{R}$- ($\mathscr{L}$-, $\mathscr{H}$-) *related in S.*

**Proof.** Suppose $a$ and $b$ are $\mathscr{R}$-related in $S$ (notation $aR_Sb$). Then $aS = bS \subseteq bT$. So $a \in bT$, $aT \subseteq bT$. Similarly $bT \subset aT$, so $aR_Tb$. Dually, $aL_Sb$ implies $aL_Tb$.

Conversely, suppose $aR_Tb$. By regularity there exist idempotents $e$ and $f$ in $S$ such that $eS = aS$ and $fS = bS$. Hence by the first part of the proof, $eT = aT = bT = fT$. Thus $e = fe \in fS$, so $aS = eS \subset fS = bS$, and similarly $bS \subset aS$. Hence $aR_Sb$; a dual argument establishes the proposition.

Thus we see that regular elements of $S$ cannot become $\mathscr{R}$- ($\mathscr{L}$-, $\mathscr{H}$-) related in an extension of $S$ unless they already have the relation in $S$. This does not hold for $\mathscr{D}$, however; in fact, any semigroup can be embedded in a $\mathscr{D}$-simple semigroup [8]. It may be possible to characterize regularity in terms of the invariance of $\mathscr{R}$- ($\mathscr{L}$-, $\mathscr{H}$-) relations under embeddings.

COROLLARY 2.1. *Let* $\mathscr{C}$ *be a bicyclic semigroup which is embedded in a semigroup S. Then* $\mathscr{C}$ *meets an* $\mathscr{H}$*-class of S in at most one point. Thus if x is a non-idempotent in* $\mathscr{C}$, *then no power of x lies in a subgroup of S.*

**Proof.** Suppose $a, b \in \mathscr{C}$ with $a\mathscr{H}_Sb$. By the proposition we have $a\mathscr{H}_{\mathscr{C}}b$, so $a = b$. The remainder of the corollary follows readily.

COROLLARY 2.2. *The bicyclic semigroup cannot be embedded in a stable semigroup.*

**Proof.** Suppose that $\mathscr{C}(p,q) \subseteq S$. Note that $qp^2 = p \notin R_q \cap L_{p^2}$ so that $S$ is not stable by (5) of Corollary 1.1.

COROLLARY 2.3. *Let S be a* 0*-simple semigroup. Then S is completely* 0*-simple if and only if S is stable.*

**Proof.** If $S$ is completely 0-simple then it follows from the well-known structure theory for such semigroups that $S$ is stable.

On the other hand suppose that $S$ is 0-simple and stable. Let $x$, $y$ be elements of $S$ such that $xy \neq 0$. (Such elements exist since $S$ is, by definition, not the zero semigroup of order 2.) By (5) of Corollary 1.1 it follows that $S$ contains a nonzero idempotent. (We recall that $\mathscr{D} = \mathscr{J}$ since $S$ is stable.) Now if $S$ were not completely 0-simple it would have to contain a copy of the bicyclic semigroup (see [1, p. 81]). This is ruled out by the stability of $S$ through Corollary 2.2. Thus, $S$ is completely 0-simple.

In the same way one proves the following

COROLLARY 2.4. *A semisimple semigroup is completely semisimple if and only if it is stable.*

COROLLARY 2.5. *If $S$ is a stable semigroup which has a 0-simple ideal $K$ then $K$ is completely 0-simple.*

It should be noted that our Corollary 2.4 is, in point of fact, equivalent to Theorem 2.3 of Munn [7]. The minimal conditions $m_l^*$ and $m_r^*$ are together, for a semigroup with identity, equivalent to stability. This last statement follows immediately from (1) of Corollary 1.1.

We note that it may be possible to embed a nonstable semigroup in a stable semigroup, in fact a group.

EXAMPLE (CROISOT [3]). Let $S$ be a countable cancellative semigroup without idempotents, and let $L$ be the free cancellative semigroup generated by the elements $a_{ij}$ and $b_{ij}$, where $i$ and $j$ are positive integers. Let $P = S * L$ be the free product of $S$ and $L$. Denoting the elements of $S$ by $s_1, s_2, \cdots$, the relations $s_i = a_{ij}s_jb_{ij}$ are imposed. Let $S$ be so chosen that some element, say $s_1$, is not a product in $S$. Then, in $P$, it is easy to see that both $s_1 \;.\!\dot{:}\; s_1$ and $s_1 \;\dot{.}.\; s_1$ are empty while $s_1 \;..\; s_1$ contains $(a_{11}, b_{11})$. Moreover, since $\{s_1\}$ is itself an $\mathscr{H}$-class it is immediate that $P$ is not stable. It also follows that the elements of $S$ form a single $\mathscr{J}$-class. Moreover, it is easy to take $S$ so that each $\mathscr{D}$-class is degenerate.

Another example illustrating this point can be found in [1, p. 51], due to Olaf Andersen.

PROPOSITION 3. *Let $S$ be a semigroup; the following are equivalent:*

(1) *$\mathscr{C}$ is not embeddable in $S$.*

(2) *$SeS/(SeS\backslash J_e)$ is completely 0-simple for each nonminimal idempotent $e \in S$.*

(3) *$eSe \backslash H_e$ is an ideal in $eSe$ for each nonminimal idempotent $e \in S$.*

**Proof.** (1) → (2). Let $I = SeS \backslash J_e$ and suppose that $I \neq \emptyset$. Then $SeS/I$ is 0-simple. Since $\mathscr{C}$ is not embeddable in $S$ it is not embeddable in $SeS/I$. Thus, $SeS/I$ is completely 0-simple.

(2) → (3). Again letting $I = SeS \backslash J_e$, we suppose that $SeS/I$ is completely 0-simple. Which is to say, $J_e \cup \{I\}$ is a completely 0-simple semigroup with $\{I\}$ as zero element. Now from the known structure of completely 0-simple semigroups, we know that $H_e$, the maximal subgroup of $J_e \cup \{I\}$, is precisely those elements in $J_e \cup \{I\}$ for which $e$ is a two-sided identity. Which is to say, $eSe \cap J_e = H_e$. Thus, $eSe \backslash H_e$ lies in $I$ which completes the argument.

(3) → (1). Since $\mathscr{C}$ fails to satisfy (3) it cannot be embedded in $S$. The proof is complete.

We may summarize some of these results in the following

COROLLARY 3.1. *If $S$ satisfies any of the following then $S$ does not contain a* *copy of* $\mathscr{C}(p,q)$, *so that* $SeS/I(e)$ *is completely* 0-*simple for each nonminimal idempotent* $e$.

(1) *$S$ is stable.*

(2) *$S$ is periodic (i.e., each element generates a finite sub-semigroup).*

(3) *Some power of every element lies in a subgroup.*

(4) *For each* $x \in S$ *there exist integers* $n = n(x)$, $m = m(x)$, $n < m$ *such that* $x^n \in x^m S$.

(1) and (2) are rather clear. To see (3) and (4) we have only to call upon Corollary 2.1.

We note that Corollary 3.1 contains certain results of Croisot [2] and Munn [7].

We should note the following: *If $S$ is periodic then $S$ is stable.* To see this, suppose $Sa \subseteq Sab$. Then $Sa \subseteq Sab^n$ for each $n$. Thus for some idempotent $e$, a power of $b$, we have $Sa \subseteq Sab \subseteq Sab^2 \subseteq \cdots \subseteq Sae$. It is then immediate that $Sa = Sae$ so that $Sa = Sab$.

Let $A$ be a subset of a semigroup $S$ having an identity. The congruence of Teissier [9] is defined as follows:

$x \equiv y(\mathscr{E}_A) \rightleftarrows$ either $x = y$ or there exist elements $x_1, x_2, x_3, \cdots, x_n, y_1, y_2, \cdots, y_n$ of $S$ such that $x \in x_1Ay_1 \between x_2Ay_2 \between \cdots \between x_nAy_n \ni y$.

From the results of [9], it is known that in order for $A$ to be a class of some congruence it is necessary and sufficient that for any $c$ and $d$ in $S$ one has $cAd \between A \rightarrow cAd \subseteq A$. As we have seen, any $\mathscr{H}$-class in a stable semigroup has this property. Thus we have the following

PROPOSITION 4. *Let $H$ be an $\mathscr{H}$-class of a stable semigroup $S$. Then there exists a congruence on $S$ in which $H$ is a single class.*

Suppose $I = S \setminus D$, where $D$ is a $\mathscr{D}$-class in $S$, a stable semigroup with identity. If $H$ is an $\mathscr{H}$-class contained in $D$ we have the following diagram, the maps being canonical.

$$\begin{array}{ccc} S & \longrightarrow & S/I = S' \\ \downarrow & & | \\ S/\mathscr{E}_H & & S'/\mathscr{E}'_{H'} \\ & \searrow \; S/SDS \; \swarrow & \end{array}$$

A more precise connection between $\mathscr{E}_H$ and $\mathscr{H}$ is given by the following

PROPOSITION 5. *Let $S$ be a semigroup with identity and let $H$ and $H'$ be $\mathscr{H}$-classes of $S$. If $H$ and $H'$ lie in the same $\mathscr{D}$-class then $\mathscr{E}_H = \mathscr{E}_{H'}$. If either $H$ or $H'$ is nondegenerate and $S$ is stable then $\mathscr{E}_H = \mathscr{E}_{H'}$ implies $H$ and $H'$ lie in the same $\mathscr{D}$-class hence both are nondegenerate. Thus if $S$ is stable and $H$ is nondegenerate*

$$H \equiv H'(\mathscr{D}) \leftrightarrows \mathscr{E}_H = \mathscr{E}_{H'}.$$

**Proof.** Suppose then that $p \equiv q(\mathscr{E}_H)$. If $p = q$ then of course $p \equiv q(\mathscr{E}_{H'})$. If $p \neq q$ then there are elements $x_1, \cdots, x_n, y_1, \cdots, y_n$ such that

$$p \in x_1 H y_1 \between x_2 H y_2 \between \cdots \between x_n H y_n \ni q.$$

Since $H$ and $H'$ lie in the same $\mathscr{D}$-class, there exist elements $r$ and $s$ such that $H = rH's$. Thus we have

$$p \in x_1 r H' s y_1 \between x_2 r H' s y_2 \between \cdots \between x_n r H' s y_n \ni q$$

which is to say, $p \equiv q(\mathscr{E}_{H'})$. In the same way $p \equiv q\,(\mathscr{E}_{H'})$ implies $p \equiv q\,(\mathscr{E}_H)$.

Now suppose that $\mathscr{E}_H = \mathscr{E}_{H'}$, that $S$ is stable and that say, $H$ contains at least two distinct elements $a$ and $b$. Since $a \neq b$, it follows from the very definition of $\mathscr{E}_{H'}$ that $H \subseteq SH'S$. If $H'$ were degenerate we would immediately conclude that $a = b$. Thus, $H'$ is nondegenerate and so just as we had $H \subseteq SH'S$, we have $H' \subseteq SHS$. Thus, we see that $H$ and $H'$ lie in the same $\mathscr{L}$-class. Since $S$ is stable, $H$ and $H'$ lie in the same $\mathscr{D}$-class and the proof is complete.

To see that nondegeneracy is needed in the second half of the above(3), we note that if $S$ is the semigroup of order 3 having elements $e, f, z$ with $z$ as zero, $e^2 = e, f^2 = f, ef = fe = 0$, i.e., the resorbing semigroup of order 3, then $H = \{e\}$ and $H' = \{f\}$ are distinct $\mathscr{D}$-classes but $\mathscr{E}_H = \mathscr{E}_{H'}$.

In studying the Teissier congruence the "double orbits" $xHy$ arise. Thus it might have been reasonable to think that in a stable, compact, or perhaps finite semigroup, $sHy \between aHb$ implies equality. However, the following example shows this is not so. Let $S$ be the full transformation semigroup on 4 elements $\mathscr{T}_4$ (see [1]). Following the description in [1] we note that if $H$ is the $\mathscr{H}$-class

$$\begin{matrix} (1341) & (3143) \\ (3413) & (4314) \\ (4134) & (1431) \end{matrix}$$

then $(1313)H(1111) \between (1313)H(1242)$. Both contain the element (1111). However, (3333) is contained in the first but not the second.

For the notion of inverse hull we follow, as usual, [1]. This inverse semigroup is defined for any right cancellative semigroup $S$ having no idempotent $\neq 1$.

PROPOSITION 6. *The only semigroups having stable inverse hulls are left groups. If $S$ is a semigroup having the property that each of its sub-semigroups has a stable inverse hull then $S$ is a torsion group.*

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(3) Our thanks to the referee for this remark.

First of all, if a semigroup $S$ has an inverse hull then it is, at any rate, right cancellative. We need only show that $S$ is left simple, which is to say, $Sx = S$ for each $x \in S$. Now let $Sa$ be a proper subset of $S$. Consider the extended regular representation of $S$, $a \to \rho_a$, where $\rho_a$ is the inner right translation defined by $a$. Now in the inverse hull, if $i$ is the identical mapping of $S^1$ and $\gamma$ is the inverse of $\rho_a$ then

$$\gamma\rho_a = i \quad \text{but} \quad \rho_a\gamma \neq i.$$

But then, $\gamma$ and $\rho_a$ generate a copy of $\mathscr{C}(p, q)$. This is impossible by Corollary 2.2. Thus, $Sa = S$ for each $a \in S$.

Now for the second statement in the proposition let $x$ be any element of $S$. Since the inverse hull of an infinite cyclic semigroup is $\mathscr{C}(p, q)$, the semigroup generated by $x$ is finite and contains an idempotent $e$. The element $e$ is then a right identity and each $x$ has an inverse with respect to $e$. Thus $S$ is a torsion group.

SOME REMARKS. In the following, let $a, x, y$ be arbitrary elements of a semigroup $S$ and let $L$ and $R$ be $\mathscr{L}$- and $\mathscr{R}$-classes contained in the $\mathscr{J}$-class $J$. Consider the following statements:

(1) $S$ is stable.
(2) $a = xay$ implies $xa\mathscr{R}a$ and $ay\mathscr{L}a$.
(3) $a = xay$ implies $xa\mathscr{H}a$ and $ay\mathscr{H}a$.
(4) $a = xay$ implies $xa\mathscr{L}a$ and $ay\mathscr{R}a$.
(5) $SL \cap J = L$ and $RS \cap J = R$.
(6) $x, y, xy \in J$ imply $xy \in R_x \cap L_y$.
(7) $S$ contains no copy of $\mathscr{C}(p, q)$.

We have already seen in the proofs that (1) $\leftrightarrow$ (2) $\leftrightarrow$ (3) $\rightarrow$ (4) $\leftrightarrow$ (5) $\rightarrow$ (6) $\rightarrow$ (7).

*Condition* (4) *could be considered to be a weakened form of stability*; *in this form also the conclusion* $\mathscr{D} = \mathscr{J}$ *holds.* To see this, suppose $a\mathscr{J}b$ so that $S^1aS^1 = S^1bS^1$. Thus, $a = xby$ and $b = cad$ for appropriate $x$, $y$, $b$, and $d$. We then have

$$a = x(cad)y = (xc)a(dy).$$

Using (4) we have $xca\mathscr{L}a$ which in turn implies $ca\mathscr{L}a$. In the same way, $ad\mathscr{R}a$. Since $\mathscr{L}$ is right regular $cad\,\mathscr{L}ad$ and since $ad\mathscr{R}a$ we have by definition $cad\mathscr{D}a$.

Needless to say, the absence of $\mathscr{C}(p, q)$ in $S$ does not imply its stability. One can however make the following observation.

*If* (7) *holds and each* $\mathscr{J}$*-class contains an idempotent then* $S$ *is stable.*

Letting $I(e)$ denote $SeS \setminus J_e$ we note that $SeS\,/\,I(e)$ is 0-simple and, since it does not contain a copy of $\mathscr{C}(p, q)$, it is completely 0-simple. Thus, we see, in particular, that if $a$ is an element such that $J_a$ contains an idempotent $e$ but contains no bicyclic semigroup then $J_a = D_a$ and $a$ is regular.

Now suppose $a$ is a regular element so that there exist idempotents $e$ and $f$ in $D_a$ such that $e\mathscr{R}a$ and $f\mathscr{L}a$. We shall show that (2) holds. To this end let $a = xay$.

Then $a = xay = x(ea)y = xe(ay)$. Dually $a = xay = x(af)y = (xa)(fy)$. Note that $xe, ay, xa, fy$ lie in $J_a$. Since $xe, ay$, and $xeay = a$ lie in $J_a$ we see that $xeay\mathscr{L}ay$ so that $a\mathscr{L}ay$. Dually, $a\mathscr{R}xa$ so that (2) holds which implies stability.

If $L$ and $R$ are $\mathscr{L}$- and $\mathscr{R}$-classes contained in a $\mathscr{D}$-class $D$ and $L \cap R$ contains an idempotent then $LR = D$. There need not be an idempotent present in $L \cap R$ to have this equality. However, we note the following: *If $S$ is stable and $D$ is a $\mathscr{D}$-class containing $\mathscr{L}$- and $\mathscr{R}$-classes $L$ and $R$ then $LR = D$ if and only if $L \cap R$ contains an idempotent. Thus if $D$ is a semigroup it is completely simple.*

Finally we note that if $S$ is stable cancellative and simple then $S$ is a group. If $S$ is stable and cancellative without idempotents then each $\mathscr{J}$-class is trivial.

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