

A THEOREM ON MAPPINGS INTO RIEMANN SURFACES OF INFINITE GENUS⁽¹⁾

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1. Consider complex analytic mappings f of a noncompact Riemann surface R into a compact or noncompact Riemann surface S of finite or infinite genus. If the integrated Euler characteristic $E(\Omega)$ of an exhausting subregion $\Omega \subset R$ does not grow more rapidly than the characteristic function $C(\Omega)$ of f , then the defect sum is finite [2]. In the general case there can be infinitely many Picard points and nothing is known about the set of defect points.

We shall show that the defect points under every f are so scarce that

$$\int_S B(k, a) d\omega(a) = O(k).$$

Here k is the value on $\partial\Omega$ of the capacity function of Ω , B stands for the proximity function to $a \in S$ on $\partial\Omega$, and $d\omega(a)$ is the area element in a conformal metric with finite total area. The significance of this result will be discussed in Nos. 12, 16 below.

The proof, based on principal functions, will be decomposed into Lemmas 1-9.

§1. CONSTRUCTION OF THE PROXIMITY FUNCTION.

2. Let H be the class of harmonic functions and let E be a compact set on a Riemann surface F . There exists [2] a constant $q \in (0, 1)$ such that $\max_E |u| \leq q \sup_F |u|$ for all $u \in H$ on F with nonconstant $\operatorname{sgn} u|_E$.

3. Let W be an arbitrary Riemann surface and $W_0 \subset W$ a regular region with border α_0 . Take a noncompact bordered region $W_1 \subset W$ with compact border $\alpha_1 \subset W_0$ and with compact complement $W - W_1 \subset W_0$. On \bar{W}_1 let $\sigma \in H$ with $\sigma|_{\alpha_1} = 0$, $\int_{\alpha_1} d\sigma^* = 0$, and set $m = \min_{\alpha_0} \sigma$, $M = \max_{\alpha_0} \sigma$. Let q be the constant of No. 2 for $E = \alpha_1$, $F = W_0$, and write $Q = (1 - q)^{-1}$. Denote by L a normal operator, i.e., a linear operator associating with each $\phi \in H$ on α_1 a unique harmonic function $L\phi$ on \bar{W}_1 with $L\phi|_{\alpha_1} = \phi$ and such that $\min \phi \leq L\phi \leq \max \phi$ and $\int_{\alpha_1} d^*(L\phi) = 0$. The subsequent construction will be based on the following general existence theorem [2], [3]:

There is a function $p \in H$ on W , called the principal function, such that

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- (1) $p|W_1 = \sigma + Lp,$
- (2) $Qm \leq p - \sigma \leq QM,$
- (3) $Qm \leq p|W_0 \leq QM.$

The function is unique up to an additive constant.

4. To construct proximity functions on an arbitrary Riemann surface S choose an arbitrary but then fixed normal operator, e.g. L_1 [1]. Take parametric disks D_0, D_1 centered at ζ_0, ζ_1 , and a boundary neighborhood D_β of S , with closures disjoint by pairs. Let t_0 be the principal function corresponding to $W_1 = (D_0 - \zeta_0) \cup (D_1 - \zeta_1) \cup D_\beta$ and $\sigma|D_0 - \zeta_0 = -2 \log|\zeta - \zeta_0|$, $\sigma|D_1 - \zeta_1 = 2 \log|\zeta - \zeta_1|$, $\sigma|D_\beta = 0$. Normalize by $t_0 - \sigma \rightarrow 0$ as $\zeta \rightarrow \zeta_0$ and set $s_0 = \log(1 + e^{t_0})$.

For $a \neq \zeta_0$, D_a centered at a , let $t(\zeta, a)$ be the principal function that corresponds to $\sigma|D_a - a = -2 \log|\zeta - a|$, $\sigma|D_0 - \zeta_0 = 2 \log|\zeta - \zeta_0|$, $\sigma|D_\beta = 0$. For the normalization we now choose $t - \sigma \rightarrow s_0(a)$ as $\zeta \rightarrow \zeta_0$. Our proximity function is $s = s_0 + t$.

§2. SYMMETRY OF THE PROXIMITY FUNCTION.

5. Suppose for a moment that S is a bordered Riemann surface with compact border β . Let V be the class of harmonic functions v on $S - \zeta_0 - a$ with the same singularities and normalization as t :

$$\begin{aligned} v|D_a - a &= -2 \log r + h, \\ v|D_0 - \zeta_0 &= 2 \log r + k, \end{aligned}$$

where $h, k \in H$ and $k(\zeta_0) = s_0(a)$. Set $c = 4\pi h(a)$. In V single out the functions v_0, v_1 determined by

$$\frac{\partial v_0}{\partial n} | \beta = 0, \quad v_1 | \beta = \text{const.},$$

and set for real λ :

$$(4) \quad v_\lambda = (1 - \lambda)v_0 + \lambda v_1.$$

The quantities h, k, c corresponding to v_λ will be denoted by $h_\lambda, k_\lambda, c_\lambda$. For $v, v' \in V$ we write $B(v) = \int_\beta v dv^*, B(v, v') = \int_\beta v dv'^*$.

LEMMA 1. The function v_λ minimizes $B(v) - (2\lambda - 1)c$ in V :

$$(5) \quad B(v) + (2\lambda - 1)c = \lambda^2 c_1 - (1 - \lambda)^2 c_0 + D(v - v_\lambda).$$

Proof. The Dirichlet integral of $v - v_\lambda$ over S is $D(v - v_\lambda) = B(v) + B(v_\lambda) - B(v, v_\lambda) - B(v_\lambda, v)$. Let $A_i(v)$, $i = 1, 2$, be the integral $\int v dv^*$ along $\partial D_a, \partial D_0$, respectively, and similarly for $A_i(v, v')$. Then

$$\begin{aligned}
 B(v_\lambda) &= (1 - \lambda)\lambda B(v_0, v_1) \\
 &= (1 - \lambda)\lambda(B(v_0, v_1) - B(v_1, v_0)) \\
 &= (1 - \lambda)\lambda \sum_1^2 (A_i(v_0, v_1) - A_i(v_1, v_0)).
 \end{aligned}$$

Here the first summand is $A_1(h_0, -2 \log r) - A_1(h_1, -2 \log r) = c_1 - c_0$. Because of the normalization at ζ_0 , the second summand vanishes and we have

$$B(v_\lambda) = (1 - \lambda)\lambda(c_1 - c_0).$$

Similarly

$$\begin{aligned}
 B(v, v_\lambda) &= \lambda B(v, v_1) = \lambda(c_1 - c), \\
 B(v_\lambda, v) &= (1 - \lambda)B(v_0, v) = (1 - \lambda)(c - c_0).
 \end{aligned}$$

Equality (5) follows.

6. Now let S be an arbitrary noncompact Riemann surface, and Ω a regular subregion with border β_Ω . Let $v_{\lambda\Omega}$ be the function (4) constructed on Ω as above.

LEMMA 2. *The directed limit*

$$(6) \quad v_\lambda = \lim_{\Omega \rightarrow S} v_{\lambda\Omega}$$

exists and the convergence is uniform in compact subsets of $S - a - \zeta_0$.

Proof. Let $\Omega \subset \Omega'$ and indicate by primes the quantities corresponding to Ω' or $\beta_{\Omega'}$. We apply (5) to $v = v'_0$, $v_\lambda = v_0$ and obtain

$$(7) \quad B(v'_0) - c'_0 = -c_0 + D(v'_0 - v_0).$$

For $v = v'_1$, $v_\lambda = v_1$ we have

$$(8) \quad B(v'_1) + c'_1 = c_1 + D(v'_1 - v_1).$$

Similarly, $v = v_1$, $v_\lambda = v_0$ gives

$$(9) \quad B(v_1) - c_1 = -c_0 + D(v_1 - v_0).$$

From these equations and from the relations $B(v'_i) \leq B'(v'_i) = 0$, $B(v_1) = 0$, $D \geq 0$, we infer that c_1 increases while c_0 decreases with increasing Ω , and $c_1 \leq c_0$ for every Ω . A fortiori the limit $c_i = \lim c_{i\Omega}$ exists, with obvious meaning of $c_{i\Omega}$. This implies

$$(10) \quad \lim_{\Omega \rightarrow S} D_\Omega(v_{i\Omega'} - v_{i\Omega}) = 0.$$

In view of the normalization $v_{i\Omega'}(\zeta_0) - v_{i\Omega}(\zeta_0) = 0$ we conclude [1, p. 147] on the asserted convergence.

For later reference (No. 10) we let $\Omega' \rightarrow S$ while keeping Ω fixed in (8), and have

$$B_{\Omega}(v_1) + c_1 = c_{1\Omega} + D_{\Omega}(v_1 - v_{1\Omega}).$$

For $\Omega \rightarrow S$ this gives

$$(11) \quad \lim_{\Omega \rightarrow S} B_{\Omega}(v_1) = 0.$$

7. While both v_0 and v_1 were needed in the convergence proof we shall only make use of v_1 in the sequel.

LEMMA 3. *The function t constructed in No. 4 and the limiting function v_1 in (6) are identical.*

Proof. By definition, $t = L_1 t$, and by virtue of the uniqueness of principal functions it suffices to show that $v_1 = L_1 v_1$. Let

$$W_1 = (D_a - a) \cup (D_0 - \zeta_0) \cup D_{\beta}$$

and let Ω contain $S - W_1$. Denote by $L_{1\Omega}$ the L_1 -operator acting on functions on $\alpha_1 = \partial W_1$ and providing us with harmonic functions on $\bar{\Omega} \cap W_1$. Then $v_{1\Omega} = L_{1\Omega} v_{1\Omega}$ and we are to prove that $\lim L_{1\Omega} v_{1\Omega} = L_1 v_1$. On α_1 we have $v_{1\Omega} \rightarrow v_1$, and consequently

$$L_{1\Omega} v_{1\Omega} - L_{1\Omega} v_1 = L_{1\Omega}(v_{1\Omega} - v_1) \rightarrow 0,$$

that is, $\lim L_{1\Omega} v_{1\Omega} = \lim L_{1\Omega} v_1$. By definition of L_1 [1], the latter expression is indeed $L_1 v_1$.

The above reasoning for t can also be applied to show that

$$(12) \quad t_0 = \lim_{\Omega \rightarrow S} t_{0\Omega},$$

where $t_{0\Omega}$ is the t_0 -function constructed on Ω .

8. It is now easy to see that $s = s_0 + t$ is symmetric:

LEMMA 4. *For any $a, b \neq \zeta_0$,*

$$(13) \quad s(a, b) = s(b, a).$$

Proof. Again first suppose S is a bordered compact surface with border β . Let $\alpha_a, \alpha_b, \alpha_0$ be peripheries of parametric disks about a, b, ζ_0 and set $t_a = t(\zeta, a)$, $t_b = t(\zeta, b)$ with $t_a|_{\beta} = \text{const.}$, $t_b|_{\beta} = \text{const.}$ Then

$$\int_{\beta - \alpha_0 - \alpha_a - \alpha_b} t_a dt_b^* - t_b dt_a^* = 0.$$

Here $\int_{\beta} = 0$, and, in the same manner as in No. 5, we obtain $\int_{\alpha_0} = 4\pi(s_0(a) - s_0(b))$, Analogous computations give $\int_{\alpha_a} = 4\pi t_b(a)$ and $\int_{\alpha_b} = -4\pi t_a(b)$. We infer that

$$s_0(a) + t(a, b) = s_0(b) + t(b, a).$$

This is (13).

If S is noncompact the statement follows from the above and the uniform convergence of the approximating functions formed on the $\Omega \subset S$.

§3. CONFORMAL METRIC.

9. In terms of s we introduce the conformal metric with area element $d\omega = \lambda^2 dS$, where

$$(14) \quad \lambda^2 = \Delta s = \Delta s_0 = \frac{e^{t_0} |\text{grad } t_0|^2}{(1 + e^{t_0})^2}.$$

For $x \in (-\infty, \infty)$ denote by $\alpha(x, t_0)$ the level line $t_0 = x$ on $S - \zeta_0 - \zeta_1$. When x is near $-\infty$ or ∞ , then $\alpha(x, t_0)$ is compact encircling ζ_1 or ζ_0 , respectively. On a noncompact S , α is noncompact for some values x .

Given a regular region $\Omega \subset S$ containing ζ_0, ζ_1 set $\alpha_\Omega(x, t_0) = \alpha(x, t_0) \cap \Omega$.

LEMMA 5. *On a noncompact S , we have for $x \in (-\infty, \infty)$,*

$$(15) \quad \lim_{\Omega \rightarrow S} \int_{\alpha_\Omega(x, t_0)} dt_0^* \leq 4\pi \quad a.e.$$

Proof. If the statement is false, then there exists a constant $\varepsilon > 0$, a regular region Ω_0 , and a value x_0 with $\alpha_0 = \alpha_{\Omega_0}(x_0, t_0)$, such that $\int_{\alpha_0} dt_0^* > 4\pi + 3\varepsilon$, $\text{grad } t_0|_{\alpha_0} \neq 0$, and α_0 is not tangent to $\partial\Omega_0$. Moreover, there is a $\delta > 0$ with $\text{grad } t_0|_G \neq 0$ and $\int_\gamma |dt_0^*| = \langle \varepsilon \rangle$, where $G = \Omega_0 \cap \{\zeta \mid |t_0(\zeta) - x_0| < \delta\}$, $\gamma = \bar{G} \cap \bar{\Omega}_0$ consists of disjoint closed arcs γ_1, γ_2 joined by α_0 , and $\langle \varepsilon \rangle$ stands for a quantity in the interval $(-\varepsilon, \varepsilon)$.

Because of the uniform convergence $t_{0\Omega} \rightarrow t_0$ there exists a regular region $\Omega \supset \Omega_0$ such that

$$\begin{aligned} \int_{\alpha_0} dt_{0\Omega}^* &= \int_{\alpha_0} dt_0^* + \langle \varepsilon \rangle, \\ \int_\gamma |dt_{0\Omega}^*| &= \int_\gamma |dt_0^*| + \langle \varepsilon \rangle = \langle 2\varepsilon \rangle, \\ |t_{0\Omega} - t_0| |G| &< \delta. \end{aligned}$$

For any arc $\alpha \subset G$ from γ_1 to γ_2 , we have

$$\int_\alpha dt_{0\Omega}^* = \int_{\alpha_0} dt_{0\Omega}^* + \langle 2\varepsilon \rangle = \int_{\alpha_0} dt_0^* + \langle 3\varepsilon \rangle > 4\pi.$$

On the other hand, the arc $\alpha = \alpha_{\Omega_0}(x_0, t_{0\Omega})$ gives

$$\int_{\alpha} dt_{0\Omega}^* \leq \int_{\alpha_{\Omega}(x_0, t_0\Omega)} dt_{0\Omega}^* = 4\pi.$$

This contradiction proves Lemma 5.

10. We are now ready to show:

LEMMA 6. *The total area of S is*

$$(16) \quad \int_S d\omega = 4\pi.$$

Proof. Let $\alpha_1: t_0 = x_1$ and $\alpha_2: t_0 = x_2$ be level lines such that the sets $\zeta_1 \cup \{\zeta | t_0(\zeta) \leq x_1\}$ and $\zeta_0 \cup \{\zeta | t_0(\zeta) \geq x_2\}$ are compact. We know from (11) that $\int_{\partial\Omega} t_0 dt_0^* \rightarrow 0$ as $\Omega \rightarrow S$. For this reason the Dirichlet integral of t_0 over the region with $x_1 < t_0 < x_2$ is

$$\begin{aligned} D(t_0) &= \lim_{\Omega \rightarrow S} \int_{\alpha_2 - \alpha_1 + \partial\Omega} t_0 dt_0^* = 4\pi(x_2 - x_1) \\ &= \lim_{\Omega \rightarrow S} \int_{x_1}^{x_2} \int_{\alpha_{\Omega}(x, t_0)} dt_0^* dx. \end{aligned}$$

From this and from Lemma 5 we conclude that

$$\lim_{\Omega \rightarrow S} \int_{\alpha_{\Omega}(x, t_0)} dt_0^* = 4\pi \text{ a.e.}$$

The lemma follows on integrating (14).

§4. ANALYTIC MAPPINGS

11. Let R be an arbitrary noncompact Riemann surface and R_0 a parametric disk with border β_0 . Take an adjacent regular region Ω with border $\beta_0 \cup \beta_{\Omega}$, $\bar{R}_0 \cap \bar{\Omega} = \beta_0$, and a harmonic function u on $\bar{\Omega}$ such that $u|_{\beta_0} = 0$, $u|_{\beta_{\Omega}} = k$ (const.) with $\int_{\beta_0} du^* = 1$. Let β_h be the level line $u^{-1}(h)$, $h \in [0, k]$ and denote by Ω_h the region $u^{-1}((0, h))$ with border $\beta_0 \cup \beta_h$.

Suppose $\zeta = f(z)$ is a complex analytic mapping of R into a compact or noncompact Riemann surface S . For $a \in S$ let $v(h, a)$ be the number of inverse images z_j of a in Ω_h , counted with their multiplicities. To study value distribution under f we use the counting function

$$A(h, a) = 4\pi \int_0^h v(h, a) dh,$$

the proximity function

$$B(h, a) = \int_{\beta_h} s(f(z), a) du^*,$$

the characteristic function

$$C(h) = \int_{\Omega_h} (h - u(z)) d\omega(f(z)),$$

and an auxiliary function

$$D(h, a) = B(0, a) + hB'(0, a).$$

An application of Stokes' formula to the functions $s(f(z), a)$ and $h - u(z)$ over Ω_h less disks about z_j shrinking to their centers gives the first main theorem [2]:

$$(17) \quad A(k, a) + B(k, a) = C(k) + D(k, a).$$

12. Mappings that offer interest are those for which $k/C(k) \rightarrow 0$ as $\Omega \rightarrow S$. For these nondegenerate mappings, $D(k, a) = O(k)$ is negligible, and the $(A + B)$ -affinity is the same for all points a . The defect of a is

$$\delta(a) = \liminf_{\Omega \rightarrow S} \frac{B(k, a)}{C(k)} = 1 - \limsup_{\Omega \rightarrow S} \frac{A(k, a)}{C(k)}.$$

For a Picard point, $\delta(a) = 1$, while for a "totally covered" point, $\delta(a) = 0$. The main question is: Are the defect points a with $\delta(a) > 0$ "exceptional" in comparison with points with $\delta(a) = 0$? No bounds independent of f exist for the defect sum, for mappings with infinitely many Picard values are known. We shall show, however, that the set of defect points is small.

13. By way of preparation, we first prove:

LEMMA 7. Under an arbitrary f ,

$$(18) \quad \int_S B(k, a) d\omega(a) = \int_S D(k, a) d\omega(a).$$

Proof. We are at liberty to choose R_0 so that $f(\beta_0)$ does not meet $\zeta_0 \cup \zeta_1$. The counting function can be written

$$A(k, a) = 4\pi \sum_{z_j \in \Omega} v(z_j),$$

where $v = k - u$. It follows that

$$\begin{aligned} \int_S A(k, a) d\omega(a) &= 4\pi \int_S \sum_{z_j \in f^{-1}(a)} v(z_j) d\omega(a) \\ &= 4\pi \int_{\Omega} v(z) d\omega(f(z)) \\ &= 4\pi C(k) = \int_S C(k) d\omega(a). \end{aligned}$$

On integrating the first main theorem with respect to $d\omega(a)$ over S we obtain the lemma.

14. We decompose $\int D d\omega$:

$$(19) \quad \int_S B(k, a) d\omega(a) = \int_S B(0, a) d\omega(a) + k \int_S B'(0, a) d\omega(a),$$

and prove first:

LEMMA 8.

$$(20) \quad \int_S B(0, a) d\omega(a) = O(1).$$

Proof. Cover $\gamma_0 = f(\beta_0)$ by parametric disks $D_i, i = 1, \dots, N$, such that $\bigcup \bar{D}_i$ is disjoint from the parametric disks D_0, D_1 about ζ_0, ζ_1 , and that slightly smaller disks D'_i concentric with D_i already cover γ_0 . Decompose γ_0 into closed arcs $\gamma_i \subset D'_i, \bigcup \gamma_i = \gamma_0$.

We know that

$$s(\zeta, a) = s(a, \zeta) = s_0(a) + t(a, \zeta).$$

For $\zeta \in \gamma_i$ the principal function $t(a, \zeta)$ is constructed from the singularities

$$\sigma | D_i = 2 \log \left| \frac{1 - a\bar{\zeta}}{a - \zeta} \right|, \quad \sigma | D_0 = 2 \log | a - \zeta_0 |.$$

On $\partial D'_i$ and on the periphery $\partial D'_0$ of a disk D'_0 slightly smaller than $D_0, \sigma = O(1)$. For this reason $t | D_i = \sigma + O(1), t | D_0 = \sigma + O(1), t | S - D_i - D_0 = O(1)$, and consequently $\int_S t(a, \zeta) d\omega(a) = O(1)$.

Similarly $t_0 | D_0 = -2 \log | a - \zeta_0 | + O(1), t_0 | D_1 = 2 \log | a - \zeta_1 | + O(1), t_0 | S - D_0 - D_1 = O(1)$, and we infer for $s_0 = \log(1 + e^{t_0})$ that $\int_S s_0(a) d\omega(a) = O(1)$. In summary, $\int_S s(a, \zeta) d\omega(a) = O(1)$ and

$$\int_S B(0, a) d\omega(a) = \int_{\gamma_0} \int_S s(a, \zeta) d\omega(a) du^* = O(1).$$

15. We turn to the second term in the decomposition of $\int D d\omega$:

LEMMA 9.

$$(21) \quad \int_S B'(0, a) d\omega(a) = O(1).$$

Proof. Since $\zeta_0 \notin \gamma_0$, we have $\int_{\gamma_0} ds_0^* = O(1)$. The integral $\int_{\gamma_0} dt^*$ gives the winding number of γ_0 about ζ_0 and is finite. Consequently

$$\int_{\gamma_0} ds^* = \int_{\gamma_0} ds_0^* + \int_{\gamma_0} dt^* = O(1)$$

and $\int_S \int_{\gamma_0} ds^* d\omega = O(1)$. This proves the lemma.

16. We have established the result we set out to find:

THEOREM. *Under a complex analytic mapping of an arbitrary Riemann surface into another arbitrary Riemann surface the set of defect points is so small that*

$$(22) \quad \int_s B(k, a) d\omega(a) = O(k).$$

In view of $A(k, a) + B(k, a) = C(k) + O(k)$ this poses a strong restriction on the $B(k, a)$ under nondegenerate mappings $C(k)/k \rightarrow \infty$. In contrast with the general affinity relation (see [2])

$$\sum \alpha + \sum \beta + \sum \gamma \leq 2 + \eta,$$

which holds for nondegenerate functions satisfying an additional growth condition, (22) is valid for every nondegenerate mapping.

Added in proof. Lemma 6 has interest in its own right: it shows that in our metric an arbitrary Riemann surface behaves much like a sphere.

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