POTENT RINGS[1]

BY

R. E. JOHNSON

A semiprime ring may be defined as a ring in which every nonzero right ideal A is potent, that is, $A^n \neq 0$ for all $n > 0$. Evidently one can weaken the condition of semiprimeness by assuming only that some class of right ideals is potent. A natural choice for such a class is the class of all nonzero closed right ideals. A right ideal of a ring R is called closed if it has no essential extension in the lattice L_r of right ideals of R. We call ring R (right) *potent* iff every nonzero closed right ideal of R is potent.

The present paper is concerned with potent rings R for which the (right) singular ideal is zero and the lattice L_r^* of closed right ideals of R is atomic. Necessary and sufficient conditions are given (3.7) for a triangular block matrix ring over a field F to be potent. Such a potent ring is shown to have a full triangular block matrix ring as a classical quotient ring under certain conditions (3.6).

If R is a finite-dimensional potent irreducible ring, then the ideals of R in L_r^* form a chain $R = T_0 > T_1 > \cdots > T_k = 0$. This fact allows us to imbed a potent triangular block matrix ring S in R and, in turn, to imbed R in a full triangular block matrix ring M. If $\dim T_i - \dim T_{i+1} > 1$ in L_r^*, $i = 1, \cdots, k-1$, then it is shown that M is a classical quotient ring of R (4.4). This generalizes Goldie's results on prime rings.

1. **Atomic potent rings.** If R is a ring, then L_r (or $L_r(R)$) denotes the lattice of right ideals and L_2 the lattice of 2-sided ideals of R. The notation A^r is used for the right annihilator of an element or subset A of R.

If L is a lattice with 0 and I and $A, B \in L$, then B is called an *essential extension* of A iff $A \subset B$ and $A \cap C \neq 0$ whenever $B \cap C \neq 0$, $C \in L$. We call $A \in L$ *closed* iff A is the only essential extension of A and *large* iff I is an essential extension of A. A minimal element of $L - \{0\}$ is called an *atom* of L; dually, a maximal element of $L - \{I\}$ is called a *coatom* of L. We call lattice L *atomic* iff each nonzero element of L contains an atom.

The set $R_r^\Delta = \{a \in R \mid a^r \text{ large in } L_r\}$ is an ideal of ring R called the *right singular ideal.* If $R_r^\Delta = 0$, then each $A \in L_r$ has a unique maximal essential extension A^*, and the set L_r^* of closed right ideals of R is a complete complemented modular lattice. If J_r^* denotes the lattice of all annihilating right ideals of R, then it is easily

Received by the editors July 30, 1964.

(1) Supported in part by grant NSFG-24155.

seen that $J_r^* \subset L_r^*$. The lattice J_r^* is not usually a sublattice of L_r^*, although intersections are set-theoretic in both lattices. For convenience, we let $L_{r2}^* = L_r^* \cap L_2$ and $J_{r2}^* = J_r^* \cap L_2$. Corresponding left properties of a ring R are indicated by replacing each "r" by an "l".

A ring R is called right *atomic* iff $R_r^\Delta = 0$ and L_r^* is atomic. The union in L_r of all atoms of L_r^* is denoted by R_r^0. A right atomic ring R is called (right) *stable* in [2] iff $(R_r^0)^r = 0$. If every nonzero closed right ideal of a right atomic ring R is potent then R is called a (right) *potent ring*, or a P-ring. It is clear that a right atomic ring R is potent iff $A^2 \neq 0$ for every atom $A \in L_r^*$. Hence, a P-ring is also a stable ring.

If R is a right atomic ring, then atoms A and B of L_r^* are called *perspective*, $A \sim B$, iff they have a common complement in L_r^*. It may be shown that if $A \neq B$, then $A \sim B$ iff either $A \cup B$ contains a third atom C or $a^r = b^r$ for some nonzero $a \in A$ and $b \in B$ [3, p. 540]. The union in L_r^* of all atoms perspective to an atom A is an atom in the *center* C_r^* of L_r^*. It is known that C_r^* is a Boolean algebra and that the elements of C_r^* are ideals of R [3, p. 541]. The ring R is called (right) *irreducible* iff $C_r^* = \{0, R\}$. We shall call a right atomic, irreducible ring an I-*ring*. Clearly a right atomic ring R is an I-ring iff $A \sim B$ for all atoms $A, B \in L_r^*$. An I-ring which is also a P-ring will be called a PI-*ring*.

1.1. LEMMA. *If R is a P-ring and $A, B \in L_r^*$ then $A^r \subset B^r$ iff $A^r \cap B = 0$.*

Proof. If $A^r \subset B^r$ then $A^r \cap B \subset B^r \subset (A^r \cap B)^r$, $(A^r \cap B)^2 = 0$, and therefore $A^r \cap B = 0$. Conversely, if $A^r \cap B = 0$ then $A^r \subset (AB)^r \subset B^r$.

It might be worth observing that the atomicity of R is not needed in 1.1.

1.2. LEMMA. *If R is a P-ring and $A \sim B$, where A and B are atoms of L_r^*, then either $AB \neq 0$ or $BA \neq 0$.*

Proof. The lemma is obvious if $A = B$, so let us assume that $A \neq B$. Suppose that $AB = BA = 0$. Then there exists an atom $C \subset A \cup B$ such that $C \cap A = C \cap B = 0$. Since $A \cap A^r = B \cap B^r = 0$, evidently $A(a+b) \neq 0$ and $B(a+b) \neq 0$ for all nonzero $a \in A$ and $b \in B$. Hence, $AC \neq 0$ and $BC \neq 0$ in view of the fact that $C \cap (A+B) \neq 0$. Therefore, $A^r \subset C^r$ and $B^r \subset C^r$ by 1.1. We cannot have either $CA \neq 0$ or $CB \neq 0$, for then either $C^r = A^r$ or $C^r = B^r$ and either $A \subset B^r \subset A^r$ or $B \subset A^r \subset B^r$ contrary to assumption. Hence, $CA = CB = 0$ and $C(A \cup B) = 0$, contrary to the fact that $C^2 \neq 0$. We conclude that either $AB \neq 0$ or $BA \neq 0$ as desired.

Perhaps we should point out that if A and B are atoms of L_r^* such that $AB \neq 0$ then necessarily $A \sim B$. For if $ab \neq 0$ for some $a \in A$ and $b \in B$, then $(ab)^r = b^r$ [3, 6.9] and $A \sim B$ by our remarks above.

1.3. LEMMA. *If R is a PI-ring and $A, B \in L_r^*$ then either $A^r \cap B = 0$ or $A \cap B^r = 0$.*

Proof. If $A^r \cap B \neq 0$ and $A \cap B^r \neq 0$, then there exist atoms $C, D \in L_r^*$ such that $C \subset A^r \cap B$ and $D \subset A \cap B^r$. However, then $CD = DC = 0$ contrary to 1.2.

1.4. THEOREM. *If R is a PI-ring and $S = \{A^r \mid A \in L_r^*\}$, then S is a chain in J_r^*. Also, $S = \{0, R\}$ iff R is a prime ring.*

Proof. The first part follows from 1.1 and 1.3. If R is not prime, then $BC = 0$ for some nonzero B, $C \in L_2$. Since $B^r \in L_r^*$, $B^r \neq R$, evidently $A \cap C = 0$ for some atom $A \in L_r^*$. Clearly $AC = 0$ and therefore $A^r \neq 0$. This proves 1.4.

2. **Finite-dimensional rings.** A ring R is said to have finite right rank iff there exists an integer n such that every independent subset of L_r has at most n elements. If $R_r^\Delta = 0$, then R has finite right rank iff the lattice L_r^* is finite dimensional. The dimension of L_r^* is called the (right) rank, or *dimension*, of R and is denoted by $\dim R$. A prefix of "F" used in designating a ring indicates that it is assumed to be finite dimensional. The case $\dim R = 1$ is uninteresting (if $R_r^\Delta = 0$, it means that RR^{-1} is a field), so we shall always tacitly assume that $\dim R > 1$.

2.1. LEMMA. *If R is an FPI-ring and $T \in L_{r2}^* - \{R\}$, then $T = A^r$ for some atom $A \in L_r^*$.*

Proof. Clearly $BT \subset B \cap T = 0$ for some atom $B \in L_r^*$. Let atom $A \in L_r^*$ be chosen so that A^r is a minimal element of $\{B^r \mid B \in L_r^*,\ B \text{ an atom},\ B^r \supset T\}$. If $T \neq A^r$, there exists an atom $C \in L_r^*$ such that $C \cap T = 0$ and $C \subset A^r$. Since $AC = 0$, necessarily $CA \neq 0$ by 1.2 and $C^r \subset A^r$, $C^r \neq A^r$, by 1.1. Since $C^r \supset T$, this is contrary to the choice of A. Hence, $T = A^r$ as desired.

An interesting consequence of 2.1 is that $A^r = 0$ for some atom $A \in L_r^*$. Actually, this is true for any finite-dimensional stable ring by [2, 2.13].

2.2. THEOREM. *If R is an FPI-ring, then $L_{r2}^* = J_{r2}^*$ and L_{r2}^* is a finite chain $R = T_0 > T_1 > \cdots > T_k = 0$. If $A \in L_r^* - \{0\}$ and $A \subset T_j$ then $A^r \subset T_{j+1}$. Conversely, if A is an atom of L_r^* and $A^r \subset T_{j+1}$ then $A \subset T_j$.*

Proof. The first part follows directly from 2.1 and 1.4. The other parts are obvious if $j = 0$, so let us assume that $j > 0$. By 2.1, there exists an atom $B \in L_r^*$ such that $B^r = T_j$. If $A \in L_r^* - \{0\}$ and $A \subset T_j$, then $A^r \cap B = 0$ by 1.3 and $A^r \subset B^r$, $A^r \neq B^r$, by 1.1. Hence, $A^r \subset T_{j+1}$. Conversely, if A is an atom and $A^r \subset T_{j+1}$, then $A^r \subset B^r$, $A^r \neq B^r$, and $A \cap B^r \neq 0$ by 1.1. Hence, $A \subset T_j$. This proves 2.2.

2.3. COROLLARY. *If $0 \leqq j < k$, then T_j is the union of all atoms $A \in L_r^*$ such that $A^r \subset T_{j+1}$.*

If R is an FPI-ring of dimension n, then the lattice J_r^* (J_l^*) is shown in [4] to be a complemented lower (upper) semimodular lattice in which every maximal chain has length n. If J_{r2}^* consists of $R = T_0 > T_1 > \cdots > T_k = 0$ as in 2.2, then

J_{l2}^* consists of $0 = T_0^l < T_1^l < \cdots < T_k^l = R$. For each atom $A \in L_r^*$, there exists an integer j, $0 \leqq j < k$, such that $A \subset T_j$ and $A^r = T_{j+1}$ by 2.1 and 2.2. Let us select $a \in A$ such that $a^2 \neq 0$, and define $B = a^{rl}$, an atom of J_l^* containing a. Clearly $B \subset T_{j+1}^l$ and $B \cap T_j^l = 0$. Hence, $T_j^l B = 0$ and $B^l \subset T_j^l$. Since $B^l \in J_{l2}^*$ and $B^l \not\supset T_{j+1}^l$ (for $B^l \supset T_{j+1}^l$ implies $B \subset T_{j+1}$ and $a \in T_{j+1}$, contrary to the fact that $A \cap T_{j+1} = 0$), evidently $B^l = T_j^l$. Clearly B is potent, and we have proved the following result.

2.4. Lemma. *If R is an FPI-ring and $J_{l2}^* = \{T_0^l, \cdots, T_k^l\}$ as above, then for each integer j, $0 \leqq j < k$, there exists a potent atom $B \in J_l^*$ such that $B \subset T_{j+1}^l$ and $B^l = T_j^l$.*

Assume that we have selected an independent set $\{B_1, \cdots, B_p\}$ of potent atoms of J_l^* (i.e., $B_{i+1} \cap (B_1 \cup \cdots \cup B_i) = 0$, $i = 1, \cdots, p-1$) such that

$$C \subset T_{j+1}^l \text{ and } C \cap T_j^l = 0 \quad \text{where } C = B_1 \cup \cdots \cup B_p.$$

If $C \cup T_j^l \neq T_{j+1}^l$, then $C^r \cap T_j \neq T_{j+1}$ and there exists an atom $A \in L_r^*$ such that $A \subset C^r \cap T_j$ and $A \cap T_{j+1} = 0$. Let $a \in A$, $a^2 \neq 0$, and $B = a^{rl}$, an atom of J_l^*. By the proof of 2.4, B is a potent atom such that $B \subset T_{j+1}^l$ and $B \cap T_j^l = 0$. If $B \subset C \cup T_j^l$ then $B^r \supset C^r \cap T_j$ and $a^2 = 0$, contrary to assumption. Hence, $B \cap (C \cup T_j^l) = 0$. By a lattice-theoretic argument (see [4, §4]), $(B \cup C) \cap T_j^l = 0$ also. The result below now follows by induction.

2.5. Lemma. *Let R be an FPI-ring and $J_{l2}^* = \{T_0^l, \cdots, T_k^l\}$ as above. Then for each integer j, $0 \leqq j < k$, there exists an independent set $\{B_1, \cdots, B_q\}$ of potent atoms of J_l^* such that*

$$(B_1 \cup \cdots \cup B_q) \cup T_j^l = T_{j+1}^l, \quad (B_1 \cup \cdots \cup B_q) \cap T_j^l = 0.$$

If R is an FPI-ring of dimension n and $J_{r2}^* = \{T_0, \cdots, T_k\}$ as above, then $\dim T_i$ in L_r^* equals $n - \dim T_i^l$ in J_l^* by [4]. For convenience, let

$$d_i = \dim T_i^l, \qquad i = 0, \cdots, k.$$

Thus, $0 = d_0 < d_1 < \cdots < d_k = n$. We shall call

$$(d_1 - d_0, d_2 - d_1, \cdots, d_k - d_{k-1})$$

the set of *block numbers* of R. By 1.4, R is prime iff (n) is its set of block numbers.

In view of 2.5, there exists an independent set $\{B_1, \cdots, B_n\}$ of potent atoms of J_l^* such that if

$$C_j = B_{d_j+1} \cup \cdots \cup B_{d_{j+1}}, \qquad j = 0, \cdots, k-1,$$

then

$$C_j \cup T_j^l = T_{j+1}^l, \quad C_j \cap T_j^l = 0, \qquad j = 0, \cdots, k-1.$$

If we define

$$A_j = \bigcap_{i=1;i\neq j}^{n} B_i^r, \qquad j = 1, \cdots, n,$$

then $\{A_1, \cdots, A_n\}$ is an atomic basis of L_r^* contained in J_r^*, as shown in [4]. It is immediate that

$$A_j^l = \bigcup_{i=1;i\neq j}^{n} B_i, \qquad j = 1, \cdots, n.$$

If i and j are selected so that $d_j < i \leqq d_{j+1}$, then $B_i \subset T_{j+1}^l$ and $B_i \cap T_j^l = 0$, so that $B_i \not\subset T_{j+1}^l$ and $B_i \subset T_j$. Since $A_i^l \not\supset B_i$, clearly $A_i \cap T_{j+1} = 0$. On the other hand, $A_i^l \supset T_j^l$ and therefore $A_i \subset T_j$. Thus by 2.2, $A_i^r = T_{j+1}$. Since $B_p \subset T_{j+1}$ iff $p > d_{j+1}$, evidently $A_i B_p \neq 0$ iff $p \leqq d_{j+1}$. We assemble these results below.

2.6. Theorem. *Let R be an FPI-ring of dimension n with block numbers $(b_1, \cdots, b_k)$. Then there exist potent atomic bases $\{B_1, \cdots, B_n\}$ for J_l^* and $\{A_1, \cdots, A_n\}$ for L_r^* such that:*

(1) $A_i = (\bigcup_{j\neq i} B_j)^r$ *and* $B_i = (\bigcup_{j\neq i} A_j)^l$, $i = 1, \cdots, n$.

(2) $J_{r2}^* = \{A_i^r \mid i = 1, \cdots, n\}$, $J_l^* = \{B_i^l \mid i = 1, \cdots, n\}$.

(3) $A_1^r \geqq A_2^r \geqq \cdots \geqq A_n^r = 0$ *and* $0 = B_1^l \leqq B_2^l \leqq \cdots \leqq B_n^l$.

(4) $A_i^r = A_j^r$ *and* $B_i^l = B_j^l$ *iff* $d_0 + \cdots + d_p < i$ *and* $j \leqq d_0 + \cdots + d_{p+1}$ *for some* p, *where* $d_0 = 0$.

(5) $A_i B_j \neq 0$ *iff* $i > d_0 + \cdots + d_p$ *and* $d_0 + \cdots + d_p < j \leqq d_0 + \cdots + d_{p+1}$ *for some* p.

3. **Triangular-block matrix rings.** We shall give examples of FPI-rings in this section. To this end, let F be a (skew) field and F_{ij}, $i, j = 1, \cdots, n$, be additive subgroups of F such that

$$F_{ij} F_{jk} \subset F_{ik}, \qquad i, j, k = 1, \cdots, n, \tag{3.1}$$

and let

$$S = \sum_{i,j=1}^{n} F_{ij} e_{ij}, \tag{3.2}$$

where the e_{ij} are the usual $n \times n$ unit matrices. Clearly S is a subring of $(F)_n$, the ring of all $n \times n$ matrices over F.

The ring S will be called a T-*ring* (triangular-block matrix ring) in $(F)_n$ iff there exist integers $0 = d_0 < d_1 < \cdots < d_k = n$ such that

$$F_{ij} \neq 0 \text{ iff } i > d_p \text{ and } d_p < j \leqq d_{p+1},\ p = 0, \cdots, k-1.$$

Associated with S is the *full* T-*ring*

$$M = \sum_{i,j=1}^{n} F'_{ij} e_{ij}, \text{ where } F'_{ij} = F \text{ whenever } F_{ij} \neq 0, \text{ and } F'_{ij} = 0 \text{ otherwise.} \tag{3.3}$$

It is clear that M is closed under inverses. We shall call M the *full cover* of S.

The T-ring S described above can be thought of as a ring of $k \times k$ matrices whose elements are rectangular matrices over F. Thus,

$$S = (S_{rs} \mid r, s = 1, \cdots, k) \tag{3.4}$$

where S_{rs} is a set of $m_r \times m_s$ matrices ($m_i = d_i - d_{i-1}$) of the form

$$(f_{ij} \mid i = d_r + 1, \cdots, d_{r+1}, j = d_s + 1, \cdots, d_{s+1}), \quad f_{ij} \in F_{ij}.$$

The $k \times k$ matrices of S are triangular, having zeros above the main diagonal. The full cover M of S has the form (M_{rs}), where M_{rs} is the set of all $m_r \times m_s$ matrices over F if $r \geqq s$, and is zero otherwise. It is easily shown that the matrix ring S_{ii} is prime for each i.

A ring R is called a (right) *quotient ring* of ring S, and we write $S \leqq R$, iff $S \subset R$ and $aS \cap S \neq 0$ for every nonzero $a \in R$. If ring R has a unit and $S \subset R$, then R is called a (right) *classical quotient ring* of S iff every regular element $b \in S$ (i.e., $b^r = b^l = 0$) has an inverse in R and $R = \{ab^{-1} \mid a, b \in S,\ b \text{ regular}\}$. If R is a classical quotient ring of S, we write $R = SS^{-1}$. Clearly $S \leqq R$ whenever $SS^{-1} = R$. If M is a full T-ring, then $MM^{-1} = M$.

3.5. Theorem. *If S is a T-ring in $(F)_n$ given by* 3.2, *then $S \leqq (F)_n$ iff $F_{11}F_{11}^{-1} = F$.*

Proof. If $B = Se_{11}$, then $B^l = 0$ and therefore $B \leqq S$. If $S \leqq (F)_n$, then $B \leqq (F)_n$ and for every nonzero $f \in F$, $(fe_{11})B \cap B \neq 0$. Hence, $fF_{11} \cap F_{11} \neq 0$ and $F \in F_{11}F_{11}^{-1}$. Thus, $F_{11}F_{11}^{-1} = F$.

Conversely, if $F_{11}F_{11}^{-1} = F$ then $F_{i1}F_{j1}^{-1} = F$ for all i and j by [4, Lemma 1.1]. Let $a = \sum a_{ij}e_{ij} \in (F)_n$, where $a_{ij} \in F$ and some $a_{rs} \neq 0$. For each i, there exists some nonzero $f_i \in F_{s1}$ such that $a_{is}f_i \in F_{i1}$. Since $\bigcap_i f_iF_{11} \neq 0$ by [4, Lemma 1.1], $f_ig_i = f \neq 0$ for some $g_i \in F_{11}$, $i = 1, \cdots, n$. Clearly $a(fe_{s1}) \in S$ and $a(fe_{s1}) \neq 0$. Hence, $S \leqq (F)_n$.

If S is a T-ring in $(F)_n$ and $S \leqq (F)_n$, then necessarily $S_r^\Delta = 0$ and $L_r^*(S) \cong L_r^*((F)_n)$. Since $(F)_n$ is an FI-ring, S is also an FI-ring.

3.6. Theorem. *Let S be a T-ring in $(F)_n$ given by* 3.2 *and M be its full cover. Then $SS^{-1} = M$ iff $F_{ii}F_{ii}^{-1} = F$, $i = 1, \cdots, n$.*

Proof. If $SS^{-1} = M$ and $S = (S_{rs})$ and $M = (M_{rs})$ are represented as in 3.4, then evidently $S_{ii}S_{ii}^{-1} = M_{ii}$ for each i. Hence, $F_{ii}F_{ii}^{-1} = F$ for each i by [4, Theorem 1.2].

Conversely, if $F_{ii}F_{ii}^{-1} = F$ for each i then $F_{ij}F_{jj}^{-1} = F_{jj}F_{ij}^{-1} = F$ for all i and j for which $F_{ij} \neq 0$ by [4, Lemma 1.1]. Let $d = (d_{ij}) \in M$, $d_{ij} \in F$, $d \neq 0$. Then $d_{ij} = a_{ij}b_{ij}^{-1}$ for some $a_{ij} \in F_{ij}$ and $b_{ij} \in F_{jj}$, $i, j = 1, \cdots, n$. Now $\bigcap_i b_{ij}F_{jj} \neq 0$ for each j by [4, Lemma 1.1], and hence there exist nonzero $b_j \in F_{ij}$ and $c_{ij} \in F_{jj}$ such that $b_j = b_{ij}c_{ij}$, $i,j = 1,\cdots,n$. Clearly $d = ab^{-1}$ where $a = \sum a_{ij}c_{ij}e_{ij}$ and $b = \sum_j b_j e_{jj}$. This proves 3.6.

3.7. THEOREM. *Let S be a T-ring in $(F)_n$ given by 3.2 such that $S \leqq (F)_n$. Then S is potent iff*

$$F_{jj}F_{kj}^{-1} = F, \qquad j < k,\ j, k = 2,\cdots,n. \tag{3.8}$$

Proof. Assume that R is potent. Let j and k be integers such that $2 \leqq j < k \leqq n$ and let $d \in F$, $d \neq 0$. Then $d = a_j a_k^{-1}$ for some $a \in F_{i1}$ by 3.5. If

$$a = a_j e_{j1} + a_k e_{k1},$$

then $a \in S$ and $A = (aR)^*$ is an atom of L_r^* (since a^r is a coatom). By assumption, $b^2 \neq 0$ for some $b \in A$. If $b = \sum b_{rs}e_{rs}$, $b_{rs} \in F_{rs}$, then $b_{rs} = 0$ if $r \neq j$ or k and $b_{js} \neq 0$ iff $b_{ks} \neq 0$. For if $b_{rs} \neq 0$ with $r \neq j$ or k, or if $b_{js} \neq 0$ and $b_{ks} = 0$, then $(be_{s1})R \cap aR \neq 0$ contrary to the atomicity of A. Hence, either $b_{jj} \neq 0$ or $b_{kk} \neq 0$.

If $b_{jj} \neq 0$, then $(be_{j1})R \cap aR \neq 0$ and $b_{jj}f = a_j g$, $b_{kj}f = a_k g$ for some nonzero $f, g \in F$. Hence, $a_j a_k^{-1} = b_{jj}b_{kj}^{-1} \in F_{jj}F_{kj}^{-1}$. If $b_{kk} \neq 0$, then $(be_{k1})R \cap aR \neq 0$ and $a_j a_k^{-1} = b_{jk}b_{kk}^{-1} \in F_{jk}F_{kk}^{-1}$ by the same reasoning. However, if both F_{jk} and F_{kj} are nonzero, then $F_{jk}F_{kk}^{-1} \subset F_{jj}F_{kj}^{-1}$. Therefore, $d \in F_{jj}F_{kj}^{-1}$. We conclude that $F_{jj}F_{kj}^{-1} = F$.

Conversely, let us assume that 3.8 holds. Every atom A of L_r^* contains a nonzero element a of the form $a = a_k e_{k1} + \cdots + a_n e_{n1}$, $a_i \in F_{i1}$, $a_k \neq 0$. If $k = 1$, then $a^2 \neq 0$ and A is potent. If $k > 1$, we claim that there exists some $b = b_k e_{kk} + \cdots + b_n e_{nk} \in A$, $b_i \in F_{ik}$, with $b_i \neq 0$ iff $a_i \neq 0$. Since $b^2 \neq 0$, this will prove that A is potent and hence will prove the theorem.

Such a $b \in A$ exists iff $a_i f = b_i g$, $i = k,\cdots,n$, for some nonzero $f, g \in F$; i.e., iff

$$a_k^{-1}b_k = a_i^{-1}b_i \quad \text{for each } i \text{ for which } a_i \neq 0. \tag{1}$$

Assume, for simplicity of notation, that $a_i \neq 0$ if $k \leqq i \leqq p$ and that $a_i = 0$ if $i > p$. If we have found nonzero $b_i \in F_{ik}$, $i = 1,\cdots,m-1$, for which (1) holds, with $m \leqq p$, then let us select nonzero $b_m \in F_{mk}$ and $c \in F_{kk}$ such that $a_i^{-1}b_i c = a_m^{-1}b_m$ (which we can do, since $F_{kk}F_{mk}^{-1} = F$). Then $a_k^{-1}b_k c = a_i^{-1}b_i c = a_m^{-1}b_m$, $i = 1,\cdots, m-1$, and (1) follows by induction. This proves 3.7.

If a T-ring is potent, then its block numbers are the obvious ones according to the result below.

3.9. THEOREM. *Let S be a T-ring in $(F)_n$ whose blocks are defined by the numbers $0 = d_0 < d_1 < \cdots < d_k$ as in 3.2. If $S \leqq (F)_n$ and S is potent, then $L_{r2}^* = \{T_0,\cdots,T_k\}$ where $T_0 = R$, $T_k = 0$, and*

$$T_i = \sum_{r>m}\sum_{j=1}^{n} F_{rj}e_{rj} \quad \textit{where } m = d_i;\ i = 1,\cdots,k-1.$$

Proof. If $A = e_{jj}S$, where $j = d_i$ for some i, $0 < i \leqq k$, then clearly $A^r = T$. Conversely, if B is an atom of L_r^*, let r be the maximum integer for which $be_{rr} \neq 0$ for some $b \in B$. We claim that $r = d_i$ for some i. Otherwise, $d_{i-1} < r < d_i$ for some i and we can find a nonzero $ce_{uu} \in S$, where $u = d_i$, and nonzero $f \in F_{11}$ $g \in F_{u1}$ such that $b(fe_{r1}) = (ce_{uu})(ge_{u1})$ just as we did in the proof of 3.7. Clearly $ce_{uu} \in B$, contrary to the choice of r. Since B contains nonzero elements of the form be_{rr} for $r = 1, \cdots, d_i$ and $Be_{ss} = 0$ if $s > d_i$, evidently $B^r = T_i$. This proves 3.9.

The block numbers of the potent ring S of 3.9 clearly are $(d_1 - d_0, \cdots, d_k - d_{k-1})$. Thus, 3.9 gives us a way of constructing FPI-rings having any prescribed block numbers. In particular, any full T-ring in $(F)_n$ is an FPI-ring. A T-ring over the ring of integers is also an FPI-ring.

As a slightly different example, let F be a field which has a nonzero subring K such that $KK^{-1} \neq F$. Then the 2×2 matrix rings

$$S = Fe_{11} + Fe_{21} + Ke_{22}, \quad M = Fe_{11} + Fe_{21} + Fe_{22}$$

are both potent. However, $SS^{-1} \neq M$, i.e., S doesn't have M as a classical quotient ring.

We point out that if S is a potent T-ring in $(F)_n$ such that $S \leqq (F)_n$ and F_{jk} and F_{kj} are nonzero for some j and k, say with $k > j$, then

$$F_{jj}F_{jj}^{-1} = F_{kk}F_{kk}^{-1} = F. \tag{1}$$

To prove (1), we have that for any nonzero $a,b,c,d \in F$ there exist $f \in F_{jj}$ and $g \in F_{kj}$ such that $a^{-1}db = fg^{-1}$, or $d = (afc)(bgc)^{-1}$. By letting $a, c \in F_{jj}$ and $b \in F_{jk}$, we see that $d \in F_{jj}F_{jj}^{-1}$; and by letting $a \in F_{kj}$, $b \in F_{kk}$, and $c \in F_{jk}$, we can see that $d \in F_{kk}F_{kk}^{-1}$. Since c is any nonzero element of F, (1) is proved.

The situation described in the preceeding paragraph will occur in T-ring S of 3.2 iff $d_{i+1} - d_i > 1$ for some i. If $S \leqq (F)_n$, then $F_{11}^{-1}F_{11} = F$ irrespective of whether or not $d_1 - d_0 > 1$. If, in addition, S is potent and $d_{i+1} - d_i > 1$ for all $i > 0$, then $F_{kk}F_{kk}^{-1} = F$ for all k by (1) above. This proves the following corollary of 3.6 and 3.7.

3.10. Corollary. *Let S be a T-ring in $(F)_n$ defined by* 3.2, $S \leqq (F)_n$, *and M be the full cover of S. If S is potent and $d_{i+1} - d_i > 1$ for all $i > 0$, then $SS^{-1} = M$.*

4. FPI-rings as matrix rings. It is well known that every n-dimensional I-ring R has a full ring Q of linear transformations of an n-dimensional vector space over a field as a quotient ring and that $L_r^*(Q) \cong L_r^*(R)$ under the correspondence $A \to A \cap R$, $A \in L_r^*(Q)$. (See [1] for references.) Let R be an FPI-ring, and the A_i and B_j be as given in 2.6. Corresponding to the basis $\{A_1, \cdots, A_n\}$ of $L_r^*(R)$ is an atomic basis $\{A_1', \cdots, A_n'\}$ of $L_r^*(Q)$. By [5, Proposition 5, p. 52], there exists a set $\{e_{ij} \mid i,j = 1, \cdots, n\}$ of matrix units in Q such that $A_i' = e_{ii}Q$ and hence $A_i = (e_{ii}Q) \cap R$, $i = 1, \cdots, n$. Clearly $B_i = (\bigcup_{j \neq i} A_j)^l = [(\sum_{j \neq i} e_{jj}Q) \cap R]^l = (Qe_{ii}) \cap R$, $i = 1, \cdots, n$.

Relative to the given set of matrix units in Q, there exists a field F such that [6; Proposition 6, p. 52]

$$Q = \sum_{i,j=1}^{n} Fe_{ij} \cong (F)_n.$$

Then

$$A_i \cap B_j = F_{ij}e_{ij}, \qquad i, j = 1, \cdots, n,$$

for some additive subgroups F_{ij} of F satisfying 3.1. If we let

$$S = \sum_{i,j=1}^{n} F_{ij}e_{ij}, \tag{4.1}$$

then S is a subring of R. By 2.6 (5), $F_{ij} \neq 0$ iff $i > d_p$ and $d_p < j \leqq d_{p+1}$ for some p. Thus, S is a T-ring in $(F)_n$ with the same block numbers as R.

Since $B_1^l = 0$, we know that $B_1 \leqq R$. Also, $\{A_1 \cap B_1, \cdots, A_n \cap B_1\}$ is an atomic basis of $L_r^*(B_1)$. Hence, $F_{11}e_{11} + \cdots + F_{n1}e_{n1} \leqq B_1 \leqq R$, and therefore (since $F_{11}e_{11} + \cdots + F_{n1}e_{n1} \subset S$) $S \leqq R \leqq Q$.

Associated with the T-ring S is the full T-ring M over F with the same block numbers as S.

4.2. Lemma. *If R is an FPI-ring and rings Q, S, and M are defined as above then $S \subset R \subset M$.*

Proof. If $b \in R$, then $b \in Q$ and $b = \sum b_{ij}e_{ij}$ for some $b_{ij} \in F$. If $b_{rs} \neq 0$ then $(e_{rr}f)b(e_{ss}g) \in R$ for any nonzero $f \in F_{rr}$ and $g \in F_{ss}$; i.e., $c = fb_{rs}ge_{rs} \in R$. Since $c \in A_r \cap B_s$, evidently $fb_{rs}g \in F_{rs}$ and $F_{rs} \neq 0$. Thus, $b \in M$. This proves 4.2.

4.3. Theorem. *If R is an FPI-ring and $S \subset R \subset M$ as in 4.2, then S is an FPI-ring having the same dimensions and same block numbers as R.*

Proof. Since $S \leqq Q$, $\dim S = \dim R$ and $F_{j1}F_{m1}^{-1} = F$ for all j and m. Let $a_je_{j1} + a_me_{m1} = a \in S$, where $1 < j < m$, $a_i \in F_{j1}$, and $a_j \neq 0$ and $a_m \neq 0$. Also let $d = a_ma_j^{-1}$ and $e = e_{jj} + de_{mj}$. Clearly $e^2 = e$ and eQ is an atom of $L_r^*(Q)$. Hence, $A = eQ \cap R$ is an atom of $L_r^*(R)$. As such, it is potent. Let $b \in A$, $b^2 \neq 0$. Now $b = \sum_i(e_{ji} + de_{mi})c_i$ for some $c_i \in F$. Since $b^2 \neq 0$, either $c_j \neq 0$ or $c_m \neq 0$. Assume that $c_j \neq 0$. Then $(e_{jj}f)b(e_{jj}g) \in F_{jj}e_{jj}$ and $(e_{mm}h)b(e_{jj}g) \in F_{mj}e_{mj}$ for all nonzero $f, g \in F_{jj}$ and $h \in F_{mm}$. Hence, $fc_jg \in F_{jj}$, $hdc_jg \in F_{mj}$, and $(hdf^{-1})(fc_jg) \in F_{mj}$, $hdf^{-1} \in F_{mj}F_{jj}^{-1}$. Since d ranges over F, so does hdf^{-1} and therefore $F_{mj}F_{jj}^{-1} = F$ (or, taking inverses, $F_{jj}F_{mj}^{-1} = F$).

If $c_j = 0$ but $c_m \neq 0$, then $(e_{mm}f)b(e_{mm}g) \in F_{mm}e_{mm}$ and $(e_{jj}h)b(e_{mm}g) \in F_{jm}e_{jm}$ for all nonzero $f, g \in F_{mm}$ and $h \in F_{jj}$. Hence, $fdc_mg \in F_{mm}$, $hc_mg \in F_{jm}$, and $(fdh^{-1})(hc_mg) \in F_{mm}$, $fdh^{-1} \in F_{mm}F_{jm}^{-1}$. Therefore, $F_{mm}F_{jm}^{-1} = F$. By previous remarks, we must also have $F_{jj}F_{mj}^{-1} = F$. Thus, S is potent by 3.7.

If the block numbers of S, and hence also of R, are all greater than 1 with the possible exception of the first one, then $F_{ii}F_{ii}^{-1} = F$ for all i and $SS^{-1} = M$ by 3.6. Clearly, then, $RR^{-1} = M$, and we have proved the following result.

4.4. THEOREM. *Let R be an FPI-ring with block numbers $(m_1, \cdots, m_k)$. If $m_i > 1$ for $i = 2, \cdots, m_k$, then $RR^{-1} = M$, the full T-ring over a field F with block numbers $(m_1, \cdots, m_k)$.*

5. **Reducible rings.** Let us call a ring R *reducible* iff $R_r^{\Delta} = 0$ and $C_r^* \neq \{0, R\}$. If $H \in C_r^*$, $H \neq 0$ or R, then $K = H^l \in C_r^*$ also and $H \cap K = 0$, $H = K^l$[3, 6.7]. Hence, $H + K \leqq R$. If R is atomic, then each atom of $L_r^*(R)$ is contained in either H or K by our remarks in §1. Therefore, the set of atoms of $H + K$ coincides with the set of atoms of R, and R is potent iff $H + K$ is potent.

This leads us to consider the direct sum $R = R_1 \oplus R_2$ of two rings R_1 and R_2 Clearly, $L_r(R_i) \subset L_r(R)$, $i = 1, 2$. Let us assume that $R_i^r = 0$ (in R_i), $i = 1, 2$. Then $C \cap R_i \neq 0$ for each right ideal C of R not contained in R_j, $i \neq j$. In particular, if C is a large right ideal of R then $C \cap R_i$ is a large right ideal of $R_i, i = 1, 2$. Conversely, if C_i is a large right ideal of R_i, $i = 1, 2$, then $C = C_1 + C_2$ is a large right ideal of R. From these remarks, it follows readily that $R_r^{\Delta} = R_{1r}^{\Delta} + R_{2r}^{\Delta}$. If $R_r^{\Delta} = 0$, then it is not difficult to show that

$$L_r^*(R) = \{A_1 + A_2 \mid A_i \in L_r^*(R_i)\}.$$

Hence, R is potent iff R_1 and R_2 are potent.

If R is an FP-ring, and $\{R_1, \cdots, R_n\}$ is the set of atoms of C_r^*, then $R_1 + \cdots + R_n$ is a direct sum of ideals of R and $R_1 + \cdots + R_n \leqq R$ [3, p. 541]. By our remarks above, each R_i is an FP-ring. Actually, each R_i is an FPI-ring. Let the T-rings S_i, M_i, and Q_i be selected as in 4.2, $S_i \subset R_i \subset M_i \subset Q_i$, $i = 1, \cdots, n$. Then $Q_1 + \cdots + Q_n$ is the maximal right quotient of $R_1 + \cdots + R_n$, so that

$$S_1 + \cdots + S_n \leqq R_1 + \cdots + R_n \leqq R \leqq Q_1 + \cdots + Q_n.$$

If $b \in R$, then $b = \sum b_i$, $b_i \in Q_i$, $bR_i \subset R_i$, and hence $b_iS_i \subset M_i$ for each i. Clearly, then, $b_i \in M_i$ for each i and $b \in M_1 + \cdots + M_n$. Thus, $R \leqq M_1 + \cdots + M_n$.

The theorem below follows from these remarks and the work of the preceeding sections.

5.1. THEOREM. *If R is an FP-ring then there exist FPI-rings $R_1, \cdots, R_n$ such that $R_1 \oplus \cdots \oplus R_n \leqq R$. Furthermore, if S_i and M_i are the associated T-rings of R_i, selected as in 4.2, $i = 1, \cdots, n$, then*

$$S_1 \oplus \cdots \oplus S_n \leqq R \leqq M_1 \oplus \cdots \oplus M_n.$$

If $M = M_1 \oplus \cdots \oplus M_n$, then $RR^{-1} = M$ iff $R_iR_i^{-1} = M_i$, $i = 1, \cdots, n$.

Bibliography

1. R. E. Johnson, *Quotient rings of rings with zero singular ideal*, Pacific J. Math. **11** (1961), 1385–1392.

2. ———, *Rings with zero right and left singular ideals*, Trans. Amer. Math. Soc. **118** (1965), 150–157.

3. ———, *Structure theory of faithful rings*. II, Trans. Amer. Math. Soc. **84** (1957), 523–544.

4. ———, *Prime matrix rings*, Proc. Amer. Math. Soc. **16** (1965), 1099–1105.

5. N. Jacobson, *Structure of rings*, Amer. Math. Soc. Colloq. Publ. Vol. 35, Amer. Math. Soc., Providence, R. I., 1956.

University of Rochester,
Rochester, New York