THE HAHN-BANACH THEOREM FOR FINITE DIMENSIONAL SPACES

BY

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1. **Introduction.** An immediate generalization of the Hahn-Banach Theorem on the dominated extension of linear functionals is obtained by replacing the real number field by a boundedly complete vector lattice as the range space of the function. The question arises: Do boundedly complete vector lattices characterize those range spaces which permit dominated extensions? It is proved in [1], [4] that an ordered linear space is a boundedly complete vector lattice if and only if both the positive wedge is lineally closed and the Hahn-Banach Theorem holds with the space taken as the range of the extensions. Thus, the above question has an affirmative answer if the positive wedge must be closed in a space for which the Hahn-Banach Theorem is valid. This paper shows that at least for finite dimensional ordered linear spaces this is indeed the case.

An example is presented in [4] of a two dimensional ordered linear space whose positive wedge is not lineally closed and it is erroneously asserted that this space permits Hahn-Banach type extensions. The error in the argument is that the extensions described may not be well defined.

§2 introduces preliminary definitions and theorems. In §3, finite dimensional boundedly complete vector lattices are characterized by the existence of a "partly positive basis" for the positive wedge. This characterization is a slight generalization of a theorem of Yudin [5] or Nagy [3]. In §4 it is proved that if a finite dimensional ordered linear space permits Hahn-Banach type extensions, then the closure of the positive wedge determines an ordered linear space which is a boundedly complete vector lattice. This restricts the class of positive wedges which must be considered in ascertaining the class which admits dominated extensions.

Examples are given in §5 which are used in §6 to complete the proof that a finite dimensional ordered linear space admits dominated extensions if and only if the space is a boundedly complete vector lattice.

2. **Preliminaries.** Let (V, C) be an *ordered linear space* (OLS) over the real number field R. That is, V is a linear space, $C = \{v \in V \mid v \geqq 0\}$ is the *positive*

Received by the editors November 24, 1963 and, in revised form, August 12, 1964 and April 5, 1965.

(1) The work of both authors was supported by the National Science Foundation [NSF G-24345].

wedge of V, where ($\geqq$) is a transitive relation such that, if $v_1 \geqq v_2$, then $tv_1 \geqq tv_2$ and $v_1 + v \geqq v_2 + v$ for every non-negative number t and all $v \in V$. The positive wedge determines the ordering: $v_1 \geqq v_2$ if and only if $v_1 - v_2 \in C$. Conversely, if C is any *wedge* (i.e., C is a nonempty set, closed under addition and multiplication by non-negative scalars) in a vector space, then C is a positive wedge relative to the ordering: $v_1 \geqq v_2$ if and only if $v_1 - v_2 \in C$.

The wedge C is *lineally closed* if every line intersects C in a set which is closed relative to the line. The *lineal closure* $\bar{C}$ of a wedge C is the union of closures (in the line) of all line segments contained in the wedge. Clearly, if C is lineally closed, $C = \bar{C}$. A *core point* of a convex set is a point in the convex set such that every line through the point contains an open segment which contains the point and is contained in the set.

If C is a finite dimensional wedge (i.e., the linear hull of C is finite dimensional), then $\bar{C}$ coincides with the closure of C relative to any topology which makes the linear hull of C into a linear topological space, a core point and an interior point are synonomous, and C has an interior (core) point relative to its linear hull. These statements are not generally true for infinite dimensional wedges. Further, the closure of C is the union of closures of those line segments which contain the core point and are contained in C.

The OLS (V, C) has the *least upper bound property* (LUBP) (or is a *boundedly complete vector lattice*) if every set of elements with an upper bound has a least upper bound (not necessarily unique). An OLS (V, C) has the *Hahn-Banach extension property* (HBEP) if given (1) a real linear space Y, (2) a linear subspace X of Y, (3) a function $p: Y \to V$ which is *sublinear*, (i.e., $p(y) + p(y') \geqq p(y + y')$ and $p(ty) = tp(y)$, $y, y' \in Y$, $t \geqq 0$) and (4) a linear function $f: X \to V$ such that $p(x) \geqq f(x)$ for all $x \in X$, implies there is a linear extension $F: Y \to V$ of f such that $p(y) \geqq F(y)$ for all $y \in Y$. It is proved in [1], [4] that (V, C) *has the LUBP if and only if* (V, C) *has the HBEP and C is lineally closed.*

The following three lemmas show that the HBEP, the LUBP, and lineal closure of the positive wedge are independent of the containing space. So henceforth we may speak of a *wedge having* the HBEP or the LUBP or being lineally closed without mention of the containing space. If D is a subset of a vector space, lin D will denote the linear hull of D.

LEMMA 2.1. *The OLS* (V, C) *has the HBEP if and only if the OLS* (lin C, C) *has the HBEP.*

Proof. Assume that (V, C) has the HBEP. Let $p: Y \to \text{lin } C$ be a sublinear function; $f: X \to \text{lin } C$ be a linear function on the linear subspace X of Y such that for all $x \in X$, $p(x) - f(x) \in C$. Then, there exists a linear extension $F: Y \to V$ of f such that for all $y \in Y$, $p(y) - F(y) \in C$. Thus

$$F(y) = p(y) - (p(y) - F(y)) \in (\text{lin } C) - C \subset \text{lin } C.$$

Assume (lin C, C) has the HBEP. Let $\{b_\alpha \mid \alpha \in A'\} \subset C$ be an algebraic basis for lin C. Extend $\{b_\alpha \mid \alpha \in A'\}$ to an algebraic basis $\{b_\alpha \mid \alpha \in A\}$ of V. Let X be a linear subspace of linear space Y, $p: Y \to V$ be sublinear with respect to C, and $f: X \to V$ be linear and satisfy $p(x) - f(x) \in C$ for $x \in X$. Consider the coefficient functionals $\{p_\alpha \mid \alpha \in A\}$ so that $p(z) = \sum_{\alpha \in A} p_\alpha(z) b_\alpha$ for $z \in Y$, and functions p' and p'' defined by $p'(z) = \sum_{\alpha \in A'} p_\alpha(z) b_\alpha$ and $p''(z) = \sum_{\alpha \notin A'} p_\alpha(z) b_\alpha$. The functions p' and p'' are well defined, positive homogeneous, and $p = p' + p''$. Moreover, from $p(z_1) + p(z_2) - p(z_1 + z_2) \in C$, it follows that $p''(z_1) + p''(z_2) - p''(z_1 + z_2) = 0$ and hence that $p'(z_1) + p'(z_2) - p'(z_1 + z_2) \in C$. Thus p'' is linear and p' is sublinear.

Also consider the coefficient functionals $\{f_\alpha \mid \alpha \in A\}$ and functions f' and f'' so that $f(x) = \sum_{\alpha \in A} f_\alpha(z) b_\alpha$, $f'(x) = \sum_{\alpha \in A'} f_\alpha(x) b_\alpha$ and $f''(x) = \sum_{\alpha \notin A'} f_\alpha(x) b_\alpha$ for $x \in X$. Then $f = f' + f''$ and f' and f'' are linear. Further, $p(x) - f(x) \in C$ for all $x \in X$ implies that $p''(x) = f''(x)$ and $p'(x) - f'(x) \in C$ for all $x \in X$. By hypothesis let $F': Y \to \text{lin } C$ be a linear extension of f' such that $p'(z) - F'(z) \in C$, $z \in Y$. Define $F'': Y \to V$ by $F''(z) = p''(z)$, $z \in Y$. Then $F = F' + F''$ on Y to V is a linear extension of f such that $p(z) - F(z) \in C$, $z \in Y$.

LEMMA 2.2. *An OLS* (V, C) *has the LUBP if and only if* (lin C, C) *has the LUBP.*

Proof. Assume (V, C) has the LUBP. Let A be a subset of lin C which has an upper bound $b_0 \in \text{lin } C$. By assumption there is a $v_1 \in V$ such that v_1 is a least upper bound of A in V, hence in lin C, if $v_1 \in \text{lin } C$. But if $a \in A$, then $v_1 - a \in C$, and $v_1 \in a + C \subset \text{lin } C$.

Conversely, if (lin C, C) has the LUBP, let b_0 be an upper bound of a subset $A \subset V$. Then $A' = \{b_0 - a \mid a \in A\}$ is a subset of C and hence has a greatest lower bound $b_1 \in \text{lin } C$. So $b_2 = b_0 - b_1 \geqq a$, $a \in A$. If $u \geqq a$ for all $a \in A$, then $u - b_0 - a + b_0 \in C$ and therefore $-b_0 + u \in \text{lin } C$ and $b_0 - u \leqq b_1$. Thus, $u \geqq b_0 - b_1 = b_2$.

LEMMA 2.3. *In an OLS* (V, C), *C is linearly closed in V if and only if C is linearly closed in* lin C.

The proof is omitted.

LEMMA 2.4. *If C is a wedge in linear space V, then* lin C = lin $\bar{C}$.

Proof. If $v \in \bar{C}$, then there is a $v_1 \in C$ such that $\lambda v + (1 - \lambda) v_1 \in C$, $0 \leqq \lambda < 1$ Thus $v \in \text{lin } C$ and $\text{lin } C = \text{lin } \bar{C}$.

An element x in a wedge C is *sharp* if $-x \notin C$. A wedge C is *sharp* (C is a *cone*) if each nonzero element in C is sharp. It is well known and easily checked that a wedge C is sharp if and only if $a \leqq b \leqq a$ implies $a = b$. In this case least upper bounds are unique if they exist.

A linear map ϕ from OLS (V, C) to OLS (V', C') is *positive* (or *monotone*) if $\phi(C) \subset C'$. Two OLS (V, C) and (V', C') are *isomorphic* and $\phi: V \to V'$ is an *isomorphism* if ϕ is linear, one to one, onto and ϕ and ϕ^{-1} are monotone. Wedges C and C' are said to be *isomorphic* if (lin C, C) and (lin C', C') are isomorphic. Then C and C' are isomorphic if and only if there exist OLS (V, C) and (V', C') which are isomorphic. It is also readily verified that the LUBP, the HBEP, and lineal closure of the positive wedge are preserved by an isomorphism from one OLS to another.

The wedge C in an OLS (V, C) is *reproducing* if $V = \text{lin } C$. Wedge C is *proper* if $C \neq V$.

THEOREM 2.5 (*Monotone Extension Theorem* [1]). *If (V, K) is an OLS, K has a core point v_0 and X is a linear subspace of V such that v_0 is in X, then any monotone linear functional defined on X can be extended to a monotone linear functional defined on V.*

Let K be a wedge in linear space V, K' the wedge of positive linear functionals on V and $V' = \text{lin } K'$. Let Q be the canonical map of V into the conjugate space V'' of linear functionals on V', $Qv(g) = g(v)$, $v \in V$, $g \in V'$. Let

$$\tilde{K} = \{v \in V \mid g(v) \geqq 0, \ g \in K'\}.$$

THEOREM 2.6. *If (V, K) is an OLS with sharp reproducing wedge K with core point, then* (1) $\tilde{K} = \bar{K}$, (2) $K' = \bar{K}'$, (3) *Q is an isomorphism from $(V, \bar{K})$ into (V'', K'') (and onto if V is finite dimensional).*

The proof of this theorem follows from Theorem 2.5 in a direct manner and is omitted.

3. **Partly positive bases.** Let V be an n-dimensional vector space, K a proper, reproducing wedge in V. A basis $B = \{b_1, \cdots, b_n\}$ of V is a *k-partly positive basis* for K if and only if $B \subset K$ and there is an integer k, $1 \leqq k \leqq n$, such that $v = \sum_{i=1}^{n} v_i b_i \in K$ if and only if $v_i \geqq 0$, $i = 1, 2, \cdots, k$. If $k = n$, the basis B is called a *positive basis* for K.

THEOREM 3.1. (1) *The wedge K has the LUBP if and only if there exists a k-partly positive basis for K.* (2) *If K has a k-partly positive basis, then K is sharp if and only if $k = n$.* (3) *The number k associated with a partly positive basis is unique.*

Because Theorem 3.1 is a well known unpublished result, and be can proven without too much difficulty from Theorem 3.2 below of Yudin [5] or Nagy [3], the proof of 3.1 is omitted.

THEOREM 3.2. *Let K be a proper, reproducing, sharp wedge in a finite dimensional vector space V so that (V, K) is a vector lattice. Then there is a positive basis for K.*

THEOREM 3.3. *A proper, reproducing wedge K in a finite dimensional space V has the LUBP if and only if K is the intersection of k closed halfspaces, $1 \leqq k \leqq n$ (i.e., there exists k linearly independent linear functionals $g_1, \cdots, g_k$ defined on V such that $v \in K$ if and only if $g_i(v) \geqq 0$, $i = 1, \cdots, k$).*

Proof. Let $B = \{b_1, \cdots, b_n\}$ be a k-partly positive basis for K. Then, as is well known, there exists linear functionals $g_1, \cdots, g_n$ such that $g_i(b_j) = \delta_{ij}$, $i, j = 1, \cdots, n$ and if $v = \sum_{i=1}^n v_i b_i$, then $v_i = g_i(v)$. Thus, $g_1, \cdots, g_k$ satisfy the corollary.

The converse follows by choosing b_i, $i = 1, \cdots, k$, so that

$$g_j(b_i) = \delta_{ji}, \quad j = 1, \cdots, k.$$

Extend $b_1, \cdots, b_k$ to a basis $B = \{b_1, \cdots, b_n\}$ of V. It follows that B is a k-partly positive basis of K.

4. **The closure of a wedge with the HBEP.** The result of this section shows that attention can be restricted to wedges whose closure has the LUBP in determining finite dimensional wedges with the HBEP.

THEOREM 4.1. *Let K be a finite dimensional wedge with the HBEP, then $\bar{K}$, the closure of K, has the HBEP and thus the LUBP.*

A definition and lemma are required. An OLS (V, K) has the WHBEP if and only if for any quadruple (Y, X, p, f) where (a) Y is a linear space, (b) X is a linear subspace of Y, (c) $p: Y \to V$, p is sublinear with respect to K, (d) $f: X \to V$, f linear, $p(x) - f(x) \in K$, $x \in X$, implies that there is an $F: Y \to V$, such that F is a linear extension of f and $p(z) - F(z) \in \bar{K}$, $z \in Y$.

LEMMA 4.2. *Let K be a finite dimensional wedge such that $\bar{K}$ is sharp and* (lin K, K) *has the WHBEP, then $\bar{K}$ has the HBEP.*

Proof of 4.2. (1) Let Y be a vector space, X a linear subspace of Y, $p: Y \to V$ such that p is sublinear with respect to $\bar{K}$, $f: X \to V$ such that f is linear and $p(x) - f(x) \in \bar{K}$, $x \in X$. Note that $V = \operatorname{lin} K$.

(2) Let $\{u_\alpha\}_{\alpha \in A}$ be an algebraic basis of Y and $z = \sum_{\alpha \in A} \zeta_\alpha u_\alpha$, $z \in Y$. Define $q: Y \to R$, the real number field, by $q(z) = (\sum_{\alpha \in A} |\zeta_\alpha|^2)^{1/2}$, $z \in Y$. The functional is well defined, sublinear and $q(z_1 + z_2) = q(z_1) + q(z_2)$ if and only if $z_1 = \lambda z_2$ or $z_2 = \lambda z_1$ for some real number $\lambda \geqq 0$. This is readily verified.

(3) Let v_0 be a core point of K. For each real number $\varepsilon > 0$, define $p_\varepsilon: Y \to V$ by $p_\varepsilon(z) = p(z) + \varepsilon q(z) v_0$. Then p_ε is sublinear with respect to K. In fact, for every $z_1, z_2 \in Y$, $-p_\varepsilon(z_1 + z_2) + p_\varepsilon(z_1) + p_\varepsilon(z_2)$ is a core point of K or is 0.

This follows easily from (1) and (2) and the fact that v_0 is a core point.

(4) For each $x \in X$, $x \neq 0$, $p_\varepsilon(x) - f(x) = (p(x) - f(x)) + \varepsilon q(x) v_0$ is a core point of K and therefore in K, since $\varepsilon q(x) v_0$ is a core point of K and $p(x) - f(x) \in \bar{K}$. If $x = 0$, $p_\varepsilon(x) - f(x) = 0$.

(5) Thus by assumption, there is a linear extension $F_\varepsilon: Y \to V$ of f such that $p_\varepsilon(z) - F_\varepsilon(z) \in \bar{K}$, for every $z \in Y$. Let $\mathscr{F}_\varepsilon$ be the collection of linear extensions of f which are $\bar{K}$-dominated by p_ε.

(6) If $\varepsilon' \leqq \varepsilon$, then $\mathscr{F}_{\varepsilon'} \subset \mathscr{F}_\varepsilon$. For $p_\varepsilon(z) \geqq p_{\varepsilon'}(z) \geqq F_{\varepsilon'}(z)$, $z \in Y$.

(7) By virtue of Theorem 2.6 on identifying V with V'' and $\bar{K}$ with K'', it may be assumed that V is the conjugate space of an ordered linear space (W, C), so that $\bar{K} = \{v \in V \mid v(w) \geqq 0,\ w \in C\}$ and C is reproducing and sharp.

(8) For every $w \in C, z \in Y, F_\varepsilon \in \mathscr{F}_\varepsilon$, it follows that $[p_\varepsilon(z) - F_\varepsilon(z)](w) \geqq 0$ and $[p_\varepsilon(-z) + F_\varepsilon(z)](w) \geqq 0$. If $u \in W$, then $u = w_1 - w_2$, where w_1, w_2 are in C. Hence, for every $F_\varepsilon \in \mathscr{F}_\varepsilon$, $z \in Y$, $u \in W$, $F_\varepsilon(z)(u)$ is a member of the closed bounded interval, $I(u,z) = [-p_\varepsilon(-z)(w_1), p_\varepsilon(z)(w_1)] + [p_\varepsilon(z)(-w_2), -p_\varepsilon(-z)(-w_2)]$.

(9) Therefore, $\mathscr{F}_\varepsilon \subset \pi\{I(u,z) \mid u \in W, z \in Y\}$, the direct product of compact sets, and hence each $\mathscr{F}_\varepsilon$ is a subset of a compact subset of the direct product of lines relative to the direct product topology. Each set $\mathscr{F}_\varepsilon$ is closed in the direct product topology. For if ϕ is the limit point of a net $\{F_\alpha\}_{\alpha \in A} \subset \mathscr{F}_\varepsilon$, it is easily verified that ϕ is linear and an extension of f. Further, ϕ is $\bar{K}$-dominated by p_ε. For if $w \in C$, $z \in Y$, $\eta > 0$, then there is an element F_α of the net such that $[p_\varepsilon(z) - \phi(z)](w) = [p_\varepsilon(z) - F_\alpha(z)](w) + [F_\alpha(z) - \phi(z)](w) \geqq -\eta$. Thus, $p_\varepsilon(z) - \phi(z) \in \bar{K}$, and $\mathscr{F}_\varepsilon$ is closed, hence, compact.

(10) By compactness, there is an $F \in \bigcap\{\mathscr{F}_\varepsilon \mid \varepsilon > 0\}$. Then F is the desired extension of f for $(p(z) - F(z))(w) \geqq [-\varepsilon q(z)v_0](w)$, $w \in C, z \in Y, \varepsilon > 0$. Hence, $p(z) - F(z) \in \bar{K}$, $z \in Y$. Lemma 4.2 is proved.

Proof of 4.1. (1) Let K be a proper wedge with core point v_0 relative to lin $K = V$ such that (V, K) has the HBEP. Let Y be a vector space, X a linear subspace of Y, $p: Y \to V$, a sublinear functional with respect to $\bar{K}$, $f: X \to V$ a linear function such that $p(x) - f(x) \in \bar{K}$, $x \in X$.

(2) Let $T = \{v \mid -v,\ v \in \bar{K}\}$. Clearly T is a linear subspace of V and $v_0 \notin T$. Choose an algebraic basis β' of T, Extend β' to a basis β of V by adjoining an appropriate linearly independent set β'' and so that $v_0 \in \beta''$.

(3) Let $S = \text{lin } \beta''$ and $K_S = S \cap K$. The wedge $\bar{K}_S$ is sharp and has v_0 as a core point in S. That v_0 is a core point is clear. Let $v, -v \in \bar{K}_S$. Then $v \in S \cap T$, so $v = 0$.

(4) The OLS (S, K_S) has the WHBEP. For let Y' be a linear space, X' a linear subspace of Y', $p': Y \to S$ such that p' is sublinear with respect to K_S; $f': X' \to S$, f' linear and $p'(x) - f'(x) \in K_S$, $x \in X$. By assumption (V, K) has the HBEP. Thus, there is a linear extension $\phi: Y' \to V$ of f' and $p'(z) - \phi(z) \in K$, $z \in Y$.

Decompose ϕ into $\phi(z) = \phi'(z) + \phi''(z)$, $z \in Y'$, where

$$\phi(z) = \textstyle\sum_{b \in \beta} \phi_b(z)b,\ \phi'(z) = \sum_{b \in \beta'} \phi_b(z)b,\ \phi''(z) = \sum_{b \in \beta''} \phi_b(z)b.$$

Then, $\phi'(z) \in T$, $\phi''(z) \in S$, $z \in Y$, and ϕ' and ϕ'' are well defined linear functions on Y. Also, $p'(z) - \phi''(z) \in S$ and $(p'(z) - \phi''(z) - \phi'(z)) + \phi'(z) \in K + T \subset \bar{K}$. Thus $p'(z) - \phi''(z) \in \bar{K}_S$ and since ϕ'' is an extension of f, (S, K_S) has the WHBEP.

(5) Therefore, by Lemma 4.2, $(S, \bar{K}_S)$ has the HBEP.

(6) Represent $p(z) = p'(z) + p''(z)$, $z \in Y$, where

$$p'(z) = \sum_{b \in \beta'} p_b(z)b \in T, \quad p''(z) = \sum_{b \in \beta''} p_b(z)b \in S$$

are well defined functions, $p' : Y \to T$, $p'' : Y \to S$. Similarly, uniquely represent f by $f(z) = f'(z) + f''(z)$, $z \in Y$, where $f' : X \to T$, $f'' : X \to S$.

(7) Now, $\pm (p'(z_1) + p'(z_2) - p'(z_1 + z_2)) \in T \subset \bar{K}$, $z_1, z_2 \in Y$. Therefore,

$$(p(z_1) + p(z_2) - p(z_1 + z_2)) - (p'(z_1) + p'(z_2) - p'(z_1 + z_2))$$
$$= p''(z_1) + p''(z_2) - p''(z_1 + z_2) \in S \cap \bar{K} \subset \bar{K}_S.$$

Thus, p'' is sublinear with respect to $\bar{K}_S$. Similarly, $\pm (p'(x) - f'(x)) \in T \cap \bar{K}$, $x \in X$. Hence, $p''(x) - f''(x) \in S \cap \bar{K} = \bar{K}_S$. Thus, by (5), there is a linear extension $F'' : Y \to S$ of f'' such that $p''(z) - F''(z) \in \bar{K}_S$, $z \in Y$.

(8) Extend f' linearly in any fashion to $F' : Y \to T$ and define $F = F' + F''$. Then, F is a linear extension of f and

$$p(z) - F(z) = (p'(z) - F'(z)) + (p''(z) - F''(z)) \in T + T + \bar{K}_S \subset \bar{K}.$$

Thus, $\bar{K}$ has the HBEP and the LUBP. Theorem 4.1 is proved.

5. **Examples.** Examples are presented in this section which will be used in the next section to show that every n-dimensional wedge with the HBEP also has the LUBP.

EXAMPLE 1.

LEMMA 5.1. *There exists a three dimensional vector space Y and a two dimensional linear subspace X of Y, with* (1) *a sublinear functional $p_2 : Y \to R$,* (2) *linear functionals $f_1 : X \to R$, $f_2 : X \to R$, such that $p_2(x) - f_1(x) \geqq 0$, $p_2(x) - f_2(x) \geqq 0$, $x \in X$ with $p_2(x) - f_2(x) = 0$ if and only if $p_2(x) - f_1(x) = 0$, and* (3) *a set $U \subset Y$ such that for any linear extension $F_2 : Y \to R$ of f_2, if $p_2(z) - F_2(z) \geqq 0$, $z \in Y$, then $p_2(u) - F_2(u) = 0$, $u \in U$, and such that for any linear extension $F_1 : Y \to R$ of f_1 there is a $u_0 \in U$ such that $p_2(u_0) - F_1(u_0) \neq 0$. Further, by the Hahn-Banach Theorem, a linear extension $F_2 : Y \to R$ of f_2 exists, such that $p_2(z) - F_2(z) \geqq 0$, $z \in Y$.*

Proof[2]. Let $C(-\pi/2, 3\pi/2)$ be the Banach space of continuous real valued functions on the closed interval $[-\pi/2, 3\pi/2]$, and let X be the linear subspace of

[2] The referee suggested the elegant example used in the proof of Lemma 5.1. Our original example, presented here, does not require sophisticated function space arguments but the computations involved are lengthy.

Let $Y = R_3$, $X = \{(0, b, a) \mid b, a \in R\}$. Define $f_2 : X \to R$ by $f_2(0, b, a) = a + b$ and $f_1 : X \to R$ by $f_1(0, b, a) = b$. Define
$p_2 : Y \to R$ by $p_2(t, b, a) = |a| + b + t$, $t \geqq a$, $b \geqq 0$;
$p_2(t, b, a) = |a| + b + t + (a-t)^2/(a-t+b)$, $a > t$, $b \geqq 0$;
$p_2(t, b, a) = |a| + t$, $t \geqq a$, $b \leqq 0$;
$p_2(t, b, a) = |a| + a$, $a \geqq t$, $b \leqq 0$.

$C(-\pi/2, 3\pi/2)$ spanned by $\phi_1(t) = \cos t$ and $\phi_2(t) = \sin(t/3)$. Let Y be the subspace of $C(-\pi/2, 3\pi/2)$ spanned by X and the function ϕ_3, where $\phi_3(t) = 1$, if $0 \leqq t \leqq 3\pi/2$, and $\phi_3(t) = 1 + \sin(t/3)$, if $-\pi/2 \leqq t \leqq 0$. Define $f_2: X \to R$ by $f_2(\phi) = \phi(0)$, $\phi \in X$, and $f_1: X \to R$ by $f_1(\phi) = -\phi(\pi)$, $\phi \in X$. Further define $p_2: Y \to R$ by $p_2(\phi) = \|\phi\|$, $\phi \in Y$.

Then $\|f_1\| = \|f_2\| = 1$ and the following three statements are equivalent for $\phi \in X$, (a) $f_1(\phi) = p_2(\phi)$, (b) $f_2(\phi) = p_2(\phi)$, (c) $\phi = \lambda\phi_1$, $\lambda \geqq 0$. Moreover from the Riesz Representation Theorem, f_2 has a unique norm preserving linear ex-extension F_2 defined on all of $C(-\pi/2, 3\pi/2)$ and hence is defined uniquely on Y. For if μ is a Stieltjes measure on the interval $[-\pi/2, 3\pi/2]$ which represents an extension of f_2 then μ is of norm 1 and satisfies

$$\text{(i)} \int_{-\pi/2}^{3\pi/2} \cos t \, d\mu(t) = 1, \qquad \text{(ii)} \int_{-\pi/2}^{3\pi/2} \sin(t/3)\, d\mu(t) = 0.$$

Statement (i) implies that the support of μ contains at most the points 0 and π, while (ii) further restricts the support of μ to consist of 0 alone. Hence the extension F_2 is unique.

Let $U = \{\phi \mid \phi \in Y, F_2(\phi) = p_2(\phi)\}$. The set U contains the functions ϕ_3 and $\phi_3 - \phi_2$. Further $p_2(\phi_3) = p_2(\phi_3 - \phi_2) = 1$. If F_1 is any linear extension to Y of f_1, $F_1(\phi_3 - \phi_2) = F_1(\phi_3) + \sin(\pi/3) \neq F_1(\phi_3)$. Thus u_0 can be taken to be one of ϕ_3 or $\phi_3 - \phi_2$, so Lemma 5.1 is proved.

EXAMPLE 2.

LEMMA 5.2. *Let $Y = R_2$, the vector space of ordered pairs of real numbers, $X = \{(0,a) \mid (0,a) \in Y\}$. Then there exists,* (1) *a sublinear functional $r_2: Y \to R$,* (2) *a positive homogeneous functional $r_1: Y \to R$ such that if $r_2(z_1 + z_2) = r_2(z_1) + r_2(z_2)$, then $r_1(z_1 + z_2) \leqq r_1(z_1) + r_1(z_2)$,* (3) *linear functionals $f_2: X \to R$, $f_1: X \to R$ such that $r_2(x) - f_2(x) \geqq 0$, $x \in X$, and if $r_2(x) = f_2(x)$, then $r_1(x) \geqq f_1(x)$, and* (4) *a set $U \subset Y$ such that if* (a) *$F_2: Y \to R$ is a linear extension of f_2 and $r_2(z) - F_2(z) \geqq 0$, $z \in Y$, then $F_2(u) = r_2(u)$, $u \in U$, and* (b) *if $F_1: Y \to R$ is a linear extension of f_1, there is a sequence $\{u_n\} \subset U$ such that $\lim_{n\to\infty}(r_1(u_n) - f_1(u_n)) = -\infty$.*

Proof. Define $r_2: Y \to R$ by
$r_2(t,a) = |a| + t$, if $t \geqq 0$;
$r_2(t,a) = a + at/(a - t)$, if $t < 0$ and $a > 0$;
$r_2(t,a) = -a$, if $t \leqq 0$ and $a \leqq 0$.
Define $r_1: Y \to R$ by
$r_1(t,a) = -(at)^{1/2}$, if $t \geqq 0$ and $a \geqq 0$;
$r_1(t,a) = 0$, if $a \leqq 0$ or $t \leqq 0$.
Define $f_2: X \to R$ by $f_2(0,a) = a$, $f_1: X \to R$ by $f_1(0,a) = 0$, $(0,a) \in X$.

It is clear that r_1 and r_2 are positive homogeneous. It requires detailed computations to show analytically that r_2 is subadditive. However, a sketch of the graph of r_2 makes it obvious, geometrically, that r_2 is indeed subadditive. Hence, the computations will not be presented.

Further, from the sketch of the graph of r_2, it is clear, geometrically, that $r_2(z_1 + z_2) = r_2(z_1) + r_2(z_2)$ where $z_i = (t_i, a_i)$, $i = 1,2$, if and only if (a) $z_1 = \lambda z_2$, $\lambda \geqq 0$, or (b) $t_1 \geqq 0$, $a_1 \geqq 0$, $t_2 \geqq 0$, $a_2 \geqq 0$, or (c) $t_1 \geqq 0$, $a_1 \leqq 0$, $t_2 \geqq 0$, $a_2 \leqq 0$, or (d) $t_1 \leqq 0$, $a_1 \leqq 0$, $t_2 \leqq 0$, $a_2 \leqq 0$. Simple direct computations show that in these cases $r_1(z_1 + z_2) \leqq r_1(z_1) + r_1(z_2)$.

Consider $(0,a) = x$. Then $r_2(x) - f_2(x) = |a| - a \geqq 0$ and $= 0$ if and only if $a \geqq 0$. If $a \geqq 0$, then $r_1(0,a) - f_1(0,a) = 0$.

By the Hahn-Banach Theorem there exists a linear extension $F_2: Y \to R$ of f_2 which is dominated by r_2.

If

$$\gamma_2 = \inf\{r_2(1,a) - f_2(0,a) \,|\, a \in R\} \text{ and}$$

$$\delta_2 = \sup\{-r_2(-1,a) + f_2(0,a) \,|\, a \in R\},$$

then $\gamma_2 \geqq F_2(1,0) \geqq \delta_2$.

Direct computation shows that $\gamma_2 = 1 = r_2(1,a) - f_2(0,a)$, $a \geqq 0$, and $\gamma_2 = \delta_2$. Let $U = \{(1,a) \,|\, a \geqq 0\}$. The set U satisfies (4) of Lemma 5.2. Then, for any linear extension $F_1: Y \to R$ of f_1 with

$$F_1(1,0) = c, \lim_{a \to \infty} (r_1(1,a) - F_1(1,a)) = \lim_{a \to \infty} (-(a)^{1/2} - c) = -\infty.$$

The lemma is proved.

6. **Finite dimensional wedges with the HBEP.** Every 0-dimensional wedge (0) and 1-dimensional wedge (line through 0 or closed half-line with end-point 0) is lineally closed and has the LUBP and the HBEP. Also, if the positive wedge of an OLS V is all of V then, trivially, V has the HBEP, the LUBP, and is lineally closed.

For ordered linear spaces of higher dimension with proper wedges the situation is more complicated. The general result for finite dimensional wedges is Theorem 6.1.

THEOREM 6.1. *A finite dimensional wedge K has the HBEP if and only if K has the LUBP.*

As remarked in the introduction, Theorem 6.1 follows from already known results and

THEOREM 6.2. *If K is a finite dimensional wedge with the HBEP then K is lineally closed.*

For 2-dimensional wedges there is a stronger result.

COROLLARY 6.3. *A 2-dimensional wedge K has the HBEP if and only if K is lineally closed.*

The corollary follows from Theorems 6.1 and 6.2 upon the following observations. Every closed 2-dimensional wedge is isomorphic to one of the following three wedges and each of these wedges has the LUBP(3).

$K_1 = \{(a,b) \in R_2 \mid a \geqq 0, b \geqq 0\}$,

$K_2 = \{(a,b) \in R_2 \mid b \geqq 0\}$,

$K_3 = R_2$, where R_2 is the linear space of ordered pairs of real numbers.

Proof of 6.2. The proof proceeds in part by induction. Nonclosed 2-dimensional wedges play a special role.

By Theorem 4.1, if the finite dimensional wedge K has the HBEP then $\bar{K}$ has the LUBP. Further, K can be assumed to be a proper reproducing wedge in its containing OLS V, and can be assumed to have nonvoid interior. Moreover, by Theorem 3.1, there is a k-partly positive basis B of dimension n for $\bar{K}$, where $1 \leqq k \leqq n$.

Let $T = \{v \mid -v, v \in \bar{K}\}$. It is easily verified that T is a closed linear subspace of V. Assume $K \neq \bar{K}$ and K has the HBEP.

Case 1. $T = \{0\}$. There exists a positive basis B for $\bar{K}$ such that

$$v = \sum_{i=1}^{n} v_i b_i \in \bar{K} \Leftrightarrow v_i \geqq 0, \; i = 1, \cdots, n.$$

Since K is not closed, there is some b', say b_1, in B such that b_1 is not in K. Let Y, X, p_2, f_2, f_1 be as in Lemma 5.1, Example 1, §5. Define:

$$p\colon Y \to V, \; p(z) = p_2(z) \sum_{i=1}^{n} b_i, \qquad z \in Y;$$

$$f\colon X \to V, f(x) = f_1(x) b_1 + f_2(x) \sum_{i=2}^{n} b_i, \quad x \in X.$$

The function p is sublinear with respect to K, for $v_0 = \sum_{i=1}^{n} b_i$ is an interior point of K and p_2 is a sublinear functional. Also, $p(x) - f(x) \in K$, $x \in X$, for $p(x) - f(x) = (p_2(x) - f_1(x)) b_1 + (p_2(x) - f_2(x)) \sum_{i=2}^{n} b_i$. By Lemma 5.1, $p_2(x) - f_1(x) \geqq 0$, $p_2(x) - f_2(x) \geqq 0$, and $p_2(x) = f_1(x)$ if and only if $p_2(x) = f_2(x)$, $x \in X$. Also, $\sum_{i=1}^{n} a_i b_i$ is in the interior of K if $a_i > 0$, $i = 1, \cdots, n$.

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(3) It is also easily verified that every other 2-dimensional wedge is isomorphic to one of

$C_1 = \{(a,b) \in R_2 \mid a > 0, \; b > 0, \; \text{or} \; a = 0, \; b = 0\}$,

$C_2 = \{(a,b) \in R_2 \mid a \geqq 0, \; b > 0, \; \text{or} \; a = 0, \; b = 0\}$,

$C_3 = \{(a,b) \in R_2 \mid b > 0, \; \text{or} \; a = 0, \; b = 0\}$,

$C_4 = \{(a,b) \in R_2 \mid b > 0, \; \text{or} \; a \geqq 0, \; b = 0\}$.

The wedge C_4 determines a lattice, but this conditon is not sufficent to guarantee that C_4 has the HBEP.

There exists no linear extension $F\colon Y \to V$ of f so that $p(z) - F(z) \in K$, $z \in Y$. For assume F is a linear extension of f, $F(z) = F_1(z)b_1 + \sum_{i=2}^{n} F_{2_i}(z)b_i$. Then the coefficient functions are linear, F_1 is an extension of f_1 and F_{2_i} is an extension of f_2, $i = 2, \cdots, n$. Further assume that $p(z) - F(z) \in K$, then $p_2(z) - F_{2_i}(z) \geqq 0$, $i = 2, \cdots, n$, $z \in Y$. By Lemma 5.1, there is a set $U \subset Y$, so that $p_2(u) - F_{2_i}(u) = 0$, $u \in U$, $i = 2, \cdots, n$ and there is a $u_0 \in U$ so that $p_2(u_0) - F_1(u_0) \neq 0$. Thus, $p(u_0) - F(u_0) = (p_2(u_0) - F_1(u_0))b_1 \in K$. This is a contradiction since $-b_1, b_1 \notin K$ and $p_2(u_0) - F_1(u_0) \neq 0$.

Case 2. $T \neq \{0\}$. There are several subcases.

Case (2i) $T \cap K = \{0\}$. In this situation if $\sum_{i=1}^{n} v_i b_i = v \in K$ and $v_i = 0$, $i = 1, \cdots, k$, then $v_i = 0$, $i = k+1, \cdots, n$. Again let Y, X, p_2, f_1, f_2, be as in Lemma 5.1. Define $p\colon Y \to V$ by $p(z) = p_2(z) \sum_{i=1}^{n} b_i$, $z \in Y$. As in Case 1, p is sublinear with respect to K. Define $f\colon X \to V$, $f(x) = f_2(x)(\sum_{i=1}^{k} b_i) + f_1(x)(\sum_{i=k+1}^{n} b_i)$, $x \in X$. Again it is easily verified that $p(x) - f(x) \in K$, $x \in X$. Assume $F\colon Y \to V$ is a linear extension of f, $F(z) = \sum_{i=1}^{k} F_{2_i}(z)b_i + \sum_{i=k+1}^{n} F_{1_i}(z)b_i$, $z \in Y$. Thus, $p(z) - F(x) \in K$ implies that $p_2(z) - F_{2_i}(z) \geqq 0$, $z \in Y$, $i = 1, \cdots, k$. Again from Lemma 5.1, choose $u_0 \in U$, so that $p_1(u_0) - F_1(u_0) \neq 0$. Thus an extension is not possible and K does not have the HBEP.

Case (2 ii) Assume $T \subset K$. Again let Y, X, p_2, f_1, f_2 be as in Lemma 5.1. Again define $p\colon Y \to V$ by $p(z) = p_2(z) \sum_{i=1}^{n} b_i$, $z \in Y$. Since $T \subset K$ there is a $b_j \in B$, $1 \leqq j \leqq k$, say $j = 1$, so that $b_1 \notin K$. Define $f\colon X \to V$, $f(x) = f_1(x)b_1 + f_2(x) \sum_{i=2}^{n} b_i$. Again it is easily verified that p is sublinear with respect to K and $p(x) - f(x) \in K$, $x \in X$. Assume $F\colon Y \to V$ is a linear extension of f such that $p(z) - F(z) \in K$, $z \in Y$, and $F(z) = F_1(z)b_1 + \sum_{1=2}^{n} F_{2_i}(z)b_i$. Choosing $u_0 \in U$ so that $p_2(u_0) - F_1(u_0) \neq 0$, then one has clearly, $(p_2(u_0) - F_1(u_0))b_1 \in K$, since $\pm \sum_{i=k+1}^{n} (p_2(u_0) - F_{2_i}(u_0))b_i \in T$ and $p_2(u_0) = F_{2_i}(u_0)$, $i = 2, \cdots, k$. Thus, a contradiction is obtained in this case and K must be closed.

Case (2 iii) Suppose $T \cap K$ is not closed. Then by induction on dimension, K does not have the HBEP, for $T \cap K$ has dimension less than that of K and it will be shown that:

If K has the HBEP, then $T \cap K$ has the HBEP.

Let Y be a linear space, X a linear subspace of Y, $p\colon Y \to V$ a function sublinear with respect to $T \cap K$, $f\colon X \to V$ a linear function such that $p(x) - f(x) \in T \cap K$, $x \in X$. Then there is a linear extension $F\colon Y \to V$ of f such that $p(z) - F(z) \in K$, $z \in Y$. Thus, $p(\pm z) - F(\pm z) = \sum_{i=1}^{k} - F_i(\pm z)b_i + \sum_{i=k+1}^{n} (p_i(\pm z) - F_i(\pm z))b_i$ $z \in Y$, where the p_i's and F_i's are the coefficient functions of p and F respectively. The functions F_i are linear. Also if $v = \sum_{i=1}^{n} a_i b_i \in K$, then $a_i \geqq 0$, $i = 1, \cdots, k$. Thus, $F_i = 0$, $i = 1, \cdots, k$ and $F(z) \in T$, $z \in Y$, and $p(z) - F(z) \in T \cap K$, $z \in Y$.

Case (2iv). Suppose $T \cap K$ is closed, $T \not\subset K$, $T \cap K \neq \{0\}$, and for every $b \in B$, if $b \in T \cap K$ then $-b \in T \cap K$. It may be assumed that $b_n \in B \cap T \cap K$.

Let $B' = \{b \mid b \in B,\ b \neq b_n\}$ and let $K' = K \cap \text{lin } B'$. Then K' is not closed since $T \not\subset K$. Further, K' has dimension less than K. Then by induction and the following lemma, K does not have the HBEP. *The wedge K' has the HBEP if K has the HBEP.*

Let Y be a linear space, and X be a linear subspace of Y. Let $p\colon Y \to V$ be a sublinear function with respect to K' and $p(z) = \sum_{i=1}^{n-1} p_i(z)b_i$. Let $f\colon X \to V$, f linear, be so that $f(x) = \sum_{i=1}^{n-1} f_i(x)b_i$ and $p(x) - f(x) \in K'$, $x \in X$.

Then there is a linear extension $G\colon Y \to V$ of f, $G(z) = \sum_{i=1}^{n} G_i(z)b_i$ and $p(z) - G(z) \in K$, $z \in Y$. Thus, $p(z) - G(z) - G_n(z)b_n \in K \cap \text{lin } B' = K'$, $z \in Y$. Letting $F(z) = G(z) - G_n(z)b_n$, $z \in Y$, then F is the desired extension and K' has the HBEP.

Case (2 v). Suppose $T \cap K$ is closed, $T \not\subset K$, $T \cap K \neq \{0\}$ and there is a $b \in B$, say b_n, so that $b_n \in T \cap K$ and $-b_n \notin T \cap K$. This case breaks down into 2 subcases.

If $n = 2$, the only wedge which need be considered is $C_4 = \{(a, b) \in R_2 \mid b > 0$, or $a \geqq 0,\ b = 0\}$. (See comment at end of Footnote 3.)

Let Y be a linear space, X a linear subspace of Y. Then, $q\colon Y \to R_2$, $q(z) = (q_1(z),\ q_2(z))$, $z \in Y$, is sublinear with respect to C_4 if and only if q_2 is a sublinear functional, q_1 is positive homogeneous, and if $q_2(z_1) + q_2(z_2) = q_2(z_1 + z_2)$, then $q_1(z_1) + q_1(z_2) \geqq q_1(z_1 + z_2)$.

Let $f\colon X \to R_2$, $f(x) = (f_1(x), f_2(x))$, $x \in X$, f linear. Then f_1 and f_2 are linear functionals. For every $x \in X$, $q(x) - f(x) \in C_4$ if and only if $q_2(x) - f_2(x) \geqq 0$, and if $q_2(x) - f_2(x) = 0$, then $q_1(x) - f_1(x) \geqq 0$. Further, if $F\colon Y \to R_2$, $F(z) = (F_1(z), F_2(z))$, $z \in Y$, is a linear extension of f, then F_1 is a linear extension of f_1 and F_2 is a linear extension of f_2. Also, if $q(z) - F(z) \in C_4$, $z \in Y$, then $q(z) - F(z)$ must satisfy the same conditions as $q(x) - f(x)$ on replacing x by z and f by F.

Let Y, X, r_1, r_2, f_1, f_2 be as in §5, Example 2, Lemma 5.2, and $q_1 = r_1$, $q_2 = r_2$, $f(x) = (f_1(x), f_2(x))$, $x \in X$. Then by Lemma 5.2, q is sublinear with respect to C_4, $q(x) - f(x) \in C_4$, $x \in X$; but by Lemma 5.2 no linear extension $F\colon Y \to R_2$ of f satisfies the condition $q(z) - F(z) \in C_4$, $z \in Y$.

If $n > 2$, let $B' = \{b \in B \mid b \neq b_1\}$ and let $K' = K \cap \text{lin } B'$. Then clearly, K' has dimension less than K. Also K' is not closed since $T \not\subset K$ and the dimension n is greater than 2. Then by induction and the following lemma, K does not have the HBEP. *The wedge K' has the HBEP if K has the HBEP.*

Let Y be a linear space; X a linear subspace of Y; $p\colon Y \to V$ a function sublinear with respect to K', so that $p(z) = \sum_{i=2}^{n} p_i(z)b_i$; $f\colon X \to V$, f linear, so that $f(x) = \sum_{i=2}^{n} f_i(x)b_i$; and $p(x) - f(x) \in K'$, $x \in X$. Then there is a linear extension $F\colon Y \to V$ of f, $F(z) = \sum_{i=1}^{n} F_i(z)b_i$, and $p(z) - F(z) \in K$, $z \in Y$. Thus, $p(\pm z) - F(\pm z) = -F_1(\pm z)b_1 + \sum_{i=2}^{n}(p_i(\pm z) - F_i(\pm z))\, b_i$. Since F_1 is linear and $v = \sum_{i=1}^{n} v_i b_i \in K$ implies $v_1 \geqq 0$, $F_1 = 0$. Thus $F(z) \in K'$ and $p(z) - F(z) \in K'$, $z \in Y$. Theorems 6.1 and 6.2 are proved.

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