GENERALIZATIONS OF THE RIEMANN DERIVATIVE([1])

BY

J. MARSHALL ASH

Introduction. In §1 of this paper a derivative generalizing the Riemann derivative is considered. The existence of this derivative on a set is shown to imply the existence of the Peano derivative almost everywhere on the set. In §2 the L^p norm $(1 \leqq p < \infty)$ replaces the L^∞ norm of §1 and the same result is proved. A special case of this result is that the existence of the Riemann L^p derivative implies the existence of the Peano L^p derivative almost everywhere. In §3 a generalization of smoothness is shown to imply smoothness almost everywhere. We consider only measurable sets of real numbers and real valued functions of a real variable.

1. **An L^∞ generalization of the Riemann derivative.** A function f is said to have a Peano derivative of order k at x, i.e., $f \in t_k(x)$, if there are constants $f_0(x)$, $f_1(x), \ldots, f_k(x)$ such that

$$f(x+t) = f_0(x)+f_1(x)t+\cdots+\frac{f_k(x)}{k!}t^k+o(t^k) \qquad \text{as } t \to 0.$$

We say f is Peano bounded of order k at x, i.e., $f \in T_k(x)$, if there are constants $f_0(x), \ldots, f_{k-1}(x)$ such that

$$f(x+t) = f_0(x)+f_1(x)t+\cdots+\frac{f_{k-1}(x)}{(k-1)!}t^{k-1}+O(t^k) \qquad \text{as } t \to 0.$$

Let $A = \{a_0, a_1, \ldots, a_{k+l}; A_0, \ldots, A_{k+l}\}$ be a set of real numbers with $a_i \neq a_j$ if $i \neq j$ satisfying

$$\begin{aligned} \sum_{i=0}^{k+l} A_i a_i^j &= 0, \qquad j = 0, 1, \ldots, k-1, \\ &= k!, \qquad j = k. \end{aligned}$$

We say that f has a kth generalized derivative with respect to A at the point x, i.e., $f \in g_k(x, A)$, if there is a constant $f_{(k)}(x) = f_{(k)}(x, A)$ such that

$$\sum_{i=0}^{k+l} A_i f(x+a_i t) = f_{(k)}(x)t^k + o(t^k) \qquad \text{as } t \to 0.$$

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A function f is generalized-bounded of order k with respect to A at x, i.e., $f \in G_k(x, A)$, if

$$\sum_{i=0}^{k+l} A_i f(x+a_i t) = O(t^k) \qquad \text{as } t \to 0.$$

To demonstrate the reason for the conditions on $\sum_{i=0}^{k+l} A_i a_i^j$, let $f \in t_k(x)$. Then

$$\begin{aligned}\sum_{i=0}^{k+l} A_i f(x+a_i t) &= \sum_{i=0}^{k+l} A_i \left[\sum_{j=0}^{k} \frac{f_j(x)}{j!}(a_i t)^j + o(t^k)\right] \\ &= \sum_{j=0}^{k} \frac{f_j(x)}{j!} t^j \left[\sum_{i=0}^{k+l} A_i a_i^j\right] + o(t^k) \\ &= f_k(x) t^k + o(t^k) \qquad \text{as } t \to 0.\end{aligned}$$

In other words, the conditions assure that if the Peano derivative exists, the generalized derivative will exist and be equal to it.

If $l=0$ and if the a_i's are given, since the $k+1$ A_i's must satisfy the $k+1$ conditions, and since the matrix $((a_i^j))$ is a Van der Monde matrix and hence invertible, it follows that the A_i's can be expressed in terms of the a_i's. To be precise (see Denjoy [1])

$$A_i = \left[\prod_{j \neq i} (a_i - a_j)\right]^{-1} \cdot k! \qquad i = 0, 1, \ldots, k.$$

If, on the other hand, $l > 0$, the a_i's and the $k+1$ conditions do not uniquely determine the A_i's. l will be called the *excess*.

Probably the most important example of the generalized derivative is the Riemann derivative. The kth Riemann derivative is obtained by setting

$$a_i = -\frac{k}{2} + i, \qquad i = 0, 1, \ldots, k.$$

Since $l=0$, we find that

$$A_i = \left(\prod_{j \neq i} \left[\left(-\frac{k}{2}+i\right) - \left(-\frac{k}{2}+j\right)\right]\right)^{-1} k! = \binom{k}{i}(-1)^{k-i}.$$

The relationships between the various derivatives which have been introduced may be displayed diagrammatically as at top of p. 183.

The arrows denote inclusion. For example, if $f \in t_k(x)$, then $f \in T_k(x)$, so that $t_k(x) \subseteq T_k(x)$. As may be shown by simple counterexample, none of the arrows may be reversed.

However, there is a classical theorem of Zygmund and Marcinkiewicz which states that if a function is k Riemann-bounded on a set, then at almost every point of that set it has a kth Peano derivative [2]. This may be generalized to

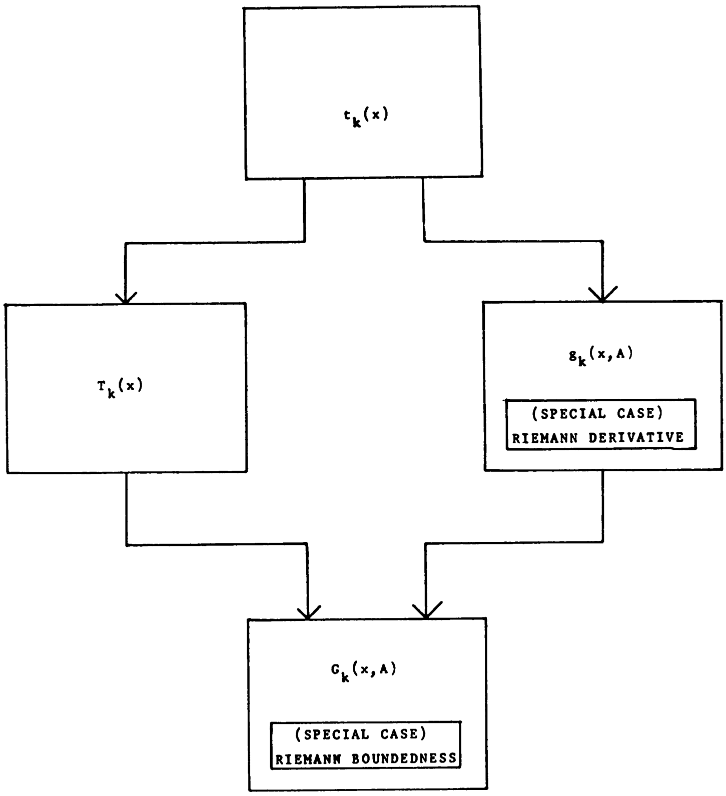

FIGURE 1 (Relationships between derivatives).

THEOREM 1. *Let $f \in G_k(x, A)$ for all $x \in E$, then $f \in t_k(x)$ for almost every $x \in E$.*

(The case $k=1$, $l=0$ is classical. The case $k=2$, $l=0$ was done by Marcinkiewicz and Zygmund [3].)

LEMMA 1. *Let 0 be a point of density of $\mathscr{E}$. Let $\{\alpha_i, \beta_i\}$, $i=1, \ldots, m$ be any set of real numbers such that $\beta_i \neq 0$ for all i. Then for all $u>0$ sufficiently small, there is a $v \in [u, 2u]$ such that*

$$\alpha_i u+\beta_i v \in \mathscr{E}, \qquad i = 1, \ldots, m.$$

Proof. Let φ be the characteristic function of $\mathscr{E}$. Let

$$A_i = \{v \in [u, 2u] \mid \alpha_i u+\beta_i v \in \mathscr{E}\}.$$

For each i,

$$|A_i| = \int_u^{2u} \varphi(\alpha_i u+\beta_i v)\, dv = \frac{1}{\beta_i}\int_{(\alpha_i+\beta_i)u}^{(\alpha_i+2\beta_i)u} \varphi(s)\, ds$$

$$\to \frac{1}{\beta_i}[(\alpha_i+2\beta_i)u-(\alpha_i+\beta_i)u] = u \qquad \text{as } u \to +0$$

since 0 is a point of density of $\mathscr{E}$. Hence by choosing u sufficiently small we can make

$$|A_i| > \left(1-\frac{1}{m}\right)u, \qquad i = 1, 2, \ldots, m.$$

For any sets A and B, let $A-B$ denote all the points of A that do not belong to B. Then

$$\begin{aligned}\left|[u, 2u]-\bigcap_{i=1}^{m} A_i\right| &= \left|\bigcup_{i=1}^{m} ([u, 2u]-A_i)\right| \\ &\leqq \sum_{i=1}^{m} |[u, 2u]-A_i| < \sum \frac{1}{m} u = u\end{aligned}$$

so that

$$[u, 2u] \cap \left(\bigcap_{i=1}^{m} A_i\right) \neq \varnothing$$

and any v of this set will satisfy the conclusion of the lemma.

LEMMA 2. *The sliding lemma: Suppose that* $\alpha \geqq 0$, $n \geqq 1$ *and*

$$\sum_{i=0}^{n} A_i f(x+a_i t) = O(t^{\alpha}) \qquad \textit{for all } x \in E,$$

then for any real a,

$$\sum_{i=0}^{n} A_i f(x+(a_i-a)t) = O(t^{\alpha}) \qquad \textit{for almost every } x \in E.$$

If "O" is replaced by "o" in the hypothesis, then the conclusion also holds with "o" in place of "O".

Proof. We may assume $a_0 \neq 0$ by reordering the terms if necessary. We may assume that $0<|E|<\infty$. Let $E_j = \{x \in E \mid |\sum_{i=0}^{n} A_i f(x+a_i t)| \leqq j|t|^{\alpha}$ if $|t|<1/j\}$. Since $|E-E_j| \to 0$, it suffices to prove the lemma at every point of E_j which is a point of density of E_j. To simplify notation, let $x=0$ be such a point. Let t be greater than 0 (the case $t<0$ is proved similarly). By Lemma 1, if t is sufficiently small, there is a $u \in [t, 2t]$ such that all of

$$(a_i-a)t-a_0 u, \qquad i = 0, 1, \ldots, n$$

and

$$-at+(a_k-a_0)u, \qquad k = 1, \ldots, n$$

belong to E_j. Since $(a_i-a)t-a_0 u \in E_j$, we have

$$\sum_{k=0}^{n} A_k f[\{(a_i-a)t-a_0 u\}+a_k u] = O(u^{\alpha}) = O(t^{\alpha}), \qquad i = 0, 1, \ldots, n.$$

Multiplying the ith equation by A_i and summing over i, we have

$$\sum_{i=0}^{n} A_i\left[\sum_{k=0}^{n} A_k f[\{(a_i-a)t-a_0u\}+a_ku]\right] = O(t^{\alpha}).$$

Rearranging the order of summation, we obtain

$$\sum_{k=0}^{n} A_k\left[\sum_{i=0}^{n} A_i f[\{-at+(a_k-a_0)u\}+a_it]\right] = O(t^{\alpha}).$$

Since the term in curly brackets is in E_j for $k>0$, each term of the outer sum except the $k=0$ term is $O(t^{\alpha})$. Hence that term is also $O(t^{\alpha})$, i.e.,

$$A_0\sum_{i=0}^{n} A_i f[\{-at+(a_0-a_0)u\}+a_it] = O(t^{\alpha}).$$

Dividing by A_0 and simplifying, we have

$$\sum_{i=0}^{n} A_i f[(a_i-a)t] = O(t^{\alpha}) \qquad \text{as } t\to +0,$$

which is the desired result. The "o" case is proved in a very similar manner.

LEMMA 3. *If* $|\sum_{i=0}^{n} A_i f(x+a_it)|=O(1)$ *for all* $x\in E$, *then* f *is bounded in a neighborhood of almost every point* $x\in E$.

We omit the proof which is similar to that of Lemma 5 on page 13 of [2].

Because of Lemma 3, without loss of generality we may add the assumption that f is bounded to the hypotheses of Theorem 1. Since f is bounded it is locally integrable; so we may define $D^{-i}f$ by

$$D^0f(x) = f(x),\ D^{-1}f(x) = \int_a^x f(t)\,dt,\ldots, D^{-i}f(x) = \int_a^x D^{-(i-1)}f(t)\,dt.$$

We come now to the cornerstone of the proof of Theorem 1.

LEMMA 4. *Suppose that* f *is bounded and*

$$\sum_{i=0}^{k+l} A_i f(x+a_it) = O(t^k) \qquad \text{as } t\to 0 \text{ for all } x\in E.$$

Then there is an integer $s\geqq 0$ *such that* $D^{-s}f$ *is* $(k+s)$ *Riemann-bounded at almost every* $x\in E$.

Proof. First let us suppose that all the a_i's are integers and that $l=0$. We may suppose that $a_0<a_1<\cdots<a_k$. From the sliding lemma it follows that we may assume $a_0=1$. If there are no gaps in the sequence $\{a_0,\ldots,a_k\}$, i.e., if $a_0=1$, $a_1=2$, $a_2=3,\ldots, a_k=k+1$, after sliding the sequence to the left by $k/2+1$, we deduce from the remarks preceding Figure 1 that f is k Riemann-bounded almost everywhere on E. If there are gaps, we fill them by integration. For example, if

$a_1 > 2$, we adjoin 2 to the set of a_i's as follows. Sliding the original derivative to the left by 2, we obtain

$$\sum_{i=0}^{k} A_i f(x+(a_i-2)t) = O(t^k) \qquad \text{for almost every } x \in E.$$

Now integrating from 0 to h where h is small, we obtain

$$\sum_{i=0}^{k} \frac{A_i}{a_i-2} D^{-1}f(x+(a_i-2)h) - \left[\sum_{i=0}^{k} \frac{A_i}{a_i-2}\right] D^{-1}f(x) = O(h^{k+1})$$

for almost every x in E.

Finally sliding this to the right by 2, we obtain

$$\sum_{i=0}^{k} \frac{A_i}{a_i-2} D^{-1}f(x+a_i h) - \left[\sum_{i=0}^{k} \frac{A_i}{a_i-2}\right] D^{-1}f(x+2h) = O(h^{k+1})$$

for almost every $x \in E$. This result shows that almost everywhere on E, $D^{-1}f$ is $k+1$ generalized-bounded with respect to a set whose a_i's have one fewer gap than had the original set of a_i's(²). Note that the excess is still 0. If $a_k = k+1+s$, there were s gaps initially, so after repeating this filling process $s-1$ more times, we obtain the conclusion of the lemma.

Next we suppose that the a_i's are integers, but that $l > 0$. Fix l. It suffices to show that for some positive integer s_1, $D^{-s_1}f$ is $k+s_1$ generalized-bounded with respect to a $(k+s_1)$th generalized derivative of excess $\leqq l-1$. For if we can do this, an at most l-fold iteration of the process will reduce this case to the $l=0$ case above.

By employing the process of filling in the gaps, we may suppose that $a_0 = 1$, $a_1 = 2, \ldots, a_{k+l} = k+l+1$. It is important to note that the process of filling never increases the excess since at each step the order of the derivative is increased by one, while the number of a_i's is increased by at most one. (The process of filling may actually decrease the excess. For example, if $a_1 > 2$ and $\sum_{i=0}^{k+l} A_i/(a_i-2) = 0$, then after sliding to the left by 2, we have a $(k+1)$st derivative based on the original $k+l+1$ a_i's so that the excess is immediately $l-1$.)

Set $r = k+l+1$. Recalling that we may suppose $a_{i-1} = i$, $1 \leqq i \leqq r$, we may now write our assumption

$$\sum_{i=1}^{r} A_{i-1} f(x+it) = O(t^k) \qquad \text{as } t \to 0, \tag{1.1}$$

for all $x \in E$.

(²) If $\sum_{i=0}^{k+l} (A_i a_i^j)$ is equal to 0 when $j < k$, is equal to $k!$ when $j = k$, then $\sum_{i=0}^{k+l} A_i(a_i - a)^j = \sum_{i=0}^{k+l} (A_i a_i^j)$ so that any slide of a kth generalized derivative is still a kth generalized derivative. Also if no $a_i = 0$, $\sum_{i=0}^{k+l} (A_i/a_i)a_i^j$ is equal to $\sum_{i=0}^{k+l} (A_i/a_i)$ when $j = 0$, is equal to 0 when $j = 1, 2, \ldots, k$ and equals $k!$ when $j = k+1$ so that integration from 0 to h yields a $(k+1)$st generalized derivative. To obtain proper normalization, each integration should be coupled with a multiplication by the constant $k+1$. We shall always assume that this has been done.

Sliding this to the left by $r+1$, we have

$$\sum_{i=1}^{r} A_{i-1}f(x+[i-(r+1)]t) = O(t^k) \quad \text{for almost every } x \in E. \tag{1.2}$$

Integrating (1.1) and (1.2), we obtain

$$\sum_{i=1}^{r} \frac{A_{i-1}}{i} D^{-1}f(x+it) - \left[\sum_{i=1}^{r} \frac{A_{i-1}}{i}\right] D^{-1}f(x) = O(t^{k+1}) \tag{1.3}$$

for almost every $x \in E$; and

$$\sum_{j=1}^{r} \frac{A_{j-1}}{j-(r+1)} D^{-1}f(x+[j-(r+1)]t) - \left[\sum_{j=1}^{r} \frac{A_{j-1}}{j-(r+1)}\right] D^{-1}f(x) = O(t^{k+1}) \tag{1.4}$$

for almost every $x \in E$. Sliding equation (1.4) to the right by r and changing indices by setting $i=j-1$, we have

$$\sum_{i=0}^{r-1} \frac{A_i}{i-r} D^{-1}f(x+it) - \left[\sum_{i=0}^{r-1} \frac{A_i}{i-r}\right] D^{-1}f(x+rt) = O(t^{k+1}) \tag{1.5}$$

for almost every $x \in E$.

We now show that the coefficients of (1.3) and (1.5) are not proportional. If the derivatives in (1.3) and (1.5) have been suitably normalized (see footnote (3)), then when they are tested on the function $g(x)=x^{k+1}/(k+1)!$, both are identically equal to t^{k+1}. Hence if the coefficients of the derivatives in (1.3) and (1.5) are proportional, they must be equal. Suppose this is the case. Equating the coefficients of $f(x+it)$, $i=1, 2, \ldots, r-1$, we have

$$\frac{A_{i-1}}{i} = \frac{A_i}{i-r}, \qquad i = 1, \ldots, r-1,$$

which yields recursively,

$$A_1 = \frac{1-r}{1} A_0 = (-1)\binom{r-1}{1} A_0,$$

$$A_2 = \frac{2-r}{2}\cdot\frac{1-r}{1} A_0 = (-1)^2\binom{r-1}{2} A_0,$$

$$\vdots$$

$$A_{r-1} = \frac{(r-1)-r}{(r-1)}\cdot\frac{(r-2)-r}{r-2}\cdots\frac{1-r}{1} A_0 = (-1)^{r-1}\binom{r-1}{r-1} A_0.$$

But $r-1=k+l>k$, so

$$\sum_{i=1}^{r} A_{i-1}i^k = A_0 \sum_{i=0}^{r-1} (-1)^i\binom{r-1}{i} i^k = 0$$

contrary to the assumption that (1.1) is a kth generalized derivative.

Hence the derivatives in (1.3) and (1.5) do not have all coefficients equal. Therefore there is an $i_0 \in \{0, 1, \ldots, r\}$ such that the coefficients of $f(x+i_0t)$, call them

a and b, are unequal. Set $\alpha=b(b-a)^{-1}$ and $\beta=-a(b-a)^{-1}$ and consider the derivative formed by adding the derivative in (1.3) multiplied by α to the derivative in (1.5) multiplied by β. Since $\alpha+\beta=1$, this is a $(k+1)$th generalized derivative which is properly normalized, as can be seen by testing it on g. Since $\alpha a+\beta b=0$, the coefficient of $f(x+i_0t)$ is equal to zero, so that this derivative is based on the set $B=\{1, 2, \ldots, i_0-1, i_0+1, \ldots, k+l+1\}$ and hence has excess $\leqq l-1$. Then from (1.3) and (1.5) we note that $D^{-1}f \in G_{k+1}(x, B)$ for almost every $x \in E$.

Finally, let the a_i's be arbitrary. Lemma 4 will be proved if we can show that there is a $C=\{C_0, \ldots, C_{k+l+s_2}; c_0, \ldots, c_{k+l+s_2}\}$ such that all the c_i's are integers and $D^{-s_2}f \in G_{k+s_2}(x, C)$ for almost every $x \in E$. Our hypothesis is that

$$\sum_{i=0}^{k+l} A_i f(x+a_i t) = O(t^k) \qquad \text{for all } x \in E. \tag{1.6}$$

Let $M \subset \{a_0, \ldots, a_{k+l}\}$ be a commensurable set, i.e., there is a real number q such that mq is an integer for every $m \in M$. Let M be of maximal cardinality, i.e., if $N \subset \{a_0, \ldots, a_{k+l}\}$ is a commensurable set, then N has no more elements than does M. Replacing t by qt, we may assume without loss of generality, that all the elements of M are integers. By the sliding lemma we may assume that no $a_i=0$. Let n be any integer $\notin \{0, a_0, \ldots, a_{k+l}\}$. Integrating equation (1.6) we have

$$\sum_{i=0}^{k+l} \frac{A_i}{a_i} D^{-1}f(x+a_i t) - \left[\sum_{i=0}^{k+l} \frac{A_i}{a_i}\right] D^{-1}f(x) = O(t^{k+1}) \tag{1.7}$$

for all $x \in E$. From equation (1.6) and the sliding lemma, we have

$$\sum_{i=0}^{k+l} A_i f(x+[a_i-n]t) = O(t^k)$$

for almost every $x \in E$. Integrating this and then sliding the result back to the right by n, we have

$$\sum_{i=0}^{k+l} \frac{A_i}{a_i-n} D^{-1}f(x+a_i t) - \left[\sum_{i=0}^{k+l} \frac{A_i}{a_i-n}\right] D^{-1}f(x+nt) = O(t^{k+1}) \tag{1.8}$$

for almost every $x \in E$.

If $M=\{a_0, \ldots, a_{k+l}\}$, the conclusion is immediate with $s_2=0$, $c_i=a_i$. If not, we pick $a_j \in \{a_0, \ldots, a_{k+l}\}$ such that $a_j \notin M$. Since $n \neq 0$, we may set $\gamma=a_j/n$ and $\delta=(n-a_j)/n$. We find that $\gamma+\delta=1$ and $(A_j/a_j)\gamma+(A_j/(a_j-n))\delta=0$. Therefore if we add the derivative in (1.7) multiplied by γ to the derivative in (1.8) multiplied by δ, by the argument preceding (1.6), the resultant $(k+1)$th derivative is normalized, and has zero for the coefficient of $D^{-1}f(x+a_jt)$. Furthermore $D^{-1}f$ is $(k+1)$ generalized-bounded with respect to this derivative at almost every $x \in E$. If

$$\{a_0, a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_{k+l}, 0, n\}$$

is a commensurable set, the conclusion has been reached with $s_2=1$. If not, we pick $a_j' \in \{a_0, \ldots, a_{k+l}\}$ such that $a_j' \notin M \cup \{a_j\}$ and repeat the argument. At least one element of the set of a_i's that are not members of M is removed at each step, so we obtain the desired result with $s_2 \leqq$ the cardinality of this set.

Proof of Theorem 1. From Lemma 4 we have that for some integer s, $D^{-s}f$ is $k+s$ Riemann-bounded for almost every $x \in E$. From Theorem 1 of [2] it follows that $D^{-s}f \in t_{k+s}(x)$ for almost every $x \in E$.

Since $D^{-s}f \in t_{k+s}(x)$ for almost every $x \in E$, there is a perfect set $\Pi \subset E$ of measure arbitrarily close to E and there are functions F_1 and F_2 such that

(1) $D^{-s}f = F_1 + F_2$,

(2) F_1 has $k+s$ ordinary continuous derivatives on E,

(3) $F_2 = 0$ on Π.

See [4] for this decomposition. Define D^i to be the ordinary ith derivative. From (2) it follows that if we set $f_1 = D^s F_1$, then f_1 has k continuous derivatives on E. Almost everywhere on E, $D^s D^{-s} f$ exists and is equal to f. Hence

$$D^s F_2 = D^s[D^{-s}f - F_1]$$

exists and is equal to $f - f_1$ almost everywhere on E. Set $f_2 = D^s F_2$. Since Π is perfect,

$$D^1F_2(x) = \lim_{t\to 0} \frac{F_2(x+t)-F_2(x)}{t} = \lim_{x+t_n \in \Pi; t_n \to 0} \frac{F_2(x+t_n)-F_2(x)}{t_n} = 0,$$

$$D^2F_2(x) = D^1[D^1F_2(x)] = 0,$$

and

$$\cdots$$

$$D^sF_2(x) = D[D^{s-1}F_2(x)] = 0$$

almost everywhere on Π. Since f and $f_1 \in G_k(x, A)$ for almost every $x \in E$, $f_2 \in G_k(x, A)$ for almost every $x \in E$. It suffices to prove that $f_2 \in t_k(x)$ for almost every $x \in \Pi$. Lemma 7 of [2] states that if a function belongs to $T_k(x)$ on a set, then almost everywhere on that set the function belongs to $t_k(x)$. Hence Theorem 1 is proved if we can show that $f_2(x+t) = O(t^k)$ when x is a point of density of Π. We may assume that

$$\left| \sum_{i=0}^{k+l} A_i f_2(x + a_i t) \right| \leqq M|t|^k \quad \text{if} \quad |t| < \delta \text{ for all } x \in \Pi.$$

Let 0 be a point of density of Π. By Lemma 1, if $t > 0$ is sufficiently small, there is a $u \in [t, 2t]$ such that all of the points

$$t - a_0 u,$$
$$t + (a_i - a_0)u, \qquad i = 1, \ldots, k+l,$$

belong to Π (if $a_0 = 0$, reorder the first two terms). Then

$$f_2(t + (a_i - a_0)u) = 0, \qquad i = 1, 2, \ldots, k+l,$$

so that

$$|A_0 f_2(t)| = \left| \sum_{i=0}^{k+l} A_i f_2([t - a_0 u] + a_i u) \right| \leqq M u^k \leqq 2^k M t^k$$

if t is sufficiently small. This shows that $f_2 \in T_k(0)$ and completes the proof of Theorem 1.

REMARK. We may strengthen Theorem 1 by weakening the hypothesis to

$$\sum_{i=0}^{k+l} A_i f(x+a_i t) = O(t^k) \qquad \text{as } t \to +0 \text{ for all } x \in E$$

while still obtaining the same conclusion that $f \in t_k(x)$ for almost every $x \in E$.

The proof of Theorem 1 may be followed line for line with the following exceptions. We must invoke Theorem 7 of [2] instead of Theorem 1 of that paper. Theorem 7 states that if f is k Riemann-bounded *as* $t \to +0$ at each x of E, then $f \in t_k(x)$ for almost every $x \in E$. Also the reference to Lemma 7 of [2] must be replaced by one to Theorem 8 of that paper which states that if f is Peano-bounded of order k *as* $t \to +0$ for all $x \in E$, then $f \in t_k(x)$ for almost every $x \in E$.

2. **An L^p generalization of the Riemann derivative.** Now let $1 \leqq p < \infty$ and let $f \in L^p[x-\varepsilon, x+\varepsilon]$ for some $\varepsilon > 0$. We may extend all of the definitions of derivatives given in §1 to definitions of derivatives in L^p.

A function f is said to have at x a kth Peano derivative in L^p, i.e., $f \in t_k^p(x)$ if there are constants $f_0(x), \ldots, f_k(x)$ such that

$$\left(\frac{1}{h}\int_0^h \left| f(x+t) - \left\{ f_0(x) + \cdots + \frac{f_k(x)}{k!} t^k \right\}\right|^p dt\right)^{1/p} = o(h^k) \qquad \text{as } h \to 0.$$

Let $A = \{A_0, \ldots, A_{k+l}; a_0, \ldots, a_{k+l}\}$ be such that

$$\begin{aligned} \sum_{i=0}^{k+l} A_i a_i^j &= 0, \qquad j = 0, 1, \ldots, k-1, \\ &= k!, \qquad j = k. \end{aligned}$$

We say that f is k-generalized-bounded in L^p with respect to A at x, i.e., $f \in G_k^p(x, A)$, if

$$\left(\frac{1}{h}\int_0^h \left|\sum_{i=0}^{k+l} A_i f(x+a_i t)\right|^p dt\right)^{1/p} = O(h^k) \qquad \text{as } h \to 0.$$

The classes $T_k^p(x)$ and $g_k^p(x, A)$ are also defined by replacing the L^∞ norm of §1 by the L^p norm. As in §1, if $f \in G_k^p(x, A)$ in the special case when

$$A = \left\{(-1)^k\binom{k}{0}, \ldots, (-1)^0\binom{k}{k}; -\frac{k}{2}, \ldots, \frac{k}{2}\right\},$$

f is said to be k-Riemann-bounded in L^p at x. All the relations depicted in Figure 1 are still valid if the superscript p is attached to the name of each class. Parallel to Theorem 1 we have

THEOREM 2. *If $f \in G_k^p(x, A)$ for all $x \in E$, then $f \in t_k^p(x)$ for almost every $x \in E$.*

Mary Weiss has proved that if f has a kth symmetric L^p derivative for all $x \in E$, then $f \in t_k^p(x)$ for almost every $x \in E$ [5]. Since the existence of the kth symmetric

L^p derivative at x implies the existence of the kth Riemann L^p derivative at x, her result is contained in Theorem 2.

Proof. If $g \in L^p$, by Hölder's inequality we have

$$\left|\frac{1}{h}\int_0^h g(t)\,dt\right| \leqq \frac{1}{h}\int_0^h |g(t)|\,dt \leqq \left(\frac{1}{h}\int_0^h |g(t)|^p\,dt\right)^{1/p}.$$

Hence from the hypothesis of Theorem 2 we may deduce that

$$\frac{1}{h}\int_0^h \sum_{i=0}^{k+l} A_i f(x+a_i t)\,dt = O(h^k) \qquad \text{at } h \to 0 \tag{2.1}$$

for every $x \in E$. If some $a_i=0$, say $a_0=0$, then

$$\frac{1}{2^k-1}\sum_{i=0}^{k+l} A_i[f(x+2a_i t)-f(x+a_i t)]$$

still is a kth generalized derivative, since

$$\begin{aligned}\frac{1}{2^k-1}\sum_{i=0}^{k+l} A_i a_i^j(2^j-1) &= 0, \qquad j<k,\\ &= k!, \qquad j=k.\end{aligned}$$

Further,

$$\begin{aligned}\frac{1}{h}\int_0^h \sum_{i=0}^{k+l} A_i f(x+2a_i t)\,dt &= \frac{1}{2h}\int_0^{2h} \sum_{i=0}^{k+l} A_i f(x+a_i s)\,ds\\ &= O((2h)^k) = O(h^k) \qquad \text{as } h\to 0\end{aligned}$$

for every $x \in E$. Hence we may assume that (2.1) holds with no $a_i=0$.

Multiplying (2.1) by h and performing the integration, we have

$$\sum_{i=0}^{k+l}\frac{A_i}{a_i}D^{-1}f(x+a_i h)-\sum_{i=0}^{k+l}\frac{A_i}{a_i}D^{-1}f(x) = O(h^{k+1})$$

as $h \to 0$ for every $x \in E$. Since

$$\sum_{i=0}^{k+l}\frac{A_i}{a_i}-\sum_{i=0}^{k+l}\frac{A_i}{a_i}=0$$

and

$$\begin{aligned}\sum_{i=0}^{k+l}\left(\frac{A_i}{a_i}\right)a_i^j &= 0, \qquad 1\leqq j\leqq k,\\ &= k!, \qquad j=k+1,\end{aligned}$$

after multiplication by $k+1$ we have that $D^{-1}f$ is $(k+1)$ generalized-bounded with respect to

$$\left\{-\sum_{i=0}^{k+l}\frac{A_i}{a_i}, \frac{A_0}{a_0},\dots,\frac{A_{k+l}}{a_{k+l}};\ 0, a_0,\dots,a_{k+l}\right\}$$

at every $x \in E$. By Theorem 1, $D^{-1}f \in t_{k+1}(x)$ for almost every $x \in E$.

As in the proof of Theorem 1, set $D^{-1}f=G+L$ where G is $k+1$ times continuously differentiable throughout the domain of $D^{-1}f$ and where $L=0$ on an arbitrarily large perfect subset Π of E. Then on E (assuming we have removed the points where $D^1D^{-1}f\neq f$), $D^{-1}f$ and G are both differentiable and hence so is $L=D^{-1}f-G$. In particular, L is differentiable on Π and since Π is perfect, $D^1L=l=0$ on Π. Also $f\in G_k^p(x, A)$ and $D^1G\in t_k(x)\subseteq G_k^p(x, A)$ if $x\in\Pi\subseteq E$, so that $l\in G_k^p(x, A)$ for every point of Π.

It suffices to prove that l, and hence f, belongs to $t_k^p(x)$ for almost every $x\in\Pi$. To do this, it suffices to prove that $l\in T_k^p(x)$ and that $l_i(x)=0$, $i=0, 1, \ldots, k-1$, at each point of density of Π. For from this result, by Theorem 10 of [6], it follows that $l\in t_k^p(x)$ at almost every x in Π. We collect what remains to be proved of Theorem 2 into a lemma.

LEMMA 5. *Let $l(x)=0$ on E, $|E|>0$, $l\in L^p$, and*

$$\int_0^h \left|\sum_{i=0}^n A_i l(x+a_i t)\right|^p dt = O(h^\alpha) \qquad \textit{as } h\to 0$$

for all $x\in E$, where $\alpha>1$, $p\geqq 1$ (we actually only need the case of $\alpha=kp+1$). Then

$$\int_{-h}^h |l(x+t)|^p\, dt = O(h^\alpha) \qquad \textit{as } h\to 0$$

for almost every $x\in E$.

(The proof follows that of a similar lemma in [5].)

Proof. As in the proof of Lemma 2, without loss of generality we may assume that

$$\int_0^h \left|\sum_{i=0}^n A_i l(x+a_i t)\right|^p dt \leqq M|h|^\alpha \qquad \text{if } |h|<\delta \tag{2.2}$$

for all $x\in E$. It suffices to prove this lemma for each point of density of E. Let $x=0$ be such a point. We must show

$$\int_{-h}^h |l(t)|^p\, dt = O(h^\alpha) \qquad \text{as } h\to 0. \tag{2.3}$$

Assume that $a_0\neq 0$ in (2.2) (if it does, reorder). If $a_0\neq 1$, we set $s=a_0t$, and divide by $|A_0|$:

$$\left|\int_0^{a_0h} \left| l(x+s)+\sum_{i=1}^n A_0^{-1}A_i l\left(x+s\frac{a_i}{a_0}\right)\right|^p ds\right| \leqq \frac{M}{|A_0a_0^{\alpha-1}|}|a_0h|^\alpha$$

if $|h|<\delta$, i.e., if $|a_0h|<|a_0|\delta$. Hence we may assume that $a_0=A_0=1$ in (2.2).

Either

$$\int_0^h |l(t)|^p\, dt \geqq \frac{1}{2}\int_{-h}^h |l(t)|^p\, dt \tag{2.4}$$

or

$$\int_0^h |l(t)|^p\, dt < \frac{1}{2}\int_{-h}^h |l(t)|^p\, dt.$$

Suppose that the former holds (the argument is essentially the same if the latter holds). For definiteness, assume $h>0$.

Let F be the complement of E. Pick $x\in[-h/2,0]\cap E$ (this can be done if h is sufficiently small since 0 is a point of density).

$$\tag{2.5} \begin{aligned} \int_0^h |l(t)|^p\,dt &\leqq \int_0^{h-x} |l(x+t)|^p\,dt \\ &\leqq \int_0^{h-x}\left|l(x+t)+\sum_{i=1}^n A_i l(x+a_i t)\right|^p dt + \int_{A(x,h)} |l(x+t)|^p\,dt \end{aligned}$$

where $A(x,h)=\{t\in[0,h-x]\mid(x+a_it)\in F \text{ for some } i\in N=\{1,\dots,n\}\}$.

$$\tag{2.6} \int_{A(x,h)} |l(x+t)|^p\,dt \leqq \sum_{i=1}^n \int_{A_i(x,h)} |l(t)|^p\,dt,$$

since

$$A(x,h)\subseteq \bigcup_{i=1}^n A_i(x,h)$$

where $A_i(x,h)=\{s\in[0,h-x]\mid x+a_is\in F\}=\{t\in[x,h]\mid x(1-a_i)+a_it\in F\}$ (the last equality coming from the substitution $s=x+t$).

Suppose that we can prove:

(2.7) There is an $x\in[-h/2,0]\cap E$ such that

$$\int_{A_i(x,h)} |l(t)|^p\,dt \leqq \frac{1}{2n}\int_0^h |l(t)|^p\,dt, \qquad i=1,2,\dots,n.$$

Then from (2.2), (2.5), (2.6), and (2.7) we will have

$$\frac{1}{2}\int_0^h |l(t)|^p\,dt \leqq M(h-x)^\alpha \leqq M(2h)^\alpha = 2^\alpha M h^\alpha$$

which when combined with (2.4), yields

$$\int_{-h}^h |l(t)|^p\,dt \leqq 2\int_0^h |l(t)|^p\,dt \leqq 2^{\alpha+2}Mh^\alpha$$

which in turn implies (2.3), as required.

Only (2.7) remains to be proved. Set $M=\max_{i\in N}\{1/2+(3/2)|a_i|\}$. Define $\varepsilon=\varepsilon(h)$ by

$$\varepsilon = \frac{1}{h}\int_{-Mh}^{Mh} \chi_F(v)\,dv$$

where χ_F is the characteristic function of F. Note that 0 is a point of rarefaction of F, so ε can be made arbitrarily small by choosing h sufficiently small. Set

$$I_i(x) = \int_{A_i(x,h)} |l(t)|^p\,dt = \int_x^h |l(t)|^p\,\chi_F(x[1-a_i]+a_it)\,dt.$$

Then

$$\int_{-h/2}^{0} I_i(x)\,dx \leqq \int_{-h/2}^{0}\left[\int_{-h}^{h} |l(t)|^p \chi_F(x[1-a_i]+a_i t)\,dt\right] dx$$
$$\leqq \frac{1}{|1-a_i|}\int_{-Mh}^{Mh}\int_{-h}^{h} |l(u)|^p \chi_F(v)\,du\,dv,$$

setting $u=t$, $v=a_i t+(1-a_i)x$ and noting $|(\partial(t,x)/\partial(u,v))|=(1/|1-a_i|)$. Hence

$$\int_{-h/2}^{0} I_i(x)\,dx \leqq \frac{1}{|1-a_i|}\left[\int_{-Mh}^{Mh} \chi_F(v)\,dv\right]\left[\int_{-h}^{h} |l(t)|^p\,dt\right]$$

using Fubini's theorem and its converse freely since all functions are positive and integrable. By (2.4) and the definition of ε,

$$\int_{-h/2}^{0} I_i(x)\,dx \leqq \frac{1}{|1-a_i|}\,\varepsilon h\cdot 2\int_{0}^{h} |l(t)|^p\,dt.$$

By Tchebycheff's inequality, if

$$B_i = \left\{x \in \left[-\frac{h}{2}, 0\right] \cap E \,\middle|\, I_i(x) > \frac{1}{2n}\int_0^h |l(t)|^p\,dt\right\},$$

then

$$|B_i|\cdot\frac{1}{2n}\int_0^h |l(t)|^p\,dt \leqq \frac{2\varepsilon h}{|1-a_i|}\int_0^h |l(t)|^p\,dt,$$

i.e.,

$$|B_i| \leqq \frac{4n\varepsilon h}{|1-a_i|}.$$

Picking h so small that

$$\varepsilon < \frac{\min\limits_{i\in N} |1-a_i|}{16n^2},$$

we have

$$|B_i| < \frac{4nh}{|1-a_i|}\cdot\frac{\min\limits_{i\in N}|1-a_i|}{16n^2} \leqq \frac{h}{4n}$$

so that

$$\left|\bigcup_{i=1}^{n} B_i\right| \leqq \sum_{i=1}^{n} |B_i| < \frac{h}{4}.$$

If we further choose h so small that $|[-h/2, 0]\cap E| \geqq h/4$, we can find an x such that

$$x \in \left[-\frac{h}{2}, 0\right] \cap E \quad\text{and}\quad x \notin \bigcup_{i=1}^{n} B_i.$$

This is the x required for (2.7). This completes the proof of Theorem 2.

REMARK. As in §1, we may weaken the hypothesis of Theorem 2 to

$$\left(\frac{1}{h}\int_0^h\left|\sum_{i=0}^{k+l}A_if(x+a_it)\right|^p dt\right)^{1/p} = O(h^k) \qquad \text{as } h\to +0$$

for all $x\in E$, while still obtaining the conclusion $f\in t_k^p(x)$ for almost every $x\in E$.

The proof of this remark follows the proof of Theorem 2 except that Theorem 1 is replaced by the remark at the end of §1, Theorem 10 of [6] is coupled with Lemma 6 below, and Lemma 5 is replaced by Lemma 7 below.

LEMMA 6. *Let $\alpha>0$. If*

$$\int_0^h |l(x+t)|^p\,dt = O(h^\alpha) \qquad \textit{as } h\to +0 \textit{ for all } x\in E,$$

then

$$\int_{-h}^h |l(x+t)|^p\,dt = O(h^\alpha) \qquad \textit{as } h\to +0 \textit{ for almost every } x\in E.$$

The same conclusion holds if our hypothesis is

$$\int_{-h}^0 |l(x+t)|^p\,dt = O(h^\alpha) \textit{ as } h\to +0 \qquad \textit{for all } x\in E.$$

Proof. We prove only the former statement, since the proof of the latter is similar. As in Lemma 5, without loss of generality we may assume

$$\int_0^h |l(x+t)|^p\,dt \leqq Mh^\alpha \qquad \text{if } 0<h<\delta \text{ for all } x\in E.$$

To simplify notation, let 0 be a point of density of E. It suffices to show

$$\int_{-h}^h |l(t)|^p\,dt \leqq (4^\alpha M)h^\alpha$$

if h is sufficiently small. Pick $h<\delta/4$ so small that $[-2h,-h]\cap E\neq\varnothing$. Pick $-k\in[-2h,-h]\cap E$. Then $0<k<\delta/2$, $2k<4h<\delta$, and

$$\begin{aligned}\int_{-h}^h |l(t)|^p\,dt &\leqq \int_{-k}^k |l(t)|^p\,dt = \int_0^{2k}|l(-k+t)|^p\,dt\\ &\leqq M(2k)^\alpha \leqq M(4h)^\alpha = (4^\alpha M)h^\alpha.\end{aligned}$$

LEMMA 7. *If $l(x)=0$ on E, $|E|>0$, $l\in L^p$ for all $x\in E$*

$$\int_0^h\left|\sum_{i=0}^{n}A_il(x+a_it)\right|^p dt = O(h^\alpha) \qquad \textit{as } h\to +0$$

where $\alpha>1$, $p\geqq 1$, then

$$\int_{-h}^h |l(x+t)|^p\,dt = O(h^\alpha) \qquad \textit{as } h\to +0 \textit{ for almost every } x\in E.$$

Proof. Because of Lemma 6, it suffices to show

$$\int_0^h |l(x+t)|^p\,dt = O(h^\alpha) \qquad \text{as } h \to +0$$

for almost every $x \in E$. Assume that $a_0 > 0$. As in the proof of Lemma 5, we may then assume $A_0 = a_0 = 1$.

(If $a_0 < 0$, the hypothesis can be reduced to

$$\int_{-h}^0 \left| l(x+t) + \sum_{i=1}^n A_i l(x+a_i t)\right|^p dt = O(h^\alpha) \qquad \text{as } h \to +0 \text{ for all } x \in E.$$

The proof then proceeds to the conclusion that

$$\int_{-h}^0 |l(x+t)|^p\,dt = O(h^\alpha) \qquad \text{as } h \to +0 \text{ for almost every } x \in E.$$

Now apply the second part of Lemma 6 to produce the desired conclusion.)

As in Lemma 5, without loss of generality we may assume

$$\int_0^h \left| l(x+t) + \sum_{i=1}^n A_i l(x+a_i t)\right|^p dt \leqq Mh^\alpha, \qquad 0 < h < \delta \text{ for all } x \in E.$$

By discarding a subset of measure zero we may further assume that every point of E is a point of differentiability for $\int_0^x |l(t)|^p\,dt$, so that for all $x \in E$,

$$D\left(\int_0^x |l(t)|^p\,dt\right) = |l(x)|^p = 0.$$

Let 0 be a point of density of E. It suffices to show

$$\int_0^h |l(t)|^p\,dt = O(h^\alpha) \qquad \text{as } h \to +0.$$

Denote $\int_a^b |l(t)|^p\,dt$ by $I(a, b)$. We divide the proof into three cases.

Case I. $I(h/2, h) > I(0, h)/2$.

In this case the proof is very much like that of Lemma 5. Pick $x \in [0, h/2] \cap E$. As in Lemma 5,

$$\begin{aligned} I\left(\frac{h}{2}, h\right) &= \int_{h/2}^h |l(t)|^p\,dt \leqq \int_x^h |l(t)|^p\,dt \\ &\leqq \int_0^{h-x} \left| l(x+t) + \sum_{i=1}^n A_i l(x+a_i t)\right|^p dt + \sum_{i=1}^n \int_{A_i(x,h)} |l(t)|^p\,dt \end{aligned} \tag{2.8}$$

where $A_i(x, h) = \{t \in [x, h] \mid [1-a_i]x + a_i t \notin E\}$. As in Lemma 5, if h is sufficiently small, an $x \in [0, h/2] \cap E$ may be found such that for all $i = 1, 2, \ldots, n$

$$\int_{A_i(x,h)} |l(t)|^p\,dt \leqq \frac{1}{4n} I(0, h) < \frac{1}{2n} I\left(\frac{h}{2}, h\right).$$

For this x, (2.8) implies

$$I(0,h) < 2I\left(\frac{h}{2},h\right) \leqq 4\int_0^{h-x} \left|l(x+t)+\sum_{i=1}^{n} A_i l(x+a_it)\right|^p dt \leqq 4M(h-x)^\alpha \leqq 4Mh^\alpha$$

if h is sufficiently small.

Case II.

$$I\left(\frac{h}{2^{i+1}},\frac{h}{2^i}\right) \leqq \frac{1}{2}I\left(0,\frac{h}{2^i}\right), \qquad i=0,1,\ldots,k-1,$$

but

$$I\left(\frac{h}{2^{k+1}},\frac{h}{2^k}\right) > \frac{1}{2}I\left(0,\frac{h}{2^k}\right). \tag{2.9}$$

In this case we have

$$I\left(0,\frac{h}{2^i}\right) = I\left(0,\frac{h}{2^{i+1}}\right)+I\left(\frac{h}{2^{i+1}},\frac{h}{2^i}\right) \qquad \text{for all } i,$$

so

$$I\left(0,\frac{h}{2^i}\right) \leqq I\left(0,\frac{h}{2^{i+1}}\right)+\frac{1}{2}I\left(0,\frac{h}{2^i}\right), \qquad i=0,1,\ldots,k-1.$$

Hence

$$I\left(0,\frac{h}{2^i}\right) \leqq 2I\left(0,\frac{h}{2^{i+1}}\right), \qquad i=0,1,\ldots,k-1,$$

so that

$$I(0,h) \leqq 2I\left(0,\frac{h}{2}\right) \leqq 2^2I\left(0,\frac{h}{2^2}\right) \leqq\cdots\leqq 2^kI\left(0,\frac{h}{2^k}\right). \tag{2.10}$$

If h is sufficiently small, applying Case I to (2.9), we have

$$I\left(0,\frac{h}{2^k}\right) < 4M\left(\frac{h}{2^k}\right)^\alpha.$$

Combining this with (2.10), we have

$$I(0,h) < 2^k\left[4M\left(\frac{h}{2^k}\right)^\alpha\right] < 4Mh^\alpha \qquad \text{since } \alpha > 1.$$

Case III. $I(h/2^{i+1}, h/2^i) \leqq (1/2)I(0, h/2^i)$ for all $i=0, 1, 2, \ldots$.

Reasoning as in Case II, we arrive at (2.10) for every k. Dividing through by h, we obtain

$$\frac{1}{h}I(0,h) \leqq \frac{1}{(h/2^k)}I\left(0,\frac{h}{2^k}\right), \qquad k=1,2,\ldots.$$

Since by assumption, $0 \in E$ is a point of differentiability of the integral, the right-hand side tends to 0 as k tends to infinity. Hence

$$\frac{1}{h} I(0, h) = 0, \qquad I(0, h) = 0,$$

which is surely $O(h^\alpha)$. Q.E.D.

3. **A generalization of smoothness.**

THEOREM 3. *Let a, b, c be any distinct real numbers, and let A, B, C be real numbers such that $A+B+C=0$, and not all of A, B, $C=0$. Suppose that for all $x \in E$,*

$$Af(x+at)+Bf(x+bt)+Cf(x+ct) = O(t) \qquad \textit{as } t \to 0.$$

Then:

(a) *if $Aa+Bb+Cc \neq 0$, $f \in t_1(x)$ for almost every $x \in E$.*

(b) *if $Aa+Bb+Cc=0$, $f \in \Lambda_1(x)$*, i.e.,

$$f(x+t)+f(x-t)-2f(x) = O(t) \qquad \textit{as } t \to 0$$

for almost every $x \in E$.

(c) *if the "O" in the hypothesis of* (b) *is replaced by "o", it may also be replaced by "o" in the conclusion, i.e., f is then smooth almost everywhere in E.*

Proof. Part (a) was first proved by J. P. Kahane. It can be rephrased: if $f \in G_1(x, \{A, B, C; a, b, c\})$ for all $x \in E$, then $f \in t_1(x)$ for almost every $x \in E$. Thus stated, it is seen to be a special case of Theorem 1.

To prove (b), assume, for example, that $A \neq 0$. By the sliding lemma, $Af(x+[a-c]t)+Bf(x+[b-c]t)+Cf(x)=O(t)$ for almost every $x \in E$. Set $h=(b-c)t$, $\alpha=(a-c)/(b-c)$:

$$Af(x+\alpha h)+Bf(x+h)+Cf(x) = O(h) \tag{3.1}$$

for almost every $x \in E$.

$$A\alpha+B = \frac{1}{b-c}[A(a-c)+B(b-c)] = \frac{1}{b-c}[aA+bB-c(A+B)]$$

$$= \frac{1}{b-c}[aA+bB+cC] \qquad (\text{since } A+B+C = 0)$$

$$= 0 \qquad (\text{since } Aa+Bb+Cc = 0).$$

Hence $B=-\alpha A$, $C=-(A+B)=(\alpha-1)A$ so that (3.1) becomes after division by A

$$f(x+\alpha h)-\alpha f(x+h)+(\alpha-1)f(x) = O(h) \tag{3.2}$$

for almost every $x \in E$. Slide this result by $-\alpha$:

$$f(x)-\alpha f(x+[1-\alpha]h)+(\alpha-1)f(x-\alpha h) = O(h)$$

for almost every $x \in E$. Replace h by $-h$:

$$f(x)-\alpha f(x+[\alpha-1]h)+(\alpha-1)f(x+\alpha h) = O(h)$$

for almost every $x \in E$. Add the product of equation (3.2) by $(1-\alpha)$ to this result:

$$-\alpha f(x+[\alpha-1]h)+(1-\alpha)(-\alpha)f(x+h)-\alpha(\alpha-2)f(x) = O(h)$$

for almost every $x \in E$. Divide[3] by $-\alpha$ and slide this by 1:

$$f(x+\alpha h)+(1-\alpha)f(x+2h)+(\alpha-2)f(x+h) = O(h)$$

for almost every $x \in E$. Subtract (3.2) from this:

$$(1-\alpha)f(x+2h)-2(1-\alpha)f(x+h)+(1-\alpha)f(x) = O(h)$$

for almost every $x \in E$. Slide this by -1 and divide by $(1-\alpha)$:

$$f(x+h)-2f(x)+f(x-h) = O(h)$$

for almost every $x \in E$.

To prove part (c), simply replace "O" by "o" throughout the proof of (b).

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University of Chicago,
Chicago, Illinois

[3] Since a, b, c are distinct, $\alpha=(a-c)/(b-c)\neq 0$ or 1, so that α and $1-\alpha$ are both not 0.