# FUNCTION ALGEBRAS, MEANS, AND FIXED POINTS

BY
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1. **Introduction.** A semigroup $S$ is said to have the *common fixed point property on compacta* if for each compact Hausdorff space $Y$, and for each homomorphic representation $\mathscr{S}$ of $S$ as a semigroup of continuous self-maps of $Y$, there is in $Y$ a common fixed point of $\mathscr{S}$. In [11, Theorem 1], it is shown that $S$ has the common fixed point property on compacta if and only if $m(S)$ has a multiplicative left invariant mean. A natural question is: does this result generalize to the case of a topological semigroup $S$?

It is shown in Corollary 1 that if $S$ is a topological semigroup such that $C(S)$ has a multiplicative left invariant mean, then $S$ has the common fixed property on compacta with respect to continuous representations of $S$. But the proof of the converse encounters difficulties of the kind observed by M. Day in [3], namely that the family of adjoints of left translations on $C(S)$ fails to be a $w^*$-continuous representation on the desired subsets of $m(S)^*$. To remedy this, we employ $E$-representations of $S$, $C(S)$ on compact Hausdorff spaces; an analogue of the slightly continuous representations of $S$ that were introduced in [3].

In the main theorems of this paper, Theorems 1 and 2, it is shown that $C(S)$ has a multiplicative left invariant mean if and only if the pair $S$, $C(S)$ has the common fixed point property on compacta with respect to $E$-representations. In these theorems, additional implications are given concerning two other types of representations. In the proofs of Theorems 1 and 2, no use is made of the fact that the algebra $C(S)$ arises from a topology on $S$, so these results are stated in terms of closed subalgebras of $m(S)$.

Multiplicative left invariant means on $m(S)$ have been studied by the author in [11], and by E. Granirer in [6] and [7]. Subalgebras of $m(S)$ that have multiplicative left invariant means were first considered in [7].

Let $Y$ be a compact convex subset of a locally convex linear topological space, and let $S$ be a semigroup of continuous affine self-maps of $Y$ such that $m(S)$ has a left invariant mean. Day [2, Theorem 1] has shown that then $Y$ contains a common fixed point of $S$. Let $S'$ be the convex hull (i.e., the set of finite convex combinations) of $S$. Since $S$ has a common fixed point in $Y$, then $S'$ must also. In the light of our earlier remarks, it is tempting to conjecture that $m(S')$ has a multiplicative left invariant mean, but this conjecture does not hold. However, a subalgebra of $m(S')$ can be constructed which does have such a multiplicative mean. This algebra

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could be obtained thus: let $y$ be a specific element of $Y$, let $\alpha: C(Y) \to m(S')$ be given by $(\alpha h)s' = h(s'y)$ for $h \in C(Y)$ and $s' \in S'$, then $\alpha(C(Y))$ can be shown to be an algebra with the desired properties. However, a somewhat more general approach than this is employed in §5, where the construction of the algebra is given.

2. **Preliminaries and nomenclature.** Let $S$ be a semigroup, $m(S)$ the space of all bounded real-valued functions on $S$, where $m(S)$ has the supremum norm. For $s \in S$, the *left translation* $l_s$ {*right translation* $r_s$} of $m(S)$ by $s$ is given by $(l_s f)s' = f(ss')$ $\{(r_s f)s' = f(s's)\}$, where $f \in m(S)$ and $s' \in S$. Let $X$ be a subspace of $m(S)$, then $X$ is *left* {*right*} *translation-invariant* if $l_s X \subseteq X$ $\{r_s X \subseteq X\}$ for all $s \in S$. If $X$ is both left and right translation-invariant, then $X$ is called *translation-invariant*.

Now let $X$ be a left translation-invariant closed subalgebra of $m(S)$ that contains $e$, the constant 1 function on $S$. An element $\mu \in X^*$ is a mean on $X$ if $\|\mu\| = 1$ and $\mu(e) = 1$. A mean $\mu$ on $X$ is *left invariant* if $\mu(l_s f) = \mu(f)$ for all $f \in X$ and $s \in S$; $\mu$ is *multiplicative* if $\mu(f) \cdot \mu(g) = \mu(f \cdot g)$ (the pointwise product) for all $f, g \in X$.

When $Y$ is a topological space, $C(Y)$ denotes the space of all bounded real-valued continuous functions on $Y$, where $C(Y)$ has the supremum norm.

Let $S$ be a semigroup, $X$ a subset of $m(S)$, and $Y$ a compact Hausdorff space. Let $\eta$ be a homomorphism of $S$ onto $\mathscr{S}$, a semigroup (under functional composition) of continuous maps of $Y$ into itself. The set of all $y \in Y$ such that $Ty(C(Y)) \subseteq X$ is denoted by $Y'$, where the map $Ty$ is given by $(Tyh)s = h((\eta s)y)$, for $h \in C(Y)$, and $s \in S$. The family $\mathscr{S}$ is an *E-representation* of $S$, $X$ on $Y$ if $Y'$ is nonempty, $\mathscr{S}$ is a *D-representation* if $Y'$ is dense in $Y$, and $\mathscr{S}$ is an *A-representation* if $Y' = Y$. (The symbol $E$ stands for exists, $D$ for dense, and $A$ for all.) The pair $S$, $X$ has the *common fixed point property on compacta* with respect to $i$-representations (where $i = E, D, A$; respectively) if, for each compact Hausdorff space $Y$, and for each $i$-representation of $S$, $X$ on $Y$, there is in $Y$ a common fixed point of the family $\mathscr{S}$.

3. **The main theorems.**

THEOREM 1. *Let $S$ be a semigroup, $X$ a translation-invariant closed subalgebra of $m(S)$ that contains the constant functions. Then the following three conditions are equivalent*:

(1) *$X$ has a multiplicative left invariant mean.*

(2) *$S$, $X$ has the common fixed point property on compacta with respect to E-representations.*

(3) *$S$, $X$ has the common fixed point property on compacta with respect to D-representations.*

*Further, each of the equivalent conditions* (1), (2), *or* (3) *implies*

(4) *$S$, $X$ has the common fixed point property on compacta with respect to A-representations.*

**Proof.** (1) → (2). Let $\eta$ be a homomorphism of $S$ onto $\mathscr{S}$, an $E$-representation of $S$, $X$ on the compact Hausdorff space $Y$. Then there exists an element $z \in Y$

such that $Tz\colon C(Y) \to X$. Let $Tz^*\colon X^* \to C(Y)^*$ be the adjoint map of $Tz$, and let $\mu$ be a multiplicative left invariant mean on $X$. Then since $\mu$ is a mean,

$$(Tz^*\mu)1 = \mu(Tz1) = \mu(e) = 1,$$

where 1 designates the constant 1 function on $Y$. Also, a computation shows that $Tz(h\cdot k) = Tzh\cdot Tzk$ (the pointwise product) for $h, k \in C(Y)$, so $Tz^*\mu$ is a nonzero multiplicative linear functional on $C(Y)$ since $\mu$ is a multiplicative mean on $X$. But $Y$ is a compact Hausdorff space, so by [5, Lemma 25, p. 278], there exists an element $z' \in Y$ such that

$$h(z') = (Tz^*\mu)h = \mu(Tzh)$$

for all $h \in C(Y)$. We will show that $z'$ is the desired common fixed point of $\mathscr{S}$.

For each $s \in S$, define a map $\theta_s\colon C(Y) \to C(Y)$ by $(\theta_s h)y = h((\eta s)y)$ for $h \in C(Y)$, $y \in Y$. Then for each $s' \in S$,

$$\begin{aligned}(Tz(\theta_s h))s' &= (\theta_s h)((\eta s')z) = h((\eta s)(\eta s')z)\\ &= h(\eta(ss')z) = (Tzh)(ss') = (l_s(Tzh))s',\end{aligned}$$

where the first and fourth equalities follow from the definition of $Tz$, and the third follows from the fact that $\eta$ is a homomorphism. Therefore $Tz(\theta_s h) = l_s(Tzh)$. Thus it follows that for all $h \in C(Y)$ and $s \in S$,

$$\begin{aligned}h((\eta s)z') &= (\theta_s h)z' = \mu(Tz(\theta_s h)) = \mu(l_s(Tzh))\\ &= \mu(Tzh) = h(z'),\end{aligned}$$

where the fourth equality holds by virtue of the left invariance of $\mu$. But $C(Y)$ distinguishes between the points of $Y$ since $Y$ is a compact Hausdorff space, so $(\eta s)z' = z'$ for all $s \in S$.

(2) $\to$ (3). Condition (2) is formally stronger than (3).

(3) $\to$ (1). Let $Z \subseteq X^*$ be the set of all multiplicative means on $X$, and $Q$ the evaluation map $Q\colon S \to Z$ given by $(Qs)f = f(s)$ for $s \in S$, $f \in X$. Let $X^*$ have the $w^*$-topology, then $Z$ is a norm-bounded ($w^*$-) closed subset of $X^*$. Choose $Y$ to be the closure of $Q(S)$ in $X^*$, then $Y$ is a compact Hausdorff space by [4, Corollary 3, p. 41], and $Q(S) \subseteq Y \subseteq Z \subseteq X^*$ where $Q(S)$ is dense in $Y$. For each $s \in S$, define a map $L_s\colon X^* \to X^*$ by $L_s = l_s^*$. Each $L_s$ is ($w^*$-) continuous by [4, Theorem 2, p. 18], carries $Q(S)$ into $Q(S)$, and thus carries $Y$, the closure of $Q(S)$, into $Y$. Let $\mathscr{S} = \{L_s;\, s \in S\}$, then $\mathscr{S}$ forms a semigroup, homomorphic to $S$, of continuous self-maps of $Y$. We wish to show that $\mathscr{S}$ is a $D$-representation of $S$, $X$ on $Y$.

The space $X$ is a closed subalgebra of $m(S)$ that contains $e$. By Kakutani's theorem on representations of abstract $(M)$-spaces (see [10, §24.6, p. 242]), the elements of such a space $X$ evaluated on the space $Z$ yield an isometry of $X$ *onto* $C(Z)$. By the Tietze extension theorem [5, Theorem 3, p. 15], each function in $C(Y)$ is the restriction to $Y$ of some function in $C(Z)$; hence to each $h \in C(Y)$, there corresponds an $f_h \in X$ such that $h(\mu)=\mu(f_h)$ for $\mu \in Y$. (It can also be shown that $\|h\|=\|f_h\|$ and consequently that $Y=Z$, but this is not needed for the proof of Theorem 1.) Now suppose that $y \in Q(S)$, so $y=Qs'$ for some $s' \in S$. Then for $h \in C(Y)$ and $s \in S$,

$$\begin{aligned}(Tyh)s &= h(L_s y) = h(l_s^*(Qs')) = (l_s^*(Qs'))f_h \\ &= (Qs')(l_s f_h) = (l_s f_h)s' = f_h(ss') = (r_{s'} f_h)s.\end{aligned}$$

Since $X$ is right translation-invariant, this gives us $Tyh=r_{s'}f_h \in X$; which means that $Ty(C(Y)) \subseteq X$ if $y \in Q(S)$. But $Q(S)$ is dense in $Y$, thus $S$ is a $D$-representation of $S$, $X$ on $Y$. By (3), there exists $\mu_0 \in Y$ such that $\mu_0=L_s\mu_0=l_s^*\mu_0$ for all $s \in S$, hence $\mu_0$ is the required multiplicative left invariant mean on $X$.

(3) $\rightarrow$ (4). Condition (3) is formally stronger than (4), which proves Theorem 1.

Suppose $S$ is a topological semigroup, that is, $S$ has a Hausdorff topology in which the semigroup product is continuous. It is known that $C(S)$ is a translation-invariant closed subalgebra of $m(S)$ that contains $e$, so Theorem 1 can be applied to the pair $S$, $C(S)$. Let $Y$ be a compact Hausdorff space, and $\eta$ an algebraic homomorphism of $S$ onto $\mathscr{S}$, a semigroup of continuous self-maps of $Y$. Then $\mathscr{S}$ is a *continuous representation* of $S$ on $Y$ if the map $S \times Y \rightarrow Y$ given by $(\eta s)y$ for $s \in S$ and $y \in Y$, is continuous(2). The topological semigroup $S$ has the *common fixed point property on compacta* with respect to continuous representations if for each compact Hausdorff space $Y$, and each continuous representation of $S$ on $Y$, there is in $Y$ a common fixed point of $\mathscr{S}$.

COROLLARY 1. *Let $S$ be a topological semigroup. If $C(S)$ has a multiplicative left invariant mean, then $S$ has the common fixed point property on compacta with respect to continuous representations.*

**Proof.** Let $\mathscr{S}$ be a continuous representation of $S$ on a compact Hausdorff space $Y$. For $y \in Y$, the map $Fy\colon S \rightarrow Y$ is continuous, where $Fy$ is given by $Fys=(\eta s)y$ for $s \in S$. So for $y \in Y$, $h \in C(Y)$, and $s \in S$,

$$(Tyh)s = h((\eta s)y) = (hFy)s,$$

thus $Tyh \in C(S)$ since $hFy$ is the composition of two continuous functions. Therefore $\mathscr{S}$ is an $A$-representation of $S$, $C(S)$ on $Y$, hence Corollary 1 follows by Theorem 1.

The next corollary generalizes [11, Corollary 1] from the case of finite semigroups to that of compact topological semigroups. An element $s_0$ of a semigroup $S$ is a *right zero* of $S$ if $Ss_0=\{s_0\}$.

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(2) Similarly, $\mathscr{S}$ is a *separately continuous representation* of $S$ on the compact Hausdorff space $Y$ if the map on $S \times Y$ is continuous in each variable. It can be shown that $\mathscr{S}$ is an $A$-representation of $S$, $C(S)$ on $Y$ if and only if $\mathscr{S}$ is a separately continuous representation of $S$ on $Y$.

COROLLARY 2. *Let $S$ be a compact semigroup. Then the following conditions are equivalent*:

(1) *$C(S)$ has a multiplicative left invariant mean.*

(2) *$S$ has the common fixed point property on compacta with respect to continuous representations.*

(3) *$S$ has a right zero.*

**Proof.** (1) $\rightarrow$ (2). This follows from Corollary 1.

(2) $\rightarrow$ (3). The left multiplications of $S$ by elements $s \in S$ form a continuous representation of $S$ on itself. Since $S$ is compact, then by (2), $S$ has a right zero.

(3) $\rightarrow$ (1). If $s_0$ is a right zero of $S$, it follows by a routine computation that $Qs_0$ is a multiplicative left invariant mean on $C(S)$.

It was shown that condition (4) of Theorem 1 is implied by each of the other three equivalent conditions of the theorem. The converse implication was not shown; I do not know if it holds. However, the implication (3) $\rightarrow$ (1) of Theorem 1 was proven by use of the right translation-invariance of $X$. If to the hypotheses of the theorem, we add the requirement that $X$ satisfies a certain property that is stronger than right translation-invariance, then conditions (1)–(4) of Theorem 1 can be shown to be equivalent.

Let $X$ be a left invariant closed subspace of $m(S)$. For each $\mu \in X^*$, there is a $\mu_l$: $X \rightarrow m(S)$ given by $(\mu_l f)s = \mu(l_s f)$ for $f \in X$ and $s \in S$. The space $X$ is *left introverted* if $\mu_l X \subseteq X$ for all $\mu \in X^*$ (see Day [1, §10, p. 540]). Now let $X$ be a closed subalgebra of $m(S)$ which contains $e$. The algebra $X$ will be called *left $M$-introverted* if $\mu_l X \subseteq X$ for every multiplicative mean $\mu \in X^*$. It is easily verified that the statements (a) $X$ is left introverted, (b) $X$ is left $M$-introverted, and (c) $X$ is right translation-invariant, satisfy the relationship (a) $\rightarrow$ (b) $\rightarrow$ (c).

THEOREM 2. *Let $S$ be a semigroup, $X$ a left translation-invariant closed subalgebra of $m(S)$ that contains the constant functions. Let $X$, in addition, be left $M$-introverted. Then conditions* (1)–(4) *of Theorem* 1 *are equivalent.*

**Proof.** Since $S$, $X$ satisfies the hypotheses of Theorem 1, we only need to show that (4) $\rightarrow$ (1). Choose the compact Hausdorff space $Y$ to be the set of all multiplicative means on $X$, where $Y$ is given the $w^*$-topology of $X^*$. Let $\mathscr{S} = \{L_s; s \in S\}$, where $L_s = l_s^*$. By use of [10, §24.6, p. 242] again, it follows that for each $h \in C(Y)$, there exists a unique $f_h \in X$ such that $h(\mu) = \mu(f_h)$ for $\mu \in Y$. Then for $h \in C(Y)$, $\mu \in Y$ and $s \in S$, we have that

$$\begin{aligned}(T\mu h)s &= h(L_s\mu) = h(l_s^*\mu) = (l_s^*\mu)f_h \\ &= \mu(l_s f_h) = (\mu_l f_h)s.\end{aligned}$$

So $T\mu h = \mu_l f_h \in X$, since $X$ is left $M$-introverted; which means that $Ty(C(Y)) \subseteq X$ for all $y \in Y$. Thus $\mathscr{S}$ is an $A$-representation of $S$, $X$ on $Y$. Since (4) asserts the pair $S$, $X$ has the common fixed point property on compacta with respect to $A$-representations, there exists $\mu_0 \in Y$ such that $\mu_0 = l_s^* \mu$ for all $s \in S$.

4. **Examples and remarks.** (a) A modified version of Theorem 1 is obtained by dropping the requirement that $X$ is translation-invariant, and replacing it by the hypotheses that $X$ is left translation-invariant, and that the semigroup $S$ contains an identity $i$. Under these circumstances, it follows that conditions (1)–(4) of Theorem 1 satisfy the relationship:

$$(1) \leftrightarrow (2) \rightarrow (3) \rightarrow (4).$$

The proof of $(1) \rightarrow (2) \rightarrow (3) \rightarrow (4)$ goes through as in Theorem 1; we will indicate the proof of $(2) \rightarrow (1)$. Let $Y$ be the space of multiplicative means on $X$, where $Y$ has the $w^*$-topology of $X^*$, and let $\mathscr{S}=\{l_s^*; s \in S\}$. It can be verified that $Y'$ is nonempty because $Qi \in Y'$, so $\mathscr{S}$ is an $E$-representation of $S$, $X$ on $Y$, hence (1) follows.

(b) An "obvious proof" of the converse to Corollary 1 fails to go through. Let $Y$ be the space of multiplicative means on $C(S)$, where $Y$ is given the $w^*$-topology of $C(S)^*$, and let $\mathscr{S}=\{l_s^*; s \in S\}$. If the topological semigroup $S$ has the common fixed point property on compacta with respect to continuous representations, and if it can be shown that $\mathscr{S}$ is a continuous representation of $S$ on $Y$, then it follows that $C(S)$ has a multiplicative left invariant mean. However, $\mathscr{S}$ need not be a continuous representation of $S$ on $Y$ even if $C(S)$ has, in fact, a multiplicative left invariant mean.

For a counterexample (cf. Day [3, Paragraph 1]), let $S$ be the semigroup of nonnegative real numbers under multiplication, where $S$ has the usual topology. Then $Q(0)$ can be verified to be a multiplicative left invariant mean on $C(S)$, so by Corollary 1, $S$ has the common fixed point property on compacta with respect to continuous representations. Let $\mu \in Y$ be a cluster point of $Qs$ as $s \rightarrow \infty$; such a point $\mu$ must exist by compactness of $Y$. Choose $f \in C(S)$ to be $f(s)=\min(s, 1)$ for $s \in S$. It is easily shown that $(l_s^*\mu)f=1$ if $s\neq 0$, and $(l_s^*\mu)f=0$ if $s=0$. Hence if $s \rightarrow 0$ in a deleted neighborhood of 0, then $l_s^*\mu$ does not converge to $l_0^*\mu$, thus $\mathscr{S}$ is not a continuous representation of $S$ on $Y$. (Additionally, the family $\mathscr{S}$ serves as an example of a $D$-representation of $S$, $C(S)$ on $Y$ that is not also an $A$-representation.)

(c) In [11, Corollary 3], it was shown that if $S$ is a semigroup such that for every $s_1, s_2 \in S$, there exists an $s_3 \in S$ which satisfies $s_1s_3=s_2s_3$, then $m(S)$ has a multiplicative left invariant mean. The converse was obtained [11, Theorems 3 and 4] for the cases where $S$ is Abelian or has left cancellation; this converse was shown to hold for the general case by E. Granirer [6, Theorem 1].

(d) Suppose $X_1$, $X_2$ are left translation-invariant closed subalgebras of $m(S)$ that contain the constant functions, where $X_1 \subseteq X_2$. If $X_2$ has a multiplicative left invariant mean, it can be shown that $X_1$ does also. This observation, together with (c) above, enables us to construct examples of pairs $S$, $X$ which have the common fixed point property on compacta with respect to the various representations. For an illustration, let $S$ be the real numbers with the usual metric topology, but with the semigroup product $s_1s_2=\max(s_1, s_2)$. Then by (c), $m(S)$ has a multiplicative left invariant mean, so $C(S)$ does also. By Corollary 1, it follows that the topological semigroup $S$ has the common fixed point property on compacta with respect to continuous representations.

(e) Let $S$ be a semigroup. An example of an algebra $X \subseteq m(S)$ which has a multiplicative left invariant mean (even if $m(S)$ does not) is given by the trivial case where $X$ consists of all the constant functions on $S$.

(f) Another example of an algebra $X \subseteq m(S)$, where $m(S)$ does not have a multiplicative left invariant mean but $X$ does, is given by the case where $S$ is the positive integer under addition, and $X$ is the space of convergent functions on $S$. (For other examples, see Granirer [7].) Define $\mu \in X^*$ by $\mu(f)=\lim_{s\to\infty} f(s)$ for $f \in X$, then $\mu$ is a multiplicative left invariant mean on $X$. The pair $S$, $X$ can be shown to satisfy the hypotheses of Theorem 2; in particular, the space $X$ is left $M$-introverted.

(g) Now let $S$ be the group of integers under addition, and $X$ the space of all $f \in m(S)$ such that $\mu(f)$ exists, where $\mu(f)=\lim_{s\to+\infty} f(s)$. As in (f), $\mu$ is a multiplicative left invariant mean on $X$. But the pair $S$, $X$ now satisfies the hypotheses of Theorem 1, but not those of Theorem 2, since $X$ is not left $M$-introverted. In fact, it can be shown that there exists a multiplicative mean $\mu' \in X^*$ such that $\mu'_l X=m(S)$.

(h) Suppose $S$ is the group of integers under addition, but now let $X$ be the space of all $f \in m(S)$ such that both $\mu(f)$ and $\mu'(f)$ exist, where $\mu(f)$ and $\mu'(f)$ are the limits of $f(s)$ as $s\to+\infty$ and $s\to-\infty$, respectively. Then the pair $S$, $X$ satisfies the hypotheses of Theorem 2, and $X$ has two distinct multiplicative left invariant means, $\mu$ and $\mu'$.

(i) Let $S$ be a nontrivial regular Hausdorff space with the property that every continuous real-valued function on $S$ is constant; such a space exists by a result of E. Hewitt [9, Theorem 1, p. 503]. As in Granirer [8, p. 108], define a product on $S$ by the rule $ss'=s$ for all $s, s' \in S$. It is shown in [8] that $S$ is a topological semigroup. By remarks (c) and (e), $S$ provides an example of a topological semigroup for which $C(S)$ has a multiplicative left invariant mean, but $m(S)$ does not. By Corollary 1, $S$ has the common fixed point property on compacta with respect to continuous representations. We can say even more than this, for by the proof of Corollary 1, if $\mathscr{S}$ is a continuous representation of $S$ on the compact Hausdorff space $Y$, then $\mathscr{S}$ is an $A$-representation of $S$ on $Y$. Thus $Tyh \in C(S)$ for $y \in Y$ and $h \in C(Y)$, so $Tyh$ is a constant function. Therefore $h((\eta s)y)=h((\eta s')y)$ for $h \in C(Y)$, $y \in Y$, and $s, s' \in S$. But $C(Y)$ distinguishes between points of $Y$, so $\eta s=\eta s'$ for all $s, s' \in S$. Hence $\mathscr{S}$ is a singleton set $\mathscr{S}=\{\sigma\}$, where $\sigma$ is a retraction on $Y$ (recall $\mathscr{S}$ is a semigroup, so $\sigma$ is an idempotent). Thus $\sigma Y$ is the set of all common fixed points of $\mathscr{S}$.

5. **Semigroups of finite means.** Throughout this section, $S$ designates a semigroup and $X$ denotes a translation-invariant closed subspace of $m(S)$ that contains

the constant functions. It will be shown that a new pair $P$, $W$ can be constructed from $S$, $X$ such that $W$ has a multiplicative left invariant mean if and only if $X$ has a left invariant mean. This construction furnishes a source of examples of pairs $P$, $W$ for which $W$ has a multiplicative left invariant mean, but for which $m(P)$ does not.

Let $P$ be the set of all real-valued functions $p$ on $S$ that satisfy (1) $p(s) \geqq 0$ for all $s \in S$, (2) $p(s) > 0$ for at most a finite number of elements of $S$, and (3) $\sum_{s \in S} p(s) = 1$. Define a product on $P$ by

$$pp'(s) = \textstyle\sum_{ab=s} p(a)p'(b)$$

for $p, p' \in P$ and $a, b, s \in S$. In other words, $P$ is the semigroup of finite means on $S$ with the convolution operation of $l^1(S)$ as a product. Let $\tau\colon X \to m(P)$ be given by

$$(\tau f)p = \textstyle\sum_{s \in S} p(s)f(s)$$

for $f \in X$ and $p \in P$. Then the closed subalgebra of $m(P)$ that is generated by $\tau X$ is denoted by $W$, i.e., $W$ is the smallest closed subalgebra of $m(P)$ that contains $\tau X$. Let $I\colon S \to P$ be the map where $Is$ is the characteristic function of $s$.

We list some useful facts about $P$ and $W$; see Day [1, §5] for some of these items.

(1) $$I(ss') = (Is)(Is') \text{ for } s, s' \in S.$$

(2) $$p = \textstyle\sum_{s \in S} p(s)Is \text{ for } p \in P.$$

(3) $$pp' = \textstyle\sum_{a \in S} \sum_{b \in S} p(a)p'(b)I(ab) \text{ for } p, p' \in P.$$

(4) $$(\tau f)(Is) = f(s) \text{ for } f \in X \text{ and } s \in S.$$

(5) $$\tau e = e', \text{ the constant 1 function on } P.$$

(6) $$\|\tau\| = 1.$$

LEMMA 1. *The algebra $W$ is translation-invariant.*

**Proof.** This is shown only for left translation-invariance; the proof is similar for right translation-invariance. For $f \in X$ and $p, p' \in P$, it follows that

$$\begin{aligned}(l_p\tau f)p' &= (\tau f)(pp') = (\tau f)\left(\textstyle\sum_{a \in S} \sum_{b \in S} p(a)p'(b)I(ab)\right) \\ &= \textstyle\sum_{a \in S} \sum_{b \in S} p(a)p(b)f(ab) = \sum_{a \in S} p(a) \sum_{b \in S} p'(b)((l_a f)b) \\ &= \textstyle\sum_{a \in S} p(a)((\tau l_a f)p') = \left(\sum_{a \in S} p(a)\tau l_a f\right)p'.\end{aligned}$$

Thus we have

(7) $$l_p\tau f = \textstyle\sum_{a \in S} p(a)\tau l_a f = \tau\left(\sum_{a \in S} p(a) l_a f\right).$$

But $l_a f \in X$ since $X$ is left translation-invariant, so $l_p \tau f \in W$ for $f \in X$ and $p \in P$. This means that $\tau X$ is left translation-invariant. However, $l_p$ is a bounded multiplicative linear operator on $m(P)$, hence the Banach algebra $W$ generated by $\tau X$ is also left translation-invariant.

Since $W$ is a translation-invariant closed subalgebra of $m(P)$ that contains the constant functions, it is meaningful to speak of a multiplicative left invariant mean on $W$.

THEOREM 3. *The algebra $W$ has a multiplicative left invariant mean if and only if $X$ has a left invariant mean.*

**Proof.** From equations (5) and (6), it follows that if $\mu$ is a mean on $W$, then $\tau^*\mu$ is a mean on $X$. For $f \in X$, $s \in S$, we have

$$l_{Is}\tau f = \tau(\textstyle\sum_{a\in S}((Is)a)l_a f) = \tau l_s f$$

by equation (7). Let $\mu$ be a left invariant mean on $W$, then

$$(\tau^*\mu)(l_s f) = \mu(\tau l_s f) = \mu(l_{Is}\tau f) = \mu(\tau f) = (\tau^*\mu)f$$

for all $f \in X$ and all $s \in S$. Hence $\tau^*\mu$ is a left invariant mean on $X$, which shows the "only if" implication.

For the converse, let $\lambda$ be a left invariant mean on $X$. Let $q: P \to X^*$ be the map given by

$$(qp)f = \textstyle\sum_{s\in S} p(s)f(s) = (\tau f)p,$$

where $p \in P$ and $f \in X$. Then $qP$ is $w^*$-dense in the set of means on $X$ since $P$ is the set of finite means on $S$ (see [1, §10, p. 540]), so there exists a net $\{qp_\gamma\}$ which is $w^*$-convergent to $\lambda$. Let $Y$ be the set of all multiplicative means on $W$, and let $Q: P \to Y$ be the evaluation map $(Qp)h = h(p)$ for $p \in P$ and $h \in W$. By compactness of $Y$ in the $w^*$-topology of $X^*$, the net $\{Qp_\gamma\}$ has a subnet $\{Qp_\delta\}$ that converges $w^*$ to some $\mu \in Y$. We will show that $\mu$ is the desired multiplicative left invariant mean on $W$.

For $p_\delta \in \{p_\delta\}$ and $f \in X$, we have

$$(\tau^* Qp_\delta)f = (Qp_\delta)(\tau f) = (\tau f)p_\delta = (qp_\delta)f,$$

hence $\tau^* Qp_\delta = qp_\delta$. But $\{Qp_\delta\}$ and $\{qp_\delta\}$ converge $w^*$ to $\mu$ and $\lambda$ respectively, so $\tau^*\mu = \lambda$ by $w^*$-continuity of $\tau^*$. If $p \in P$ and $f \in X$, we obtain

$$\begin{aligned}\mu(l_p\tau f) &= \mu(\textstyle\sum_{a\in S} p(a)\tau l_a f) = \textstyle\sum_{a\in S} p(a)((\tau^*\mu)l_a f)\\ &= \textstyle\sum_{a\in S} p(a)\lambda(l_a f) = \textstyle\sum_{a\in S} p(a)\lambda(f)\\ &= \lambda(f) = (\tau^*\mu)f = \mu(\tau f),\end{aligned}$$

where the first equality follows from equation (7), and the fourth from the left invariance of $\lambda$. This means that $\mu$ is left invariant when restricted to $\tau X$. But $\mu$

and $l_p$ are multiplicative, continuous, and linear; hence $\mu$ is left invariant on $W$, the Banach algebra generated by $\tau X$, which proves Theorem 3.

It can also be shown that $\tau^*$ is a $w^*$-$w^*$ homeomorphism of $Y$ onto the set of means of $X$, but this is not needed for the proof above.

By use of Theorem 3, examples can be obtained of pairs $P$, $W$ which have a multiplicative left invariant mean, and so by Theorem 1, have the common fixed point property on compacta with respect to $E$, $D$, and $A$-representations. Any topological semigroup $S$ for which $C(S)$ has a left invariant mean will serve admirably as a source of a suitable pair $S$, $X$ if $X$ is taken to be $C(S)$. Some well-known examples are Abelian topological semigroups, and compact topological groups. Another possibility is the use of appropriate subspaces of $C(S)$; for example, if $S$ is a topological group, then $X$ can be taken to be the space of almost periodic functions on $S$.

If $S$ is a semigroup such that $m(S)$ has a left invariant mean, then the space $m(P)$ need not have a multiplicative left invariant mean. To illustrate this, let $S$ be the group of integers under addition, and let $X=m(S)$. A computation shows that $P$ is an infinite Abelian cancellation semigroup. But if $m(S')$ has a multiplicative left invariant mean, where $S'$ is a cancellation semigroup, then $S'$ is a singleton set by [11, Theorem 2]; hence $m(P)$ does not have a multiplicative left invariant mean. It follows that $W$ must be a proper subset of $m(P)$, since $W$ does have such a mean by Theorem 3.

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