I-BISIMPLE SEMIGROUPS

BY
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Let S be a semigroup and let E_S denote the set of idempotents of S. As usual E_S is partially ordered in the following fashion: if $e, f \in E_S$, $e \leqq f$ if and only if $ef = fe = e$. Let I denote the set of all integers and let I^0 denote the set of nonnegative integers. A bisimple semigroup S is called an I-bisimple semigroup if and only if E_S is order isomorphic to I under the reverse of the usual order. We show that S is an I-bisimple semigroup if and only if $S \cong G \times I \times I$, where G is a group, under the multiplication

$$\begin{aligned}(g, a, b)(h, c, d) &= (g f_{b-c,c}^{-1} h \alpha^{b-c} f_{b-c,d}, a, b+d-c) \quad \text{if } b \geqq c,\\ &= (f_{c-b,a}^{-1} g \alpha^{c-b} f_{c-b,b} h, a+c-b, d) \quad \text{if } c \geqq b,\end{aligned}$$

where α is an endomorphism of G, α^0 denoting the identity automorphism of G, and for $m \in I^0$, $n \in I$,

$f_{0,n} = e$, the identity of G while if $m > 0$,

$f_{m,n} = u_{n+1}\alpha^{m-1} u_{n+2}\alpha^{m-2} \cdots u_{n+(m-1)}\alpha u_{n+m}$, where $\{u_n : n \in I\}$ is a collection of elements of G with $u_n = e$, the identity of G, if $n > 0$.

If we let $G = \{e\}$, the one element group, in the above multiplication we obtain $S = I \times I$ under the multiplication $(a, b)(c, d) = (a+c-r, b+d-r)$.

We will denote S under this multiplication by C^*, and we will call C^* the extended bicyclic semigroup. C^* is the union of the chain I of bicyclic semigroups C.

If S is an I-bisimple semigroup, we will write $S = (G, C^*, \alpha, u_i)$ where G is the structure group of S, α is the structure endomorphism of G, and $\{u_i\}$ is the sequence of "distinguished elements" of G.

An I-bisimple semigroup is a bisimple inverse semigroup without identity as contrasted to a bisimple ω-semigroup (a bisimple semigroup T such that E_T is order isomorphic to I^0 under the reverse of the usual order [7], [12]) which is a bisimple inverse semigroup with identity. If $S = (G, C^*, \alpha, u_i)$, the inverse of (g, m, n) is (g^{-1}, n, m) and $E_S = ((e, n, n) : n \in I)$. If $\mathscr{H}$ is Green's relation, $S/\mathscr{H} \cong C^*$, the extended bicyclic semigroup.

Necessary and sufficient conditions for two I-bisimple semigroups to be isomorphic are established, and an explicit determination of the homomorphisms of one I-bisimple semigroup onto another is given.

A complete description of the maximal group homomorphic image of an I-bisimple semigroup is given. Since we are also able to give an explicit description of the defining homomorphism, this result should have applications to the matrix

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representation of I-bisimple semigroups over fields. To perform the construction, we first determine the maximal cancellative homomorphic image of an ω-right cancellative semigroup (a right cancellative semigroup with identity whose ideal structure is order isomorphic to I^0 under the reverse of the usual order). We then utilize this result in conjunction with the description of the maximal group homomorphic image of a bisimple inverse semigroup with identity [8] to describe the maximal group homomorphic image of a bisimple ω-semigroup. Finally, this description, structural properties of I-bisimple semigroups, and a determination of the homomorphisms of a bisimple ω-semigroup into a group are used to give the desired construction.

It is shown that if ρ is a congruence relation on an I-bisimple semigroup $S=(G, C^*, \alpha, u_i)$, ρ is a group congruence (S/ρ is a group) or ρ is an idempotent separating congruence (each ρ-class contains at most one idempotent). The group congruences are in a one-to-one correspondence with the normal subgroups of the maximal group homomorphic image while the idempotent separating congruences are determined in terms of the α-invariant subgroups of G.

The ideal extensions of an I-bisimple semigroup are studied in [15], [16].

Unless otherwise specified, we will use the terminology, definitions, and notation of [2].

1. **The structure theory.** In this section we determine the structure of I-bisimple semigroups, and we also give an example of an I-bisimple semigroup with nontrivial distinguished elements.

$\mathscr{R}$, $\mathscr{L}$, $\mathscr{H}$, and $\mathscr{D}$ will denote Green's relations [2]. R_a will denote the $\mathscr{R}$-class containing a.

THEOREM 1.1 (REILLY [7]; SEE ALSO WARNE [12]). *S is a bisimple ω-semigroup if and only if $S\cong G\times I^0\times I^0$, where G is a group, under the multiplication*

$$(g, a, b)(h, c, d) = (g\alpha^{c-r}h\alpha^{b-r}, a+c-r, b+d-r)$$

where $r=\min(b, c)$ and α is an endomorphism of G, α^0 denoting the identity automorphism.

THEOREM 1.2 (WARNE [10], [11]). *Let S be a regular bisimple semigroup and let $e\in E_S$. Then, eSe is a regular bisimple semigroup with identity e. If E_S is linearly ordered, $S=U(eSe : e\in E_S)$ with $eSe\subset fSf$ if and only if $e\leqq f$ and each eSe is a bisimple inverse semigroup with identity e with $E_{eSe}=\{f\in E_S \mid e\geqq f\}$.*

THEOREM 1.3. *S is an I-bisimple semigroup if and only if $S\cong G\times I\times I$, where G is a group, under the multiplication*

$$(1.1)\quad \begin{aligned}(g, a, b)(h, c, d) &= (gf^{-1}_{b-c,c}h\alpha^{b-c}f_{b-c,d}, a, b+d-c) \quad \text{if } b\geqq c,\\ &= (f^{-1}_{c-b,a}g\alpha^{c-b}f_{c-b,b}h, a+c-b, d) \quad \text{if } c \geqq b,\end{aligned}$$

where α is an endomorphism of G, α^0 denoting the identity automorphism of G, and for $m \in I^0$, $n \in I$,

$f_{0,n} = e$, the identity of G, and for $m > 0$,

$f_{m,n} = u_{n+1}\alpha^{m-1}u_{n+2}\alpha^{m-2}\cdots u_{n+(m-1)}\alpha u_{n+m}$, where $\{u_n : n \in I\}$ is a collection of elements of G with $u_n = e$ if $n > 0$.

Proof. Let us first consider $S = G \times I \times I$ under the multiplication (1.1). Since $f_{m,n}\alpha^c = f_{m+c,n}f_{c,m+n}^{-1}$ for $m \in I^0$, $n \in I$, and $c \in I^0$, the associative law may be verified by a routine calculation. Since $(g, m, n)\mathscr{R}(\mathscr{L})(h, p, q)$ if and only if $m = p$ $(n = q)$, S is bisimple. Since $E_S = \{(e, n, n) : n \in I\}$, S is an I-bisimple semigroup.

Let S^* be an I-bisimple semigroup and let $E_{S^*} = (e_i : i \in I)$ with $e_i < e_j$ if and only if $i > j$. If we let $S_i^* = e_j S e_i$, $S^* = \bigcup (S_i^* : i \in I$ and $i \leqq 0)$ with $S_i^* \subseteq S_i^*$ if and only if $i \geqq j$ by Theorem 1.2. Each S_i^* is a bisimple ω-semigroup with

$$E_{S_i^*} = (e_{i+n} : n \in I^0)$$

by Theorem 1.1 and Theorem 1.2. Thus, we may set $S_i^* = G_i \times I^0 \times I^0$, where G_i is a group under the multiplication

$$(g, m, n)_i(h, p, q)_i = (g\beta_i^{p-r}h\beta_i^{n-r}, m+p-r, n+q-r)_i \text{ where } r = \min(n, p) \tag{1.2}$$

and β_i is an endomorphism of G_i by Theorem 1.1. Let us write $S_i^* = (G_i, \beta_i)_i$. We note that $e_{i+n} = (e, n, n)_i$ for $n \in I^0$. Let $\beta_0 = \alpha_0$ and $G = G_0$. Thus, we may write $(G_0, \beta_0)_0 = [G, \alpha_0]_0$. Suppose that $S_{i+1}^* = [G, \alpha_{i+1}]_{i+1}$ while $S_i^* = (G_i, \beta_i)_i$. Since $[g, 0, 0]_{i+1} \in R_{e_{i+1}} \cap L_{e_{i+1}}$, $[g, 0, 0]_{i+1} = (gf_i, 1, 1)_i$ where f_i is a one-to-one mapping of G onto G_i. Utilizing (1.2) we see that f_i is an isomorphism. For $g \in G$, let $g\alpha_i = gf_i\beta_i f_i^{-1}$. Clearly, α_i is an endomorphism of G. If $g' \in G_i$, then $g' = gf_i$ for some $g \in G$. Thus, $g'\beta_i f_i^{-1} = g' f_i^{-1}\alpha_i$ or $\beta_i f_i^{-1} = f_i^{-1}\alpha_i$. Hence, by a straightforward calculation, $(g, m, n)_i\phi_i = [gf_i^{-1}, m, n]_i$ is an isomorphism of $(G_i, \beta_i)_i$ onto $[G, \alpha_i]_i$. Thus, we may set $(g, m, n)_i = [gf_i^{-1}, m, n]_i$. Hence, $[g, 0, 0]_{i+1} = [g, 1, 1]_i$. Thus, we may let $S^* = U(S_i' : i \in I, i \leqq 0)$ where $S_i' = [G, \alpha_i]_i$ and $[g, 0, 0]_{i+1} = [g, 1, 1]_i$. Since $[e, 0, 1]_{i+1} \in R_{e_{i+1}} \cap L_{e_{i+2}}$, $[e, 0, 1]_{i+1} = [z_i, 1, 2]_i$ for some $z_i \in G$. We may deduce from (1.2) that

$$\begin{aligned}[g, m, n]_{i+1} &= [e, m, 0]_{i+1}[g, 0, 0]_{i+1}[e, 0, n]_{i+1} \\ &= [z_i^{-1}\alpha_i^{m-1}\cdots z_i^{-1}\alpha_i z_i^{-1} g z_i \cdot z_i\alpha_i \cdots z_i\alpha_i^{n-1}, m+1, n+1]_i\end{aligned} \tag{1.3}$$

where if $m = 0$ $(n = 0)$, the left (right) multiplier of g becomes e. We note that

$$\begin{aligned}[z_i^{-1}g\alpha_{i+1}z_i, 2, 2]_i &= [g\alpha_{i+1}, 1, 1]_{i+1} = [e, 1, 1]_{i+1}[g, 0, 0]_{i+1} \\ &= [e, 2, 2]_i[g, 1, 1]_i = [g\alpha_i, 2, 2]_i.\end{aligned}$$

Hence, $g\alpha_i = z_i^{-1}g\alpha_{i+1}z_i$. Thus, by a straightforward calculation,

$$g\alpha_i = (z_{-1}\cdots z_i)^{-1}g\alpha_0(z_{-1}\cdots z_i).$$

For convenience, let

$$u_{i+1} = z_{-1}\cdots z_i \tag{1.4}$$

for $i \geqq -1$. For $n>0$, we set $u_n=e$, the identity of G. Thus,

$$g\alpha_i = u_{i+1}^{-1}g\alpha_0 u_{i+1}. \tag{1.5}$$

We now set $\alpha_0=\alpha$. We will show that $S^*\cong S$ where $S=G\times I\times I$ under the multiplication (1.1). Utilizing (1.1),

$$S_i = (e, i, i)S(e, i, i) = \{(g, m, n) : g \in G, m, n \in I, m \geqq i, n \geqq i\}.$$

As above, S_i is a bisimple ω-semigroup. Let $a_i=(e, i, i+1)$. Then each element of S_i may be uniquely expressed in the form $x=a_i^{-m}ga_i^n \in H_{(e,i+m,i+n)}$ where $g=a_i^m x a_i^{-n} \in H_{(e,i,i)}$. Thus, $(a_i^{-m}ga_i^n)\varphi_i=\langle g, m, n\rangle_i$ where $g=(g, i, i)$ and $a_i g=g\gamma_i a_i$ defines an isomorphism of S_i onto $\langle G, \gamma_i\rangle_i$ (the proof of the last statement is essentially given in [7, p. 164] and will thus be omitted). Since $a_i g=g\gamma_i a_i$,

$$\begin{aligned}(e, i, i+1)(g, i, i) &= (f_{1,i}^{-1}gf_{1,i}, i, i+1) = (u_{i+1}^{-1}gu_{i+1}, i, i+1)\\ &= (g\gamma_i, i, i)(e, i, i+1) = (g\gamma_i, i, i+1).\end{aligned}$$

Hence, by (1.5),

$$g\gamma_i = u_{i+1}^{-1}g\alpha u_{i+1} = g\alpha_i, \quad \text{i.e. } \gamma_i = \alpha_i. \tag{1.6}$$

We note that $a_{i+1}^{-0}ga_{i+1}^0=a_i^{-1}ga_i^1$. Thus, $\langle g, 0, 0\rangle_{i+1}=\langle g, 1, 1\rangle_i$.

We also have $a_{i+1}=a_i^{-1}za_i^2$ where $z=a_i a_{i+1}a_i^{-2}$. Therefore,

$$\begin{aligned}z &= (e, i, i+1)(e, i+1, i+2)(e, i+1, i)(e, i+1, i)\\ &= (u_{i+2}^{-1}u_{i+1}, i, i).\end{aligned}$$

Hence,

$$\langle e, 0, 1\rangle_{i+1} = \langle u_{i+2}^{-1}u_{i+1}, 1, 2\rangle_i = \langle z_i, 1, 2\rangle_i.$$

The last equality is valid by virtue of (1.4). Hence,

$$\langle g, m, n\rangle_{i+1} = \langle z_i^{-1}\alpha_i^{m-1}\cdots z_i^{-1}\alpha_i z_i^{-1}gz_i\cdot z_i\alpha_i\cdots z_i\alpha_i^{n-1}, m+1, n+1\rangle_i \tag{1.7}$$

where if $m=0$ ($n=0$), the left- (right-) hand multiplier of g is e.

By virtue of (1.6) $[g, m, n]_i\theta_i=\langle g, m, n\rangle_i$ defines an isomorphism of S_i' onto S_i.

Let $x\theta=x\theta_i$ if $x \in S_i'$.

If $x \in S_{i+1}\subseteq S_i$, $x\theta_{i+1}=x\theta_i$ by virtue of (1.3) and (1.7). Thus, it follows easily that θ is an isomorphism of S onto S^*. Q.E.D.

COROLLARY 1.1. *An I-bisimple semigroup S contains an $\mathscr{H}$-class consisting of a single element if and only if $S\cong I\times I$ under the multiplication*

$$(a, b)(c, d) = (a+c-r, b+d-r) \tag{1.8}$$

where $r=\min(b, c)$.

Proof. Let S be an I-bisimple semigroup with the above property. Thus,

$$S \cong G\times I\times I,$$

where G is a group, under the multiplication (1.1). Since $(g, a, b)\mathcal{H}(h, c, d)$ if and only if $a=c$ and $b=d$, $G=\{e\}$ and $S\cong I\times I$ under (1.8). The converse follows from Theorem 1.1.

COROLLARY 1.2. *An I-bisimple semigroup S contains an $\mathcal{H}$-class consisting of a single element if and only if $S\cong I\times I$ under the multiplication*

$$(1.9)\qquad (a, b)(c, d) = (a+c, \max(b+c, d)).$$

Proof. Let S be an I-bisimple semigroup with the above property. Thus, $S\cong I\times I$ under (1.8) by Corollary 1.1. However, $(a, b)\varphi=(b-a, b)$ defines an isomorphism of $S=I\times I$ under (1.8) onto $S=I\times I$ under (1.9). By virtue of the above isomorphism the converse is a consequence of Corollary 1.1.

Thus, the only I-bisimple semigroup containing an $\mathcal{H}$-class consisting of a single element is the extended bicyclic semigroup.

COROLLARY 1.3. *Let S be an I-bisimple semigroup. Thus $\mathcal{H}$ is a congruence on S and $S/\mathcal{H}\cong C^*$.*

Proof. Let $S=(G, C^*, \alpha, u_i)$. By Theorem 1.1, $(g, a, b)\mathcal{H}(h, c, d)$ in S if and only if $a=c$ and $b=d$. Thus, it is easily seen that $\mathcal{H}$ is a congruence on S. Clearly, $S/\mathcal{H}$ contains a trivial $\mathcal{H}$-class. Thus the result is a consequence of Corollary 1.1 or Corollary 1.2.

EXAMPLE. An example of an I-bisimple semigroup with nontrivial distinguished elements.

First suppose that $S\cong G\times I\times I$ under the multiplication

$$(1.10)\qquad (g, m, n)(h, p, q) = (g\alpha^{p-r}h\alpha^{n-r}, m+p-r, n+q-r)$$

where $r=\min(n, p)$ and α is an endomorphism of G. Thus, $S_i=(e, i, i)S(e, i, i) =\{(g, m, n) : g\in G, m, n\in I, m\geqq i, n\geqq i\}$ and $S=\bigcup\{S_i : i\in I \text{ and } i\leqq 0\}$. Let $a_i=(e, i, i+1)$. Thus, as in the proof of Theorem 1.1, $(a_i^{-m}ga_i^n)\varphi_i=(g, m, n)_i$ where $g=(g, i, i)$ and $a_ig=g\alpha_ia_i$ is an isomorphism of S_i onto $(G, \alpha_i)_i$. Hence, since $a_ig=g\alpha_ia_i$,

$$(e, i, i+1)(g, i, i) = (g\alpha_i, i, i)(e, i, i+1),$$

$$(g\alpha, i, i+1) = (g\alpha_i, i, i+1).$$

Thus, $\alpha=\alpha_i$ and $S_i\cong(G, \alpha)_i$. We note that $a_{i+1}^{-0}ga_{i+1}^0=a_i^{-1}ga_i$. Thus

$$(1.11)\qquad (g, 0, 0)_{i+1} = (g, 1, 1)_i.$$

Furthermore, $a_{i+1}=a_i^{-1}za_i^2$ where $z=a_ia_{i+1}a_i^{-2}$. Thus,

$$z = (e, i, i+1)(e, i+1, i+2)(e, i+1, i)(e, i+1, i) = (e, i, i).$$

Hence,

$$(1.12)\qquad (e, 0, 1)_{i+1} = (e, 1, 2)_i.$$

Thus, utilizing (1.10), (1.11), and (1.12), we obtain

$$(1.13)\qquad (g, m, n)_{i+1} = (g, m+1, n+1)_i.$$

By a suitable choice of representative elements, any I-bisimple semigroup with $u_i = e$ must be reduced to an "inverse limit" of the above type. We will give an example of an I-bisimple semigroup where this reduction is not possible.

Let G be the group of integers under addition. Let $S = G \times I \times I$ under the multiplication

$$(1.14)\qquad \begin{aligned}(g, a, b)(h, c, d) &= (g + h2^{b-c} + f_{b-c,d} - f_{b-c,c}, a, b+d-c) \quad \text{if } b \geqq c,\\ &= (g2^{c-b} + h + f_{c-b,b} - f_{c-b,a}, c+a-b, d) \quad \text{if } c \geqq b,\end{aligned}$$

where

$f_{0,n} = 0$ for $n \in I$ while if $m \in I^0$ and $m > 0$,

$f_{m,n} = a_{n+1}2^{m-1} + a_{n+2}2^{m-2} + \cdots + a_{m+n-1}2 + a_{m+n}$, where $a_n = 0$ for $n > 0$ and for $n \leqq 0$

$$\begin{aligned} a_n &= 1 \quad \text{if } n \text{ is odd},\\ &= 0 \quad \text{if } n \text{ is even}.\end{aligned}$$

By Theorem 1.1, S is an I-bisimple semigroup. As in the proof of Theorem 1.1, $S_i = (e, i, i)S(e, i, i) = \{(g, m, n) : g \in G, m, n \in I, m \geqq i, n \geqq i\}$ and

$$S = \bigcup (S_i : i \in I, i \leqq 0).$$

Let $(y_i : i \in I, i \leqq 0)$ be any sequence of elements of G and let $a_i = (y_i, i, i+1)$. Thus, as usual, $(a_i^{-m} g a_i^n)\varphi_i = (g, m, n)_i$ where $a_i g = g\gamma_i a_i$ and $g = (g, i, i)$ defines an isomorphism of S_i onto $(G, \gamma_i)_i$.

We again note that

$$(1.15)\qquad a_{i+1} = a_i^{-1} z a_i^2$$

where $z = a_i a_{i+1} a_i^{-2}$. Thus, by (1.14),

$$\begin{aligned} z &= (y_i, i, i+1)(y_{i+1}, i+1, i+2)(-y_i, i+1, i)(-y_i, i+1, i)\\ &= (y_{i+1} - 2y_i + a_{i+1} - a_{i+2}, i, i)_i.\end{aligned}$$

Hence, by (1.15),

$$(0, 0, 1)_{i+1} = (y_{i+1} - 2y_i + a_{i+1} - a_{i+2}, 1, 2).$$

Thus, by (1.12), if S is an I-bisimple semigroup "without factor terms", there must exist a sequence $\{y_i : i \in I, i \leqq 0\}$ of elements of G such that $y_{i+1} - 2y_i + a_{i+1} - a_{i+2} = 0$ for all $i \in I$ where $i \leqq 0$. Hence, $y_0 = 2y_{-1}$ while if $i \leqq -2$

$$\begin{aligned} y_{i+1} &= 2y_i - 1 \quad \text{if } i \text{ is even},\\ &= 2y_i + 1 \quad \text{if } i \text{ is odd}.\end{aligned}$$

To simplify the notation, let $b_n = y_{-n}$. Thus, $b_0 = 2b_1$ while if $n \geqq 1$

$$\begin{aligned} b_n &= 2b_{n+1}-1 \quad \text{if } n \text{ is odd,} \\ &= 2b_{n+1}+1 \quad \text{if } n \text{ is even.} \end{aligned}$$

Let $b_0 = 2x_0$ where x_0 is chosen arbitrarily. Hence, $b_1 = x_0$. Thus, for $n \geqq 1$,

$$\begin{aligned} b_{n+1} &= \frac{x_0+1-2+2^2+\cdots+(-1)^{n-1}2^{n-1}}{2^n}, \\ &= \frac{\frac{1}{3}(1-(-2)^n)+x_0}{2^n}. \end{aligned}$$

Hence, for $n \geqq 1$,

$$b_{2n+1} = \frac{\frac{1}{3}(1-2^{2n})+x_0}{2^{2n}}.$$

Thus, $\lim_{n\to\infty} b_{2n+1} = -\frac{1}{3}$ which is impossible since each b_n is an integer.

REMARK. The above example is related to an example communicated to the author by Professor A. H. Clifford.

2. **The homomorphism theory.** In this section, we give necessary and sufficient conditions for two I-bisimple semigroups to be isomorphic, and we determine the homomorphisms of an I-bisimple semigroup onto an I-bisimple semigroup. The following theorem is obtained from [9, Theorem 2.3, Theorem 1.2, and Theorem 1.1]. A proof will be given elsewhere [14].

THEOREM 2.1 *Let $S=(G, C, \alpha)$ and $S^*=(G^*, C, \beta)$ be bisimple ω-semigroups. Let f be a homomorphism of G onto G^* and $z \in G^*$ such that $\alpha f = f\beta C_z$ where $xC_z = zxz^{-1}$ for $x \in G^*$. For each $(g, m, n) \in S$ define*

$$(2.1) \qquad (g, m, n)\theta = (z^{-1}\beta^{m-1}\cdots z^{-1}\beta z^{-1}(gf)z\cdot z\beta\cdots z\beta^{n-1}, m, n)$$

if $m>0$, $n>0$. If $m=0$ ($n=0$), the left- (right-) multiplier of gf is e^*, the identity of G^*.

Then, θ is a homomorphism of S onto S^* and conversely every such homomorphism is obtained in this fashion. θ is an isomorphism if and only if f is an isomorphism.

The condition for two bisimple ω-semigroups to be isomorphic (homomorphic) was given by Reilly [7] (Munn and Reilly [4]), although the isomorphism (homomorphism) was not exhibited.

If G is a group and $y \in G$, we will denote the inner automorphism of G determined by y by C_y, i.e. $xC_y = yxy^{-1}$ $(x \in G)$.

THEOREM 2.2. *Let $S=(G, C^*, \alpha, u_i)$ and $S^*=(G^*, C^*, \beta, v_i)$ be I-bisimple semigroups. Then, S is isomorphic to S^* if and only if there exists a sequence*

$$\{z_i : i \in I, i \leqq 0\}$$

of elements of G^*, *a sequence* $\{f_i : i \in I, i \leqq 0\}$ *of isomorphisms of* G *onto* G^*, *and* $a \in I$ *such that for all* $i \in I$, $i \leqq 0$

$$z_{i+1}v_{i+a+2}^{-1}v_{i+a+1} = ((u_{i+2}^{-1}u_{i+1})f_iC_{z_i^{-1}})(z_i\beta C_{v_{i+a+1}^{-1}}), \tag{2.2}$$

$$f_i = f_{i+1}C_{z_i}, \tag{2.3}$$

$$\alpha C_{u_{i+1}^{-1}}f_i = f_i\beta C_{z_iv_{i+a+1}^{-1}}. \tag{2.4}$$

Proof. As in the proof of Theorem 1.1, $S = U(S_i : i \in I, i \leqq 0)$ where $S_i = (G, \alpha_i)_i$,

$$\alpha_i = \alpha C_{u_{i+1}^{-1}}, \tag{2.5}$$

$$(g, m, n)_{i+1} = (s_i^{-1}\alpha_i^{m-1}\cdots s_i^{-1}\alpha_i s_i^{-1}gs_i\cdot s_i\alpha_i\cdots s_i\alpha_i^{n-1}, m+1, n+1)_i, \tag{2.6}$$

where if $m=0$ $(n=0)$ the left (right) multiplier of g is e, the identity of G, and

$$s_i = u_{i+2}^{-1}u_{i+1}. \tag{2.7}$$

Similarly, $S^* = U(S_i^* : i \in I)$ where $S_i^* = [G^*, \beta_i]_i$ (also see p. 371)

$$\beta_i = \beta C_{v_{i+1}^{-1}}, \tag{2.8}$$

$$[g, m, n]_{i+1} = [t_i^{-1}\beta_i^{m-1}\cdots t_i^{-1}\beta_i t_i^{-1}gt_i\cdot t_i\beta_i\cdots t_i\beta_i^{n-1}, m+1, n+1]_i, \tag{2.9}$$

where if $m=0$ $(n=0)$, the left (right) multiplier of g is e^*, the identity of G^*, and

$$t_i = v_{i+2}^{-1}v_{i+1}. \tag{2.10}$$

First suppose that θ is an isomorphism of S onto S^*. Suppose that $(e, 0, 0)_0\theta = [e, 0, 0]_a$. Thus, θ induces an isomorphism θ_0 of $S_0 = (e, 0, 0)_0S(e, 0, 0)_0$ onto $[e, 0, 0]_aS^*[e, 0, 0]_a = S_a^*$. Hence, θ induces an isomorphism θ_i of S_i onto S_{i+a}^* for each $i \in I$ with $i \leqq 0$. Thus, by virtue of Theorem 2.1, for each i there exists an isomorphism f_i of G onto G^* and $z_i \in G^*$ such that

$$\alpha_i f_i = f_i\beta_{i+a}C_{z_i}, \tag{2.11}$$

and

$$(g, m, n)_i\theta_i = [z_i^{-1}\beta_{i+a}^{m-1}\cdots z_i^{-1}\beta_{i+a}z_i^{-1}gf_iz_i\cdot z_i\beta_{i+a}\cdots z_i\beta_{i+a}^{n-1}, m, n]_{i+a}, \tag{2.12}$$

where if $m=0$ $(n=0)$ the left (right) multiplier of gf is e^*.

Combining (2.5), (2.8), and (2.11), we obtain (2.4).

If $x \in S_{i+1} \subseteq S_i$, $x\theta = x\theta_{i+1} = x\theta_i$. Thus, since $(e, 0, 1)_{i+1} = [s_i, 1, 2]_i$ by (2.6),

$$[z_{i+1}, 0, 1]_{i+a+1} = [z_i^{-1}(s_if_i)z_i(z_i\beta_{i+a}), 1, 2]_{i+a} \tag{2.13}$$

by virtue of (2.12). However, by (2.9),

$$[z_{i+1}, 0, 1]_{i+a+1} = [z_{i+1}t_{i+a}, 1, 2]_{i+a}. \tag{2.14}$$

Thus, combining (2.13), (2.14), (2.10), (2.7), and (2.8), we obtain (2.2).

Furthermore, by (2.6), $(g, 0, 0)_{i+1}=(g, 1, 1)_i$. Hence, $(g, 0, 0)_{i+1}\theta_{i+1}=(g, 1, 1)_i\theta_i$. Thus, by (2.12),

$$[gf_{i+1}, 0, 0]_{i+a+1} = [z_i^{-1}gf_iz_i, 1, 1]_{i+a}. \tag{2.15}$$

However, by (2.9), we have

$$[gf_{i+1}, 0, 0]_{i+a+1} = [gf_{i+1}, 1, 1]_{i+a}. \tag{2.16}$$

Thus, combining (2.15) and (2.16) ,we obtain (2.3).

Let us now assume the conditions of the theorem are valid. By (2.4), (2.5), (2.8), and Theorem 2.1, (2.12) defines an isomorphism of S_i onto S^*_{i+a}.

By (2.12), (2.9), (2.10), (2.2), (2.7), (2.8), and (2.12),

$$\begin{aligned}(e, 0, 1)_{i+1}\theta_{i+1} &= [z_{i+1}, 0, 1]_{i+a+1} = [z_{i+1}t_{i+a}, 1, 2]_{i+a}\\ &= [z_{i+1}v_{i+a+2}^{-1}v_{i+a+1}, 1, 2]_{i+a}\\ &= [((u_{i+2}^{-1}u_{i+1})f_iC_{z_i^{-1}})z_i\beta C_{v_{i+a+1}^{-1}}, 1, 2]_{i+a}\\ &= [(s_if_iC_{z_i^{-1}})(z_i\beta_{i+a}), 1, 2]_{i+a}\\ &= [z_i^{-1}(s_if_i)z_i(z_i\beta_{i+a}), 1, 2]_{i+a} = (s_i, 1, 2)_i\theta_i.\end{aligned}$$

Thus,

$$(e, 0, n)_{i+1}\theta_{i+1} = (s_i\cdot s_i\alpha_i\cdots s_i\alpha_i^{n-1}, 1, n+1)_i\theta_i \text{ if } n \geqq 1. \tag{2.17}$$

By taking inverses, we obtain

$$(e, n, 0)_{i+1}\theta_{i+1} = (s_i^{-1}\alpha_i^{n-1}\cdots s_i^{-1}\alpha_is_i^{-1}, n+1, 1)_i\theta_i. \tag{2.18}$$

By (2.12), (2.9), (2.3), and (2.12),

$$\begin{aligned}(g, 0, 0)_{i+1}\theta_{i+1} &= [gf_{i+1}, 0, 0]_{i+a+1} = [gf_{i+1}, 1, 1]_{i+a}\\ &= [z_i^{-1}gf_iz_i, 1, 1]_{i+a} = (g, 1, 1)_i\theta_i.\end{aligned} \tag{2.19}$$

Thus, combining (2.17), (2.18), and (2.19), we obtain

$$\begin{aligned}(g, m, n)_{i+1}\theta_{i+1} &= (e, m, 0)_{i+1}\theta_{i+1}(g, 0, 0)_{i+1}\theta_{i+1}(e, 0, n)_{i+1}\theta_{i+1}\\ &= (s_i^{-1}\alpha_i^{m-1}\cdots s_i^{-1}\alpha_is_i^{-1}gs_i\cdot s_i\alpha_i\cdots s_i\alpha_i^{n-1}, m+1, n+1)_i\theta_i.\end{aligned} \tag{2.20}$$

Let us define

$$x\theta = x\theta_i \quad \text{if } x\in S_i. \tag{2.21}$$

Hence, by (2.6) and (2.20), θ defines an isomorphism of S onto S^*. Let N denote the natural numbers.

THEOREM 2.3. *Let $S=(G, C^*, \alpha, u_i)$ and $S^*=(G^*, C^*, \beta, v_i)$ be I-bisimple semigroups. Let $\{z_i : i\in I, i\leqq 0\}$ be a sequence of elements of G^*, and let $\{f_i : i\in I, i\leqq 0\}$*

be a sequence of homomorphisms of G onto G^*, and let a be an element of I such that for all $i \in I \backslash N$

$$z_{i+1}v_{i+a+2}^{-1}v_{i+a+1} = ((u_{i+2}^{-1}u_{i+1})f_iC_{z_i^{-1}})(z_i\beta C_{v_{i+a+1}^{-1}}), \tag{2.22}$$

$$f_i = f_{i+1}C_{z_i}, \tag{2.23}$$

$$\alpha C_{u_{i+1}^{-1}}f_i = f_i\beta C_{z_iv_{i+a+1}^{-1}}. \tag{2.24}$$

For each element $(g, m, n)_i \in S_i (i \in I \backslash N)$, define

$$(g, m, n)_i\theta = [z_i^{-1}\beta_{i+a}^{m-1}\cdots z_i^{-1}\beta_{i+a}z_i^{-1}gf_iz_i\cdot z_i\beta_{i+a}\cdots z_i\beta_{i+a}^{n-1}, m, n]_{i+a}$$

where the square brackets denote an element of S^* and where if $m=0$ $(n=0)$ the left (right) multiplier of gf_i is e^*, the identity of G^*.

Then, θ is a homomorphism of S onto S^* and conversely every such homomorphism is obtained in this fashion.

Proof. Let θ be a homomorphism of S onto S^*. Let us suppose that $(e, 0, 0)_i\theta = (e, 0, 0)_{a_i}$. Hence θ induces a homomorphism θ_i of S_i onto $S^*_{a_i}$. By Theorem 2.1, θ_i is given by

$$(g, m, n)_i\theta_i = [z_i^{-1}\beta_{a_i}^{m-1}\cdots z_i^{-1}\beta_{a_i}z_i^{-1}gf_iz_i\cdot z_i\beta_{a_i}\cdots z_i\beta_{a_i}^{n-1}m, n]_{a_i}, \tag{2.25}$$

where if $m=0$ $(n=0)$ the left (right) multiplier of gf_i is e^*, the identity of G^* and where, $z_i \in G^*$ and f_i is a homomorphism of G onto G^* such that

$$\alpha_i f_i = f_i\beta_{a_i}C_{z_i}. \tag{2.26}$$

As usual, α_i and β_{a_i} are given by (2.5) and (2.8).

Since $(e, 0, 0)_{i+1} = (e, 1, 1)_i$ by (2.6), $(e, 0, 0)_{i+1}\theta_{i+1} = (e, 1, 1)_i\theta_i$. Thus, by (2.25),

$$[e, 0, 0]_{a_{i+1}} = [z_i^{-1}gf_iz_i, 1, 1]_{a_i}. \tag{2.27}$$

Clearly, $a_{i+1} \geqq a_i$, i.e., $a_{i+1} = a_i + b_i$ for some $b_i \in I^0$. Hence, by (2.9),

$$[e, 0, 0]_{a_{i+1}} = [e, 0, 0]_{a_i+b_i} = [e, b_i, b_i]_{a_i}.$$

Thus, by (2.27), $b_i = 1$ and $a_{i+1} = a_i + 1$. Hence, if we let $a_0 = a$, $a_i = a+i$ for all $i \in I$, $i \leqq 0$. Hence the remainder of the proof parallels that of Theorem 2.2.

3. **The maximal group homomorphic image.** In this section we describe the maximal cancellative homomorphic image of an ω-right cancellative semigroup(1), the maximal group homomorphic image of a bisimple ω-semigroup and finally the maximal group homomorphic image of an I-bisimple semigroup.

We first review constructions of Clifford [1] and of the author [8].

Let S be a semigroup with identity 1. The set of elements of S having a right inverse with respect to 1 is called the right unit subsemigroup of S. Let S be a

(1) The terminology of "ω-right cancellative semigroup" and "bisimple ω-semigroup" leads to some confusion. Thus, in [7], we employ the term "ω-bisimple semigroup".

bisimple inverse semigroup with identity and let P denote the right unit subsemigroup of S. The principal left ideals of P form a semilattice (with respect to inclusion). From each $\mathscr{L}$-class of P pick a fixed representative element and let $a \vee b$ ($a, b \in P$) denote the representative element of the $\mathscr{L}$-class containing c where $Pa \cap Pb = Pc$. Define $(a^*b)b = a \vee b$. Then, $S \cong P \times P$ under the following definition of equality and multiplication:

(3.1) $(a, b) = (c, d)$ if $a = uc$ and $b = ud$ where u is a unit of P (an element of P which has a two-sided inverse with respect to 1, the identity of P).

(3.2) $(a, b)(c, d) = ((c^*b)a, (b^*c)d)$.

This construction is due to Clifford [1].

Let us now review the construction of the author [8] for describing the maximal group homomorphic image of a bisimple inverse semigroup with identity.

Let us define the following relation on P:

(3.3) If $a, b \in P$, $a\eta b$ if and only if there exists $h \in P$ such that $ha = hb$.

Then η is the minimal cancellative congruence on P or $\bar{P} = P/\eta$ is the maximal cancellative homomorphic image of P. Let $p \to \bar{p}$ denote the canonical homomorphism of P onto $\bar{P}$. Let $\bar{a}, \bar{b} \in \bar{P}$. We consider the set F of all pairs of elements of $\bar{P}$ writing them as fractions $\bar{b}/\bar{a}$. The relation $=$ between these fractions shall be defined thus:

$$\bar{b}/\bar{a} = \bar{d}/\bar{c} \tag{3.4}$$

shall mean that elements $\bar{x}$ and $\bar{y}$ exist in $\bar{P}$ such that $\bar{x}\bar{a} = \bar{y}\bar{c}$ and $\bar{x}\bar{b} = \bar{y}\bar{d}$.

The definition of the product is

$$\bar{b}/\bar{a} \cdot \bar{d}/\bar{c} = \bar{k}\bar{d}/\bar{h}\bar{a} \quad \text{where } \bar{h}\bar{b} = \bar{k}\bar{c}. \tag{3.5}$$

F is a group and the isomorphism of $\bar{P}$ into F is given by $\bar{a} \to \bar{a}/\bar{1}$.

THEOREM 3.1 (WARNE [8]). *With S, P, $\bar{P}$, and F as above the mapping ϕ: $(a, b) \to \bar{b}/\bar{a}$ is a homomorphism of S onto F, and F is thereby the maximal group homomorphic image of S.*

In [14], we called a right cancellative semigroup P with identity an ω-right cancellative semigroup if and only if its ideal structure (the set of principal left ideals of P ordered by inclusion) is order isomorphic to I^0 under the reverse of the usual order. The structure of ω-right cancellative semigroups was given by Rees.

THEOREM 3.2 (REES [6]). *S is an ω-right cancellative semigroup if and only if $S \cong G \times I^0$, where G is a group, under the multiplication*

$$(g, m)(h, n) = (g(h\alpha^m), m+n) \tag{3.6}$$

where α is an endomorphism of G, α^0 being interpreted as the identity transformation.

If P is an ω-right cancellative semigroup, we will write $P = (G, I^0, \alpha)$ where G is the structure group and α is the structure endomorphism of P.

THEOREM 3.3. *Let $P=(G, I^0, \alpha)$ be an ω-right cancellative semigroup and let e be the identity of G. If $N=\{g \in G \mid g\alpha^n=e$ for some $n \in I^0\}$, N is a normal subgroup of G. If $(xN)\theta=(x\alpha)N$, $x \in G$, θ is an endomorphism of G/N. Let $g \to \bar{g}$ denote the natural homomorphism of G onto G/N. The maximal cancellative homomorphic image $\bar{P}$ of P is $(G/N, I^0, \theta)$ and the canonical homomorphism of P onto $\bar{P}$ is given by $(g, m)\eta=(\bar{g}, m)$.*

Proof. Let η be the minimum cancellative congruence relation on P. By (3.6) and (3.3), $(g, k)\eta(h, j)$ if and only if $k=j$ and there exists $s \in I^0$ such that $g\alpha^s=h\alpha^s$. If we define $A\rho B$ ($A, B \in G$) if and only if $A\alpha^c=B\alpha^c$ for some $c \in I^0$, ρ is a congruence relation on G. Let N denote the congruence class containing the identity, i.e., $N=\{A \in G \mid A\alpha^c=e$ for some $c \in I^0\}$. It is easy to see that the mapping $(AN)\theta=(A\alpha)N$ is an endomorphism of G/N. Let $\bar{P}$ be the maximal cancellative homomorphic image of P under the natural homomorphism

$$(g, j) \to \overline{(g, j)}.$$

If we define

$$\overline{(g, j)}\delta = (\bar{g}, j)$$

it is easily seen that δ is an isomorphism of $\bar{P}$ onto $(G/N, I^0, \theta)$.

REMARK 3.1. By the proof of [12, Theorem 3.1], S is a bisimple ω-semigroup (G, C, α) if and only if its right unit subsemigroup P is the ω-right cancellative semigroup (G, I^0, α).

We now completely describe the maximal group homomorphic image of a bisimple ω-semigroup (including the defining homomorphism). If σ is an equivalence relation on a set X, we let x_σ denote the equivalence class containing the element x of X.

THEOREM 3.4. *Let $S=(G, C, \alpha)$ be a bisimple ω-semigroup and let e denote the identity of G. If $N=\{g \in G \mid g\alpha^n=e$ for some $n \in I^0\}$, N is a normal subgroup of G. If $(xN)\theta=(x\alpha)N$, $x \in G$, θ is an endomorphism of G/N. Let $g \to \bar{g}$ denote the natural homomorphism of G onto G/N. Let us define a relation σ on $G/N \times (I^0)^2$ by the rule*

$$((\bar{g}, a, b), (\bar{h}, c, d)) \in \sigma \tag{3.7}$$

if and only if there exists $x, y \in I^0$ such that $x+a=y+c$, $x+b=y+d$, and $\bar{g}\theta^x=\bar{h}\theta^y$. Then, σ is an equivalence relation on $G/N \times (I^0)^2$. Furthermore, the rule

$$(\bar{g}, a, b)_\sigma(\bar{h}, c, d)_\sigma = (\bar{g}\theta^c\bar{h}\theta^b, a+c, b+d)_\sigma \tag{3.8}$$

defines a binary operation on $G/N \times (I^0)^2/\sigma = V$ whereby V becomes a group which is the maximal group homomorphic image of S.

The canonical homomorphism of S onto V is given by $(g, a, b)\gamma=(\bar{g}, a, b)_\sigma$.

Proof. By Theorem 3.3 and Remark 3.1, $\bar{P}=(G/N, I^0, \theta)$. We will utilize (3.4) and (3.5) to determine the group of fractions F of $\bar{P}$. Utilizing (3.4), it is easily seen that

$$(\bar{B}, b)/(\bar{A}, a) = (\bar{D}, d)/(\bar{C}, c) \tag{3.9}$$

if and only if there exists $x, y \in I^0$ such that $x+a=y+c$, $x+b=y+d$, and

$$(\bar{A}^{-1}\bar{B})\theta^x = (\bar{C}^{-1}\bar{D})\theta^y.$$

By (3.5),

$$(\bar{B}, b)/(\bar{A}, a)\cdot(\bar{D}, d)/(\bar{C}, c) = (\bar{K}(\bar{D}\theta^k), k+d)/(\bar{H}(\bar{A}\theta^h), h+a)$$

where $\bar{H}(\bar{B}\theta^h)=\bar{K}(\bar{C}\theta^k)$ and $h+b=k+c$. Thus, applying (3.9) with $x=c$ and $y=h$, we obtain

$$(3.9)' \qquad (\bar{B}, b)/(\bar{A}, a)\cdot(\bar{D}, d)/(\bar{C}, c) = ((\bar{C}^{-1}\bar{D})\theta^b, b+d)/((\bar{B}^{-1}\bar{A})\theta^c, a+c).$$

It is easy to see that σ is an equivalence relation and that V is a groupoid.

Hence by (3.9) and (3.9)′

$$(3.10) \qquad ((\bar{B}, b)/(\bar{A}, a))\varphi = (\bar{A}^{-1}\bar{B}, a, b)_\sigma$$

defines an isomorphism of F onto V.

Let S^* be the semigroup constructed from $P=(G, I^0, \alpha)$ (see Remark 3.1) by means of the Clifford construction. Thus, utilizing Theorem 3.2, (3.1), and [12, p. 572, Equation 3.4], it is easily seen that $(g, a, b)\lambda=((A, a), (B, b))$ where $g=A^{-1}B$, defines an isomorphism of $S=(G, C, \alpha)$ onto S^*.

By Theorem 3.1, F is the maximal group homomorphic image of S^* under the homomorphism

$$((A, a), (B, b))\phi = (\bar{B}, b)/(\bar{A}, a).$$

Hence, V is the maximal group homomorphic image of S under the homomorphism $(g, a, b)\gamma=(g, a, b)\lambda\phi\varphi=(\bar{g}, a, b)$.

Note. Another construction of the maximal group homomorphic image of S is given in [4].

The following result is obtained from [9, Theorem 2.3 and Theorem 1.1] and its proof will be given elsewhere [14].

Theorem 3.5 *Let $S=(G, C, \alpha)$ be a bisimple ω-semigroup and let G^* be a group. Let f be a homomorphism of G into G^* such that $fC_z=\alpha f$ where $xC_z=zxz^{-1}$ for $x \in G^*$. Then, $(g, m, n)\phi=z^{-m}gfz^n$ is a homomorphism of S into G^* and, conversely, every such homomorphism is obtained in this fashion.*

Theorem 3.6. *Let $S=(G, C^*, \alpha, u_i)$ be an I-bisimple semigroup and let e be the identity of G. If $N=\{g \in G \mid g\alpha^n=e$ for some $n \in I^0\}$, N is a normal subgroup of G. If $(xN)\theta=(x\alpha)N$, θ is an endomorphism of G/N. Let $g \to \bar{g}$ be the natural homomorphism of G onto G/N. Let us define a relation σ on $G/N\times(I^0)^2$ by the rule $((\bar{g}, a, b), (\bar{h}, c, d)) \in \sigma$ if and only if there exists $x, y \in I^0$ such that $x+a=y+c$, $x+b=y+d$, and $\bar{g}\theta^x=\bar{h}\theta^y$. Then, σ is an equivalence relation on $G/N\times(I^0)^2$. Furthermore, the rule $(\bar{g}, a, b)_\sigma(\bar{h}, c, d)_\sigma=(\bar{g}\theta^c\bar{h}\theta^b, a+c, b+d)_\sigma$ defines a binary operation on $G/N\times(I^0)^2/\sigma=H$ whereby H becomes a group which is the maximal group homomorphic image of S. The homomorphism of S onto H is given by*

$$(3.11) \qquad (g, m, n)_i\phi = (x_i^{-1}\theta^{m-1}\cdots x_i^{-1}\theta x_i^{-1}\bar{g}\delta_i x_i\cdot x_i\theta\cdots x_i\theta^{n-1}, m, n)_\sigma$$

where if $m=0$ $(n=0)$ *the left* (*right*) *multiplier of* $\bar{g}\delta_i$ *is* $\bar{e}$ *and where*

$$x_0 = \bar{e},$$
$$x_{-1} = \bar{u}_0^{-1} \quad \textit{while for } i \leqq -2,$$
$$x_i = \bar{u}_0^{-1}(\bar{u}_{-1}^{-1}\theta)\cdots\bar{u}_{i+1}^{-1}\theta^{-(i+1)}\bar{u}_{i+2}\theta^{-(i+1)}\bar{u}_{i+3}\theta^{-(i+2)}\cdots\bar{u}_0\theta,$$
$$\bar{g}\delta_0 = \bar{g} \quad \textit{while if } i \leqq -1,$$
$$\bar{g}\delta_i = \bar{u}_0^{-1}\cdot\bar{u}_{-1}^{-1}\theta\cdots\bar{u}_{i+1}^{-1}\theta^{-(i+1)}\bar{g}\theta^{-i}\bar{u}_{i+1}\theta^{-(i+1)}\cdots\bar{u}_{-1}\theta\bar{u}_0.$$

Proof. We first use Theorem 3.5 to determine a homomorphism ϕ_i of S_i into H for each $i \in I$ with $i \leqq 0$. Let x_i snd δ_i be defined as in the statement of the theorem. In the notation of Theorem 3.5, let $G^* = H$, $z_i = (x_i, 0, 1)_\sigma$, and $gf_i = (\bar{g}\delta_i, 0, 0)_\sigma$. Clearly, f_i is a homomorphism of G into H. Let us first verify the condition of Theorem 3.5.

$$\begin{aligned} z_i g f_i z_i^{-1} &= (x_i, 0, 1)_\sigma(\bar{g}\delta_i, 0, 0)_\sigma(x_i^{-1}, 1, 0)_\sigma \\ &= (x_i\bar{g}\delta_i\theta, 0, 1)_\sigma(x_i^{-1}, 1, 0)_\sigma \\ &= ((x_i\bar{g}\delta_i\theta)\theta x_i^{-1}\theta, 1, 1)_\sigma \\ &= (x_i\bar{g}\delta_i\theta x_i^{-1}, 0, 0)_\sigma \\ &= (\bar{u}_0^{-1}\cdot\bar{u}_{-1}^{-1}\theta\cdots\bar{u}_{i+1}^{-1}\,\theta^{-(i+1)}\{\bar{u}_{i+2}\theta^{-(i+1)}\bar{u}_{i+3}\theta^{-(i+2)}\cdots\bar{u}_0\theta \\ &\qquad \cdot\bar{u}_0^{-1}\theta\bar{u}_{-1}^{-1}\theta^2\cdots\bar{u}_{i+2}^{-1}\theta^{-(i+1)}\}\bar{u}_{i+1}^{-1}\theta^{-i}\bar{g}\theta^{-i+1} \\ &\quad \cdot\bar{u}_{i+1}\theta^{-i}\{\bar{u}_{i+2}\theta^{-(i+1)}\cdots\bar{u}_{-1}\theta^2\bar{u}_0\theta \\ &\qquad \cdot\bar{u}_0^{-1}\theta\cdots\bar{u}_{i+2}^{-1}\theta^{-(i+1)}\}\bar{u}_{i+1}\theta^{-(i+1)}\cdots\bar{u}_{-1}\theta\bar{u}_0, 0, 0)_\sigma \\ &= (\bar{u}_0^{-1}\cdot\bar{u}_{-1}^{-1}\theta\cdots\bar{u}_{i+1}^{-1}\theta^{-(i+1)}\bar{u}_{i+1}^{-1}\theta^{-i}\bar{g}\theta^{-i+1} \\ &\qquad \cdot\bar{u}_{i+1}\theta^{-i}\bar{u}_{i+1}\theta^{-(i+1)}\cdots\bar{u}_{-1}\theta\bar{u}_0, 0, 0)_\sigma. \end{aligned} \tag{3.12}$$

By (2.5), $g\alpha_i = u_{i+1}^{-1}g\alpha u_{i+1}$. Thus,

$$\overline{g\alpha_i} = \bar{u}_{i+1}^{-1}\overline{g\alpha}\bar{u}_{i+1} = \bar{u}_{i+1}^{-1}\bar{g}\theta\bar{u}_{i+1}. \tag{3.13}$$

The last equality follows from the statement of the theorem. Thus, using (3.13),

$$\begin{aligned} g\alpha_i f_i &= (\overline{g\alpha_i}\delta_i, 0, 0)_\sigma = ((\bar{u}_{i+1}^{-1}\bar{g}\theta\bar{u}_{i+1})\delta_i, 0, 0)_\sigma \\ &= (\bar{u}_0^{-1}\cdot\bar{u}_1^{-1}\theta\cdots\bar{u}_{i+1}^{-1}\theta^{-(i+1)}(\bar{u}_{i+1}^{-1}\bar{g}\theta\bar{u}_{i+1})\theta^{-i}\bar{u}_{i+1}\theta^{-(i+1)}\cdots\bar{u}_{-1}\theta\bar{u}_0, 0, 0)_\sigma \\ &= (\bar{u}_0^{-1}\cdot\bar{u}_{-1}^{-1}\theta\cdots\bar{u}_{i+1}^{-1}\theta^{-(i+1)}\bar{u}_{i+1}^{-1}\theta^{-i}\bar{g}\theta^{-i+1}\bar{u}_{i+1}\theta^{-i} \\ &\qquad \cdot\bar{u}_{i+1}\theta^{-(i+1)}\cdots\bar{u}_{-1}\theta\bar{u}_0, 0, 0)_\sigma. \end{aligned} \tag{3.14}$$

Thus, comparing (3.12) and (3.14), we see that $z_i g f_i z_i^{-1} = g\alpha_i f_i$ as desired. Hence, by Theorem 3.5,

$$\begin{aligned} (g, m, n)_i\phi_i &= (x_i^{-1}, 1, 0)_\sigma^m(\bar{g}\delta_i, 0, 0)_\sigma(x_i, 0, 1)_\sigma^n \\ &= (x_i^{-1}\theta^{m-1}\cdots x_i^{-1}\theta x_i^{-1}\bar{g}\delta_i x_i\cdot x_i\theta\cdots x_i\theta^{n-1}, m, n)_\sigma, \end{aligned}$$

where if $m=0$ $(n=0)$ the left (right) multiplier of $\bar{g}\delta_i$ is $\bar{e}$, defines a homomorphism of S_i into H.

We note that $(g, m, n)_0\phi_0=(\bar{g}, m, n)_\sigma$. Hence, by Theorem 3.4, ϕ_0 is a homomorphism of S_0 onto H.

Let us define $x\phi=x\phi_i$ if $x\in S_i$. We will show that ϕ is a homomorphism of S onto H. We note that

$$\begin{aligned}(g,1,1)_i\phi_i &= (x_i^{-1}\bar{g}\delta_i x_i, 1, 1)_\sigma\\ &= (\bar{u}_0^{-1}\theta\cdots\bar{u}_{i+2}^{-1}\theta^{-(i+1)}\{\bar{u}_{i+1}\theta^{-(i+1)}\cdots\bar{u}_{-1}\theta\bar{u}_0\bar{u}_0^{-1}\cdot\bar{u}_{-1}^{-1}\theta\cdots\bar{u}_{i+1}^{-1}\theta^{-(i+1)}\}\\ &\quad\cdot\bar{g}\theta^{-i}\{\bar{u}_{i+1}\theta^{-(i+1)}\cdots\bar{u}_{-1}\theta\bar{u}_0\cdot\bar{u}_0^{-1}\cdot\bar{u}_{-1}^{-1}\theta\cdots\bar{u}_{i+1}^{-1}\theta^{-(i+1)}\}\\ &\qquad\qquad\cdot\bar{u}_{i+2}\theta^{-(i+1)}\cdots\bar{u}_0\theta, 1, 1)_\sigma\\ &= (\bar{u}_0^{-1}\theta\cdots\bar{u}_{i+2}^{-1}\theta^{-(i+1)}\bar{g}\theta^{-i}\bar{u}_{i+2}\theta^{-(i+1)}\cdots\bar{u}_0\theta, 1, 1)_\sigma\\ &= (\bar{g}\delta_{i+1}\theta, 1, 1)_\sigma = (\bar{g}\delta_{i+1}, 0, 0)_\sigma = (g, 0, 0)_{i+1}\phi_{i+1}.\end{aligned}$$

Let $s_i=u_{i+2}^{-1}u_{i+1}$. Thus,

$$\begin{aligned}(s_i,1,2)_i\phi_i &= (x_i^{-1}\bar{s}_i\delta_i x_i\cdot x_i\theta, 1, 2)_\sigma\\ &= (\bar{u}_0^{-1}\theta\cdots\bar{u}_{i+2}^{-1}\theta^{-(i+1)}\{\bar{u}_{i+1}\theta^{-(i+1)}\cdots\bar{u}_{-1}\theta\bar{u}_0\cdot\bar{u}_0^{-1}\bar{u}_{-1}^{-1}\theta\cdots\bar{u}_{i+1}^{-1}\theta^{-(i+1)}\}\\ &\quad\cdot\bar{s}_i\theta^{-i}\{\bar{u}_{i+1}\theta^{-(i+1)}\cdots\bar{u}_{-1}\theta\bar{u}_0\cdot\bar{u}_0^{-1}\bar{u}_{-1}^{-1}\theta\cdots\bar{u}_{i+1}^{-1}\theta^{-(i+1)}\}\\ &\quad\cdot\{\bar{u}_{i+2}\theta^{-(i+1)}\cdots\bar{u}_0\theta\cdot\bar{u}_0^{-1}\theta\cdots\bar{u}_{i+2}^{-1}\theta^{-(i+1)}\}\\ &\qquad\qquad\cdot\bar{u}_{i+1}^{-1}\theta^{-i}\bar{u}_{i+2}\theta^{-i}\cdots\bar{u}_0\theta^2, 1, 2)_\sigma\\ &= (\bar{u}_0^{-1}\theta\cdots\bar{u}_{i+2}^{-1}\theta^{-(i+1)}\{\bar{u}_{i+2}^{-1}\theta^{-i}\bar{u}_{i+1}\theta^{-i}\bar{u}_{i+1}^{-1}\theta^{-i}\bar{u}_{i+2}\theta^{-i}\}\cdots\bar{u}_0\theta^2, 1, 2)_\sigma\\ &= (\bar{u}_0^{-1}\theta\cdots\bar{u}_{i+2}^{-1}\theta^{-(i+1)}\bar{u}_{i+3}\theta^{-(i+1)}\cdots\bar{u}_0\theta^2, 1, 2)_\sigma\\ &= ((\bar{u}_0^{-1}\cdots\bar{u}_{i+2}^{-1}\theta^{-(i+2)}\bar{u}_{i+3}\theta^{-(i+2)}\cdots\bar{u}_0\theta)\theta, 1, 2)_\sigma\\ &= (x_{i+1}\theta, 1, 2)_\sigma = (x_{i+1}, 0, 1)_\sigma = (e, 0, 1)_{i+1}\phi_{i+1}\end{aligned}$$

Hence,

$$\begin{aligned}(g,m,n)_{i+1}\phi_{i+1} &= ((e,m,0)_{i+1}(g,0,0)_{i+1}(e,0,n)_{i+1})\phi_{i+1}\\ &= (s_i^{-1}\alpha_i^{m-1}\cdots s_i^{-1}\alpha_i s_i^{-1}gs_i\cdot s_i\alpha_i\cdots s_i\alpha_i^{n-1}, m+1, n+1)_i\phi_i\end{aligned}$$

where if $m=0$ ($n=0$) the left- (right-) hand multiplier of g is e. Hence, if $x\in S_{i+1}\subseteq S_i$, i.e.,

$$x = (g,m,n)_{i+1} = (s_i^{-1}\alpha_i^{m-1}\cdots s_i^{-1}\alpha_i s_i^{-1}gs_i\cdot s_i\alpha_i\cdots s_i\alpha_i^{n-1}, m+1, n+1)_i,$$

then $x\phi_{i+1}=x\phi_i$. Thus, ϕ is a homomorphism of S onto H.

We now will show that H is the maximal group homomorphic image of S under the homomorphism ϕ.

Let G^* be an arbitrary group and let ρ be a homomorphism of S onto G^*. We denote ρ/S_i by ρ_i. Thus, ρ_i is a homomorphism of S_i into G^*. Since H is the maximal group homomorphic image of S_0 under the homomorphism ϕ_0 by virtue of Theorem 3.4, there exists a homomorphism γ of H onto the subgroup $S_0\rho_0$ of G^* such that $(g,m,n)_0\phi_0\gamma=(g,m,n)_0\rho_0$ for all $(g,m,n)_0\in S_0$.

Next, suppose that $(g,m,n)_{i+1}\phi_{i+1}\gamma=(g,m,n)_{i+1}\rho_{i+1}$ where γ is a homomorphism of H onto $S_{i+1}\rho_{i+1}$.

By virtue of Theorem 3.5, there exists $v_i \in G^*$ and a homomorphism η_i of G into G^* such that $v_i g\eta_i v_i^{-1} = g\alpha_i\eta_i$ for all $g \in G$. Furthermore $(g, a, b)_i\rho_i = v_i^{-a} g\eta_i v_i^b$ for $(g, a, b)_i \in S_i$.

Since $(g, 0, 0)_{i+1} = (g, 1, 1)_i$, $(g, 0, 0)_{i+1}\rho_{i+1} = (g, 1, 1)_i\rho_i$. Thus, $g\eta_{i+1} = v_i^{-1} g\eta_i v_i$. Hence, $g\eta_i = v_i g\eta_{i+1} v_i^{-1}$.

Since $(e, 0, 1)_{i+1} = (s_i, 1, 2)_i$, $(e, 0, 1)_{i+1}\rho_{i+1} = (s_i, 1, 2)_i\rho_i$. Thus,

$$\begin{aligned} v_{i+1} &= v_i^{-1}(s_i\eta_i)v_iv_i \\ &= v_i^{-1}(v_i(s_i\eta_{i+1})v_i^{-1})v_iv_i \\ &= s_i\eta_{i+1}v_i. \end{aligned}$$

Hence, $v_i = s_i^{-1}\eta_{i+1}v_{i+1}$. Thus,

$$\begin{aligned} g\eta_i &= (s_i^{-1}\eta_{i+1}v_{i+1})g\eta_{i+1}(v_{i+1}^{-1}s_i\eta_{i+1}) \\ &= s_i^{-1}\eta_{i+1}(v_{i+1}g\eta_{i+1}v_{i+1}^{-1})s_i\eta_{i+1} \\ &= s_i^{-1}\eta_{i+1}(g\alpha_{i+1}\eta_{i+1})s_i\eta_{i+1} \\ &= (s_i^{-1}g\alpha_{i+1}s_i)\eta_{i+1}. \end{aligned}$$

We recall that by p. 369 $s_i^{-1}g\alpha_{i+1}s_i = g\alpha_i = u_{i+1}^{-1}g\alpha u_{i+1}$. Thus,

$$\overline{s_i^{-1}g\alpha_{i+1}s_i} = \bar{u}_{i+1}^{-1}\overline{g\alpha}\bar{u}_{i+1} = \bar{u}_{i+1}^{-1}\bar{g}\theta\bar{u}_{i+1}.$$

Hence,

$$\begin{aligned} (s_i^{-1}g\alpha_{i+1}s_i, 0, 0)_{i+1}\phi_{i+1} &= ((\overline{s_i^{-1}g\alpha_{i+1}s_i})\delta_{i+1}, 0, 0)_\sigma \\ &= (\bar{u}_0^{-1}\bar{u}_{-1}^{-1}\theta\cdots\bar{u}_{i+2}^{-1}\theta^{-(i+2)}(\overline{s_i^{-1}g\alpha_{i+1}s_i})\theta^{-(i+1)} \\ &\qquad \cdot\bar{u}_{i+2}\theta^{-(i+2)}\cdots\bar{u}_{-1}\theta\bar{u}_0, 0, 0)_\sigma \\ &= (\bar{u}_0^{-1}\bar{u}_{-1}^{-1}\theta\cdots\bar{u}_{i+2}^{-1}\theta^{-(i+2)}(\bar{u}_{i+1}^{-1}\bar{g}\theta\bar{u}_{i+1})\theta^{-(i+1)} \\ &\qquad \cdot\bar{u}_{i+2}\theta^{-(i+2)}\cdots\bar{u}_{-1}\theta\bar{u}_0, 0, 0)_\sigma \\ &= (\bar{u}_0^{-1}\bar{u}_{-1}^{-1}\theta\cdots\bar{u}_{i+1}^{-1}\theta^{-(i+1)}\bar{g}\theta^{-i}\bar{u}_{i+1}\theta^{-(i+1)}\cdots\bar{u}_{-1}\theta\bar{u}_0, 0, 0)_\sigma \\ &= (\bar{g}\delta_i, 0, 0)_\sigma = (g, 0, 0)_i\phi_i. \end{aligned}$$

Thus,

$$\begin{aligned} (g, 0, 0)_i\rho_i &= g\eta_i = (s_i^{-1}g\alpha_{i+1}s_i)\eta_{i+1} \\ &= (s_i^{-1}g\alpha_{i+1}s_i, 0, 0)_{i+1}\rho_{i+1} \\ &= (s_i^{-1}g\alpha_{i+1}s_i, 0, 0)_{i+1}\phi_{i+1}\gamma = (g, 0, 0)_i\phi_i\gamma. \end{aligned}$$

We next note that

$$\begin{aligned} (s_i^{-1}, 0, 1)_{i+1}\phi_{i+1} &= (\bar{s}_i^{-1}\delta_{i+1}x_{i+1}, 0, 1)_\sigma \\ &= (\bar{u}_0^{-1}\cdot\bar{u}_{-1}^{-1}\theta\cdots\bar{u}_{i+2}^{-1}\theta^{-(i+2)}(\bar{u}_{i+1}^{-1}\bar{u}_{i+2})\theta^{-(i+1)} \\ &\qquad \cdot\bar{u}_{i+2}\theta^{-(i+2)}\cdots\bar{u}_{-1}\theta\bar{u}_0x_{i+1}, 0, 1)_\sigma \\ &= (\bar{u}_0^{-1}\bar{u}_{-1}\theta\cdots\bar{u}_{i+1}^{-1}\theta^{-(i+1)}\bar{u}_{i+2}\theta^{-(i+1)}\{\bar{u}_{i+2}\theta^{-(i+2)} \\ &\qquad \cdots\bar{u}_{-1}\theta\bar{u}_0\bar{u}_0^{-1}\cdot\bar{u}_{-1}^{-1}\theta\cdots\bar{u}_{i+2}^{-1}\theta^{-(i+2)}\}\bar{u}_{i+3}\theta^{-(i+2)}\cdots\bar{u}_0\theta, 0, 1)_\sigma \\ &= (\bar{u}_0^{-1}\bar{u}_{-1}\theta\cdots\bar{u}_{i+1}^{-1}\theta^{-(i+1)}\bar{u}_{i+2}\theta^{-(i+1)}\bar{u}_{i+3}\theta^{-(i+2)}\cdots\bar{u}_0\theta, 0, 1)_\sigma \\ &= (x_i, 0, 1)_\sigma = (e, 0, 1)_i\phi_i. \end{aligned}$$

Thus,

$$(e, 0, 1)_i\rho_i = s_i^{-1}\eta_{i+1}v_{i+1} = (s_i^{-1}, 0, 1)_{i+1}\rho_{i+1}$$
$$= (s_i^{-1}, 0, 1)_{i+1}\phi_{i+1}\gamma = (e, 0, 1)_i\phi_i\gamma.$$

Hence, $(g, m, n)_i\phi_i\gamma=(g, m, n)_i\rho_i$ for all $(g, m, n) \in S_i$. Thus, if $x \in S$, $x\phi\gamma=x\rho$.

Clearly, $H\gamma=G^*$. Thus, H is the maximal group homomorphic image of S under the homomorphism ϕ.

4. **The congruences.** In this section, we will determine the congruence relations on an I-bisimple semigroup $S=(G, C^*, \alpha, u_i)$. We first show that every congruence relation ρ on S is either an idempotent separating congruence (each ρ-class of S contains at most one idempotent) or a group congruence (S/ρ is a group). The idempotent separating congruences are uniquely determined by the α-invariant subgroups of G. Clearly, the group congruences are uniquely determined by the normal subgroups of the maximal group homomorphic image of S (see §3).

We first show that every congruence on $S=(G, C^*, \alpha, u_i)$ is either an idempotent separating congruence or a group congruence.

Let S be an inverse semigroup and let ρ be a congruence relation on S. Let $\{N_\alpha : \alpha \in J\}$ denote the collection of idempotent ρ-classes of S and let $N_\alpha \cap E_S=E_\alpha$. Thus $E_S=U(E_\alpha : \alpha \in J)$ (each N_α contains an idempotent [5]) and $E_\alpha \cap E_\beta=\square$ if $\alpha\neq\beta$.

Furthermore,

(4.1) $E_\alpha E_\beta\subseteq E_\gamma$ for some $\gamma \in J$.

(4.2) If $a \in S$ and $\alpha \in J$, there exists a $\gamma \in J$ such that $a^{-1}E_\alpha a\subseteq E_\gamma$.

THEOREM 4.1. *If S is a bisimple ω-semigroup each congruence on S is either an idempotent separating congruence or a group congruence.*

Proof. Let $S=(G, C, \alpha)$ and let ρ be a congruence relation on S. Let E_0 denote the class containing $(e, 0, 0)$. If $E_0\neq E_S$, let $(e, k+1, k+1)$ be the first element of E_S not contained in E_0. Thus, $E_0=\{(e, j, j) : 0\leqq j\leqq k\}$. Suppose that $k>0$. Hence, by (4.2), $(e, 1, 1+k)(e, \mathrm{k}+1, k+1)(e, k+1, 1)=(e, 1, 1) \in E_0$ and

$$(e, 1, 1+k)(e, 2k+1, 2k+1)(e, k+1, 1) = (e, k+1, k+1) \in E_0$$

since $(e, k+1, 0)(e, 0, 0)(e, 0, k+1)=(e, k+1, k+1)$ and

$$(e, k+1, 0)(e, k, k)(e, 0, k+1) = (e, 2k+1, 2k+1)$$

are contained in the same class by (4.2). Hence, we have a contradiction. Thus, $k=0$, and $E_0=\{(e, 0, 0)\}$. Let us next consider E_γ, say. Let (e, n, n) denote the first element of E_γ and suppose that $(e, n+1, n+1) \in E_\gamma$. Thus,

$$(e, 0, n)(e, n, n)(e, n, 0) = (e, 0, 0) \in E_0$$

and $(e, 0, n)(e, n+1, n+1)(e, n, 0)=(e, 1, 1) \in E_0$ and we again have a contradiction, i.e., $(e, n+1, n+1) \notin E_\gamma$. If $(e, n+s, n+s) \in E_\gamma$ with $s>1$,

$$(e, n+s, n+s)(e, n+1, n+1) = (e, n+s, n+s) \in E_\gamma$$

and hence $(e, n, n)(e, n+1, n+1)=(e, n+1, n+1) \in E_\gamma$, a contradiction. Thus, each E_γ consists of a single point, i.e., ρ is idempotent separating. If $E_S=E_0$, S/ρ is an inverse semigroup [5] with a single idempotent, i.e., S/ρ is a group.

Theorem 4.1 has been established by Munn and Reilly [4] by different methods.

THEOREM 4.2. *If S is an I-bisimple semigroup, each congruence relation on S is either a group congruence or an idempotent separating congruence.*

Proof. Let ρ be a congruence relation on S. Clearly, $\rho \mid S_i \times S_i$ where $S_i=e_iSe_i$ is a congruence relation ρ_i on S_i. Thus, since S_i is a bisimple ω-semigroup, ρ_i is an idempotent separating congruence or a group congruence by Theorem 4.1. Let us suppose that ρ_0 is an idempotent separating congruence. Assume that ρ_{i+1} is idempotent separating. Let e and f be distinct idempotents of $S_{i+1} \subseteq S_i$. If ρ_i is a group congruence, $e\rho_i f$. Thus, $e\rho f$ and hence $e\rho_{i+1} f$, a contradiction. Therefore, ρ_i is idempotent separating. Hence, since $S=U(S_i : i \in I, i \leqq 0)$ by Theorem 1.2, ρ is an idempotent separating congruence by induction. Similarly, if ρ_0 is a group congruence, ρ is a group congruence.

We next will determine the idempotent separating congruences of an I-bisimple semigroup S.

We will make use of the determination of the idempotent separating congruences for an arbitrary inverse semigroup.

If ρ is a congruence relation on an inverse semigroup S, the kernel of ρ is the inverse image of $E_{S/\rho}$ under the canonical homomorphism.

THEOREM 4.3 (PRESTON [5]). *Let $\{N_e : e \in E_S\}$ be a collection of disjoint subgroups of the inverse semigroup S and let $N=U(N_e : e \in E_S)$. Furthermore, suppose that*

(4.3) $N_eN_f \subseteq N_{ef}$,

(4.4) $a N_f a^{-1} \subseteq N_g$ *where* $a \in S$ *and* $g=afa^{-1}$.

Define the relation ρ_N over S by $a\rho_N b$ if and only if for some $e \in E_S$, $aa^{-1}=e=bb^{-1}$ and $ab^{-1} \in N_e$. Then ρ_N is an idempotent separating congruence over S with kernel N.

Conversely, every idempotent separating congruence ρ over S has a kernel N of the above type such that ρ_N is ρ.

THEOREM 4.4. *Let $S=(G, C^*, \alpha, u_i)$ be an I-bisimple semigroup. There exists a 1-1 correspondence between the idempotent separating congruences on S and the α-invariant subgroups of G. If ρ^V is the congruence corresponding to the α-invariant subgroup V, $\rho^V_{(g,a,b)}=((vg, a, b) : v \in V)$, i.e., $(g, a, b)\rho^V(h, c, d)$ if and only if $a=c$, $b=d$, and $Vg=Vh$. If V_1, V_2 are α-invariant subgroups of G, $V_1 \subseteq V_2$ if and only if $\rho^{V_1} \subseteq \rho^{V_2}$.*

Proof. Let V be an α-invariant subgroup of G and let $N_{(e,a,a)}=\{(v, a, a) : v \in V\}$ and let $N=U(N_{(e,a,a)} : a \in I)$. It follows by routine calculation that $N_{(e,a,a)}$ is a subgroup of S isomorphic to V, $N_{(e,a,a)}N_{(e,b,b)} \subseteq N_{(e,a,a)(e,b,b)}$ and

$$(g, a, b)N_{(e,c,c)}(g^{-1}, b, a) \subseteq N_{(e,t,t)}$$

where $(e, t, t)=(g, a, b)(e, c, c)(g^{-1}, b, a)$. Thus, ρ_N is an idempotent separating congruence of S by Theorem 4.3. We will denote ρ_N by ρ^V.

Conversely, suppose that ρ is an idempotent separating congruence of S. Thus, by Theorem 4.3, $\rho=\rho_N$ where N is given in the statement of Theorem 4.3 and $N_{(e,0,0)}=((v, 0, 0) : v \in V)$ where V is a subgroup of G. Since

$$(h, 0, 1)(e, 0, 0)(h^{-1}, 1, 0) = (e, 0, 0), \quad (h, 0, 1)N_{(e,0,0)}(h^{-1}, 1, 0) \subseteq N_{(e,0,0)}$$

by (4.4). Thus, if $v \in V$, $(h, 0, 1)(v, 0, 0)(h^{-1}, 1, 0)=(h(v\alpha)h^{-1}, 0, 0)$ and V is an α-invariant subgroup of G. Let $N_{(e,b,b)}$ denote the subgroup of N containing (e, b, b). Thus, $N_{(e,b,b)}=\{(w, b, b) : w \in W\}$ where W is a subgroup of G. Since $(e, 0, b)(e, b, b)(e, b, 0)=(e, 0, 0)$, if $w \in W$,

$$(e, 0, b)(w, b, b)(e, b, 0) = (w, 0, 0) \in N_{(e,0,0)}$$

by (4.4). Hence $W \subseteq V$, and similarly, $V \subseteq W$. Thus, $\rho=\rho^V$ and we have the desired correspondence. If $(g, c, d) \in S$, we next show that $\rho^V_{(g,c,d)}=\{(vg, c, d) : v \in V\}$. If $(h, a, b) \in \rho^V_{(g,c,d)}$, $a=c$ and $b=d$ and $(h, c, d)(g^{-1}, d, c)=(hg^{-1}, c, c) \in N_{(e,c,c)}$ by Theorem 4.3. Thus, $\rho^V_{(g,c,d)} \subseteq \{(v, g, c, d) : v \in V\}$. Using Theorem 4.3, the desired equality follows by a routine calculation. If V_1 and V_2 are α-invariant subgroups of G, clearly $V_1 \subseteq V_2$ implies that $\rho^{V_1} \subseteq \rho^{V_2}$. If $\rho^{V_1} \subseteq \rho^{V_2}$, $v \in V_1$ implies that $(v, 0, 0)\rho^{V_2}(e, 0, 0)$, i.e., $v \in V_2$.

COROLLARY 4.1. *If S is an I-bisimple semigroup, $\mathscr{H}$ is the maximal idempotent separating congruence of S.*

Proof. By Corollary 1.3, $\mathscr{H}$ is a congruence on S. By [3, p. 389, Theorem 2], every idempotent separating congruence of S is contained in $\mathscr{H}$.

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