CLASSIFICATION OF THE ACTIONS OF THE CIRCLE ON 3-MANIFOLDS(1)

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Introduction. Effective actions of the circle group, $S^1 = \mathrm{SO}(2)$, on connected 3-manifolds are classified in this paper. For each action a complete set of invariants is obtained. For an action on a compact manifold this set of invariants is a collection of tuples of integers. We prove

THEOREM. *The set of all distinct (i.e. inequivalent) effective actions of the circle on compact connected 3-manifolds is in one to one correspondence with the set of unordered tuples*

$$(b; (\varepsilon, g, h, t); (\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots, (\alpha_n, \beta_n)).$$

(The permissible values of the integers will be described later and the statement appears more precisely as Corollaries 2a and 2b in §6.)

Corresponding to each set of invariants is a "standard" differentiable action with the same invariants. As a consequence this yields

THEOREM 6. *Each effective action of the circle on a 3-manifold is topologically equivalent to a differentiable action. Furthermore, each differentiable action is differentiably equivalent to a "standard" action.*

When the set of fixed points is *not* empty (i.e. $h \neq 0$), then the set of invariants determines the manifold's prime decomposition. In the compact case, we have the corollary to Theorem 4, namely:

M is equivariantly homeomorphic to the "equivariant" connected sum

$$M_{\varepsilon,g,h,t} \# L'(\mu_1, \nu_1) \# \cdots \# L'(\mu_n, \nu_n).$$

(The manifold $M_{\varepsilon,g,h,t}$ is an explicit sum of handles and $P^2 \times S^1$'s. The $L'(\mu_i, \nu_i)$'s are lens spaces each with a specific, or "standard," action. This is described more precisely in the text.)

Since each allowable unordered tuple corresponds uniquely to a distinct class of equivalent actions, we can, if we are able to "identify" the 3-manifold, determine all the inequivalent actions on a given 3-manifold. The corollary to Theorem 4 does just this for actions *with fixed points*. (For example, the manifold admitting the action (0; (*o*, 6, 3, 5); (9, 2), (7, 5), (8, 3), (15, 6)) admits exactly 5,632 strictly inequivalent actions with fixed points and no actions without fixed points.)

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The actions without fixed points are not nearly as tractable. In the presence of fixed points one can construct a cross-section to the orbit map outside of the exceptional orbits and use the fixed point sets to build up the manifold and its action as an "equivariant connected sum." Unlike manifolds which admit actions with fixed points, the manifolds which admit actions without fixed points have fundamental groups that can never be expressed as the nontrivial free product of two groups, see §9. If both fixed points and special exceptional orbits are absent ($h=t=0$), we shall show (6.1) that such an action gives a Seifert singular fibering of type (O, o) or (N, n, I), [18]. Furthermore it will follow that classification of such actions up to equivalence coincides with Seifert's classification of singular fiber spaces up to "fiber" preserving homeomorphism. Recently Orlik, Vogt, and Zieschang [14], and Waldhausen [20] have shown that if M admits a Seifert fibering, under certain mild restrictions, then it admits at most one such fibering. Translated into actions this means if on M, $h=t=0$, then, under certain mild restrictions, *there exists at most one such action of the circle.* This along with special exceptional orbits and no fixed points ($t>0$, but $h=0$) is discussed in much more detail in §9. See Added in proof at the end of this paper.

R. Jacoby in [7] classified the actions of the circle, without fixed points, on the 3-sphere, S^3. He coupled the techniques of Seifert with those of transformation groups. In [6], H. Gluck studied untangled manifolds. Any orientable compact manifold which admits an action of the circle with fixed points must be untangled. The converse is not true. He showed modulo the Poincaré conjecture, that an untangled orientable compact manifold must have a handle or a lens space in its prime decomposition. Our techniques are completely different and our results in this direction are much more specific because we are actually assuming an action of the circle with fixed points. We completely avoid the Poincaré conjecture. In fact, the present investigation began from the (nontrivial) observation, which can now be deduced from Theorem 1 and techniques and results of [7], [12] and [18].

If M is a compact, simply connected 3-manifold which admits an effective action of a compact connected group G, $G\neq e$, then M is homeomorphic to the 3-sphere and the action is topologically equivalent to one of the standard linear actions.

P. Mostert classified the actions of compact connected Lie groups on 3-manifolds where the dimension of the principal orbit was at least two, [13]. A number of cases were omitted from this list but they can easily be supplied. The case where the dimension of a principal orbit is one coincides with the actions of the circle. This paper completes the list then of all actions of connected groups on 3-manifolds. It completes the classification of those with fixed points and partially classifies the actions of the circle without fixed points.

Some of the arguments would be easier if we restricted ourselves solely to differentiable actions. However, in dimension three not much simplification is really gained and so we have cast everything in a topological framework.

We have adhered to the notation of transformation groups as employed by [12] and [1] throughout. In some instances, when we could have quoted a general result from the literature, we have given a specific argument for the special case at hand. This has been done only when tracking down the relevant material in the literature would take the reader far afield.

1. **The orbit types.** Let G, a compact Lie group, operate on a space (completely regular) X. If $x \in X$, G_x will denote the *stability group* of G at x. The *orbit space* will be denoted by X^* or X/G and the orbit map by π. For each $x \in X$, one can find a certain subset S_x called the *slice* at x, [1, Chapter VIII] with the following properties:

(i) S_x is invariant under G_x.

(ii) If $g \in G$, $y, y' \in S_x$, and $g(y)=y'$, then $g \in G_x$.

(iii) There exists a "cell neighborhood" C of G/G_x such that $C \times S_x$ is homeomorphic to a neighborhood of x. That is if $\Gamma: C \to G$ is a local cross section in G/G_x then the map $F: C \times S_x \to X$ defined by $F(c, s) = \Gamma(c)s$ is a homeomorphism of $C \times S_x$ onto an open set containing S_x in X.

Also if X is a cohomology manifold (cm) over a principal ideal domain then by (iii) and [1, Chapter 1] or [15, Theorem 6] S_x is a cm of the appropriate dimension. In particular, if $X=M$ is a 3-manifold (and hence a 3-cm) and if the orbit $G(x)$ is 1-dimensional then S_x, a 2-dimensional factor of a 3-cm, is a 2-cm and hence a 2-manifold. With care, in this case, the slice S_x can actually be chosen to be an open or closed 2-cell, see [2, p. 115].

The closed subgroups of the circle consist of the finite cyclic groups Z_p, the identity e, and the entire group $G=S^1$. The orbits corresponding to stability groups isomorphic to these subgroups are either the circle or a fixed point. The *ordinary* or *principal orbits* are those for which the stability groups are the identity. The set of all such, O, is open in M. If we assume M is connected and the action effective, then O is an open, connected and dense subset of M. Furthermore the restriction of the orbit map $\pi: M \to M/G = M^*$ to $\pi|_O: O \to O^*$ is a principal S^1 fibering, [1, Chapter IX].

Let G_x be a finite cyclic group, Z_p. The finite groups which operate on a closed 2-cell are all equivalent (equivariantly homeomorphic) to a rotation of period p or a reflection through a fixed line through the origin, see [4] or [8]. The latter case holds only when $p=2$ and Z_2 reverses the orientation at x in S_x. Let E be the set of points on the orbits for which $G_x \cong Z_p$, $p \neq 2$ or $G_x \cong Z_2$ but does not reverse the orientation locally. Clearly, the image E^* under π of the set of such orbits is isolated in M^*. Furthermore, $(G(S_x))^*$, $x \in E$, is again a closed 2-cell with x^* in the interior. That is M^* is a 2-manifold without boundary near x^*. The orbits which make up E are called the *exceptional orbits*.

The *special exceptional orbits* are those orbits for which $G_x \cong Z_2$ and for which G_x acts on S_x as an orientation reversing reflection. Denote the set of points which lie on special exceptional orbits by SE. In this case choose S_x to be an open

2-cell. Then S_x^* is homeomorphic to the upper half plane with x^* lying on the boundary. Each component of SE is either a torus or a cylinder since each component C^* of S_{x^*} is a one-manifold and the set $C=\pi^{-1}(C^*)$ is a principal S^1-bundle over C^*. Furthermore a neighborhood N^* of C^* can be chosen to be a half open annulus or a half 2-space with C^* corresponding to the boundary. All points of N^*-C^* are images of principal orbits. In case C^* is a line then we may regard $N=\pi^{-1}(N^*)$ as an open Moebius-band bundle over C^*. This bundle is trivial since C^* is contractible, and so the action on N has a cross-section. In case C^* is a circle we shall assume, for convenience, that N^* is a closed neighborhood of C^* and hence an annulus. The set C is a principal S^1-fibering over S^1 and therefore a torus. We can take a cross-section h over C^* to C of the action. In addition, as N^* is $C^* \times I$, we can lift the interval $c^* \times I$ to a cross-section of the action over $c^* \times I$ with initial point $h(c^*)=c$, $c^* \in C^*$, $c \in C$. We obviously then can regard N as a closed Moebius-band bundle over C^* and such that the center line of each Moebius-band is just the orbit through $h(C^*)$. There are two possible Moebius-band bundles over the circle C^*. We claim that N must be a trivial bundle. For if not, then the transition function must reverse the orientation of the "center line" as we traverse the curve $h(C^*)$. But this is impossible as the bundle N must restrict to a principal S^1-bundle on $\pi^{-1}(C^*)$. Thus, in particular, h may be extended to all of N^* to give a *cross-section of the action over* N^*.

The set of fixed points will be denoted by F and its homeomorphic image under the orbit map by F^*. It is well known that F is a 1-manifold.

Throughout the remaining part of the paper G will denote the circle group and M a connected 3-manifold.

2. The orbit space.

LEMMA 1. *The orbit space M^* of an effective action of a circle on a 3-manifold M is a 2-manifold with boundary $F^* \cup (SE)^*$. Furthermore all orbits near E^*, F^* or $(SE)^*$ are principal orbits.*

We have already seen in §1 that the lemma is true for $M-F$. It only remains to check the validity when the action has fixed points. We appeal to a general theorem of Bredon [2, Chapter XV, Theorem 1.4]. Under our special hypothesis, Bredon's theorem yields the desired conclusion. However, since Bredon's argument is rather complicated due to the very general situation he is treating we shall sketch a shorter alternative argument.

There are two essential steps. The first step is to show that near a fixed point there are only two types of orbits. We begin by lifting the action of G to the universal covering space $\tilde{M}$ of $M-(E \cup SE)$, see [5]. We now show that there exist only fixed points and principal orbits on $\tilde{M}$. Suppose $y' \in \tilde{M}$ and $y \in F(G, \tilde{M})$. Let I be an arc from y' to y which meets an orbit at most once and such that the interior of the arc lies in the principal orbits of $\tilde{M}$. (It is easy to show that such arcs exist.) Let $G(I)$ be the G-image of I. Now, using Alexander-Spanier cohomology with compact supports, $H_c^2(G(I); Z)=0$ if y' lies on an ordinary orbit, and is a finite cyclic group isomorphic to Z_p if $G_{y'} \cong Z_p$. Since $H_c^2(\tilde{M}, Z)=0$, $H_c^3(\tilde{M}-G(I); Z)$ is free and the sequence

$$H_c^2(\tilde{M}; Z) \to H_c^2(G(I); Z) \to H_c^3(\tilde{M}-G(I); Z)$$

is exact, $H_c^2(G(I); Z)=0$. Hence, $\tilde{M}$ has only two kinds of orbits, principal orbits and fixed points. Now we need to know that $\tilde{M}^*$ is a 2-manifold with nonempty boundary. This point, which merely follows from an examination of the local co-Betti numbers of $\tilde{M}^*$ at F^*, is a very special part of Bredon's argument. However in this situation it can be easily extracted from the general context and so we refer the reader to page 217 of [1].

Since $\tilde{M}$ is simply connected and S^1 is connected it follows that $\tilde{M}^*$ is simply connected [11]. Hence $\tilde{M}^*$ is a 2-cell with a closed subset of the boundary deleted. Now with a little effort it can be seen that the fundamental group of $M-(E \cup SE)$ induces a free action on $\tilde{M}^*$ such that the orbit space of $\tilde{M}^*$ under this action is homeomorphic to the orbit space of $M-(E \cup SE)$. The map from $\tilde{M}^*$ to $M^*-(E \cup SE)^*$ is clearly a covering map which implies that $M^*-(E \cup SE)^*$ is a 2-manifold with boundary corresponding to the fixed points. (This is the second step.) We have seen in §1 that $(SE)^*$ fits onto O^* (in fact onto M^*) as a manifold with boundary. Since E^* is an isolated set in the interior of M^* it must also be closed. For, otherwise all the accumulation points would then have to be on the boundary which is impossible because $F^* \subset M^*-(E \cup SE)^*$. The lemma is proved.

3. Cross sections to the orbit map when $E=\varnothing$.

LEMMA 2. *Let $G=S^1$ act effectively on a connected 3-manifold M such that there exist no exceptional orbits. Then there exists a cross section $h: M^* \to M$ to the action unless M is compact and $F=\varnothing$.*

Proof. The mapping $\pi|_O: O \to O^*$ is a principal S^1-fibering. The principal S^1-bundles over the 2-complex O^* are in one-one correspondence with the elements of $H^2(O^*; Z)$. Since O^* is a connected 2-manifold the group is 0, if and only if O is not compact; i.e. nonzero, if $O=M$ and M is compact. Thus O is equivalent to $O^* \times S^1$ since we have assumed that $F \cup SE \neq \varnothing$ when M is compact.

For each connected component B_i^* of the boundary of M^* choose a closed neighborhood U_i^* to have the form $B_i^* \times I$. We can also assume that all the U_i^* are pair-wise disjoint. It is easy to see from the earlier sections that for each U_i^* there exist cross sections

$$h_i: U_i^* \to \pi^{-1}(U_i^*)$$

to the action over each U_i^*. Set $U^*=\bigcup U_i^*$, $B^*=\bigcup B_i^*$. Let $h: U^* \to \pi^{-1}(U^*)$ be defined by $h|_{U_i^*}=h_i$. Since M^* is a 2-manifold with boundary B^* we can choose (M^*, U^*, B^*) to be a triangulated triple. (This is not really necessary but we wish to use obstruction theory in its familiar form.) The obstruction to extending the cross section $h|_{U^*-B^*}$ to all of M^*-B^* is a cohomology class in $H^2(M^*, U^*; Z)$, [17, 34.4]. If M is *noncompact*, then U^*, a closed neighborhood of the boundary, can be chosen so that M^*-U^* does not have compact closure and M^*-U^* is connected. Hence, $H^2(M^*, U^*; Z)=0$ and the cross section over U^*-B^* can be extended to all of M^*-B^*. Therefore a cross section to the action on M exists. Note that as long as M is not compact, B^* may be empty and we still get a cross section.

If M is compact a more delicate argument is needed since $H^2(M^*, U^*; Z)$ does not vanish. For definiteness assume that $F_1 \subset U_1$ which we can assume is not empty by our hypothesis. Since M is compact and M^* is a 2-manifold with boundary we may assume that the U_i^* are disjoint smooth (or triangulable) annular neighborhoods of the boundary components of M^*. Let

$$h' = h|_{U^*-U_1^*}: U^*-U_1^* \to \pi^{-1}(U^*-U_1^*),$$

and recall

$$h_1 = h|_{U_1^*}: U_1^* \to \pi^{-1}(U_1^*).$$

Extend h' to all of M^*-V^* where the closure of V^* is an annular neighborhood of F_1^* lying in the interior of U_1^*. In the next paragraph we shall modify h_1 to h_1' so that h_1' restricted to the closure of V^* agrees with h' on the overlapping domain of definition. After this modification one need only piece together the two cross sections.

The solid torus $\pi^{-1}(V^*)$ can be "coordinatized" by just applying the group to the image of the cross section $h_1(V^*)$. V^* can be written as $F\times I$ with $F\times 0$ and $F\times 1$ corresponding to the image of the fixed point set F_1 and the other boundary D^* of the annular region V^*, respectively. Let us fix a point on $h_1(D^*)$ and let H denote the oriented orbit through this point. The two circles $h_1(D^*)=D$ and H form a conjugate system of curves on the torus $\pi^{-1}(D^*)=T$. Now h' sends the boundary circle D^* onto a circle Q lying on T. The orbit H and the circle Q also form a conjugate pair of circles for T which satisfy the homology relation

$$Q \sim nH+D,$$

where n is some positive integer. (We need to have oriented the curves properly so that n can be chosen positive.) The curves D and Q intersect each orbit of T exactly once. Now, parametrize the fixed point set by θ, $0\leqq\theta\leqq 2\pi$. For each point $h_1(\theta\times 1)\in D$, there exists an element $g(\theta)$ of the group S^1 such that $g(\theta)(h_1(\theta\times 1))$ meets Q. Clearly, the function determining g depends continuously on θ. Define

$$h_1'(\theta\times t) = g(\theta)(h_1(\theta\times 1))\times t, \quad \text{where } 0 \leqq t \leqq 1.$$

The map $h_1'\colon V^* \to \pi^{-1}(V^*)$ is a cross section to the action which agrees with h on D^*.

In Remark 2 at the end of §4, two more instances of global cross sections are treated.

4. Classifications of the actions when $E=\varnothing$, $F\neq\varnothing$.

LEMMA 3. *If S^1 acts effectively on the connected 3-manifolds M_1 and M_2 such that there are cross sections to the actions, then M_1 and M_2 are equivariantly homeomorphic if and only if the orbit spaces are homeomorphic with the homeomorphism carrying the components of the fixed point set F_1^* of M_1^* homeomorphically onto the components of the fixed point set F_2^* of M_2^*.*

Proof. If the two actions are equivariantly homeomorphic then the orbit spaces must be homeomorphic regardless of whether or not there exists a cross section. Furthermore the homeomorphism sends the orbit structure of one isomorphically onto the orbit structure of the other.

If $E\neq\varnothing$ for an action, then it is easy to see that there is no cross section. In Lemma 2 we showed that all the remaining cases of an action have a cross section except possibly when $F=\varnothing$ and M was compact.

Let $h_1\colon M_1^*\to M_1$ and $h_2\colon M_2^*\to M_2$ be cross sections to the actions on M_1 and M_2. Let $\psi\colon M_1^*\to M_2^*$ be a homeomorphism which carries F_1^* homeomorphically onto F_2^*. This set may be empty but in any case $(SE)_1^*$ is carried homeomorphically onto $(SE)_2^*$ (which also may be empty). Then $h_2\psi h_1^{-1}$ sends the image of the first cross section onto the image of the second cross section. Any point $x\in M_1$ is uniquely represented by $g'(h_1(x^*))$ for some $g'\in S^1/S^1_{h_1(x^*)}$. The mapping

$$(g, h_1(x^*))\to(g, h_2\psi h_1^{-1}(x))$$

of $S^1\times h_1(M_1^*)$ onto $S^1\times h_2(M_2^*)$ induces the desired equivariant homeomorphism of M_1 onto M_2.

The equivariant classification of the actions of Lemma 3 can be described by three integers when the manifold is compact. They are the number h of components of the fixed point set, the number t of components of the special exceptional orbits and the number g where g denotes the genus of M^* if M^* is orientable (respectively, g denotes the nonorientable genus if M^* is nonorientable). As we shall see later g and t will be determined by the fundamental group of M and h. Thus we can state: *if the M_i are compact then they are equivariantly homeomorphic if their fundamental groups are isomorphic and both have the same number of components for their fixed point sets.*

We turn now to classifying the number of *inequivalent* actions on a *given* compact 3-manifold. We begin by recalling some facts concerning the connected sum of 3-manifolds. The connected sum of a combinatorial n-manifold M_1 with a combinatorial n-manifold M_2 is obtained by removing a combinatorial n-ball from each matching the resulting spaces by a piecewise linear homeomorphism of their boundaries. If the manifolds are orientable this sum is not always uniquely determined unless one uses, say, an orientation reversing homeomorphism. The resulting manifold does not depend upon which balls are removed. An analogous definition can be given in the differentiable case. We shall be concerned only with 2- and 3-dimensional manifolds and so it will suffice to remove tame balls and to match the boundaries by means of a homeomorphism (or an orientation reversing homeomorphism) if both manifolds are orientable. In any case, the constructions that we shall make can just as well be done piecewise linearly or differentiably.

A 3-manifold is *prime* if it can not be written as the connected sum of two manifolds each different from the 3-sphere. In [10] Milnor proved that every compact connected *orientable* 3-manifold is homeomorphic to a sum of $P_1 \# \cdots \# P_k$ of prime manifolds. The summands P_i are *uniquely* determined up to order and homeomorphism.

A 3-manifold M is *irreducible*, according to Kneser, if every 2-sphere in M bounds a 3-cell. Milnor easily proved that with the exception of $S^1 \times S^2$ and the 3-sphere S^3 every orientable manifold is prime, if and only if it is irreducible [10, Lemma 1]. Milnor's Lemma 1 is also true for all connected 3-manifolds if the exceptions also include the analogue N of the orientable handle $S^2 \times S^1$. The proof is essentially the same except that in his proof when one adds a handle it may be necessary to add the nonorientable handle N.

Actually, Milnor's theorem was "proved" earlier by Kneser modulo the truth of the Dehn Lemma. In fact, Kneser's theorem yields a unique decomposition theorem for compact manifolds orientable or not, see [9, p. 256], and [10, p. 6, 3]. Because $N \# N = N \# (S^2 \times S^1)$, the uniqueness statement must be stated in a normal form. (This is analogous to writing the 2-manifold $P^2 \# P^2 \# P^2$ also by $P^2 \# T^2$, P^2 and T^2 being the projective plane and torus respectively. Both P^2 and T^2 are "prime" 2-manifolds.)

Kneser's theorem states that *every compact 3-manifold can be written uniquely as the sum of irreducible manifolds, handles and at most one nonorientable handle.* In this normal form, if any irreducible manifold is nonorientable, then N can not appear.

Let ε be a symbol which can take on two values o or n, (standing for orientable or nonorientable orbit spaces). Let t be a nonnegative integer (standing for the number of components of special exceptional orbits), h an integer greater or equal to 1 (standing for the number of connected components of the fixed point set), g a nonnegative integer if $\varepsilon = 0$, and an integer greater or equal to 1 if $\varepsilon = n$ (standing for the orientable or nonorientable genus of the orbit space). The symbol $[x]$ will denote the greatest integer less than or equal to x.

THEOREM 1. (i) *There exists a one-to-one correspondence between quadruples* (ε, g, h, t) *and equivariant homeomorphism types of connected compact 3-manifolds which admit effective actions of the circle group with fixed points and no exceptional orbits.*

(ii) *If* $M_{\varepsilon,g,h,t}$ *is a manifold corresponding to a quadruple* (ε, g, h, t) *then its prime* *decomposition* (*unique up to order*) *is*

(a) $S^3 \# (S^2\times S^1)_1 \# \cdots \# (S^2\times S^1)_{2g+h-1} \# (P^2\times S^1)_1 \# \cdots \# (P^2\times S^1)_t$ *if* $\varepsilon = o$,

(b) $N \# (S^2\times S^1)_1 \# \cdots \# (S^2\times S^1)_{g+h-2}$ *if* $\varepsilon = n$ *and* $t=0$,

(c) $(S^2\times S^1)_1 \# \cdots \# (S^2\times S^1)_{g+h-1} \# (P^2\times S^1)_1 \# \cdots \# (P^2\times S^1)_t$ *if* $\varepsilon=n$ *and* $t>0$.

(iii) *If the manifold* $M_{\varepsilon,g,h,t}$ *has b summands homeomorphic to* $S^2\times S^1$ *in its prime decomposition then there exist exactly*

$$\begin{array}{ll} 1+[b/2] & \textit{if } (\varepsilon, g, h, t) = (o, g, h, 0), \\ 1+b & \textit{if } (\varepsilon, g, h, t) = (n, g, h, 0), \textit{ and} \\ 1+[b/2]+b & \textit{if } t > 0 \end{array}$$

topologically distinct actions of S^1 *on* $M_{\varepsilon,g,h,t}$ *with fixed points.*

Proof. (i) We have seen that all such actions of the circle on compact connected 3-manifolds are obtained by first taking the set of compact connected 2-manifolds with boundary for orbit spaces. The orbit structure on the orbit space then determines the equivariant homeomorphism type by Lemma 3. For a fixed compact connected 2-manifold with nonempty boundary, $M^*_{\varepsilon,g,h,t}$, let the quadruple (ε, g, h, t) represent

$\varepsilon = o$ if the 2-manifold is orientable and n if it is not orientable;

g = the genus of the 2-manifold;

h = the number >0 of connected boundary components which will correspond to the number of components of the fixed point set;

t = the number of the remaining boundary components which will correspond to the number of components of the set $(SE)^*$.

The triple $(\varepsilon, g, h+t)$ determines up to homeomorphism the 2-manifold $M^*_{\varepsilon,g,h,t}$. The quadruple (ε, g, h, t) completely determines the orbit structure of an action on a 3-manifold $M_{\varepsilon,g,h,t}$ which has $M^*_{\varepsilon,g,h,t}$ as orbit space (and with orbit structure determined by the symbol (ε, g, h, t)). Now, in this context Lemma 3 becomes (i) of Theorem 1 provided that we show that for each (ε, g, h, t) *there exists* an action corresponding to this symbol. To see this simply take $M^*_{\varepsilon,g,h,t}$ and form $M' = M^*_{\varepsilon,g,h,t}\times S^1$. Over h components of the boundary of $M^*_{\varepsilon,g,h,t}$ collapse all the circles (i.e. the second coordinates) to points and over the remaining t components identify opposite pairs of points of each circle in the second factor by reflection through the origin of the second factor. Call this 3-manifold $M_{\varepsilon,g,h,t}$. Clearly the circle operates on $M_{\varepsilon,g,h,t}$ with fixed points and no exceptional orbits $M^*_{\varepsilon,g,h,t}$ is the orbit space with the prescribed orbit structure determined by (ε, g, h, t). Obviously, there exists a cross section to this action. This completes the proof of (i).

(ii) First consider the case $t=0$. Let M^* be a compact 2-manifold with h boundary components. Suppose g is the genus of this 2-manifold and let ε denote whether or not M^* is orientable. Thus M^* has a weighting and we denote this by $M^*_{\varepsilon,g,h,0}$. Consider $(M^*_{\varepsilon,g,h,0}\times I^2) = M'$ and let the circle, G, operate trivially on the first factor and linearly on the unit disk, $I^2 = \{(x, y) \mid x^2+y^2=1\}$. The fixed point set of this action is precisely $M^*_{\varepsilon,g,h,0}$, the orbit space is $M^*_{\varepsilon,g,h,0}\times I$. If we now restrict the action to the boundary of M', $\partial(M')$, a connected 3-manifold, we see that the orbit space is now $M^*_{\varepsilon,g,h,0}$ and the fixed point set is precisely $\partial(M^*_{\varepsilon,g,h,0})$ and all the other orbits are principal orbits. But, $\partial M'$ may be succinctly described as the double of $(M^*_{\varepsilon,g,h,0}\times I)$. But $M^*_{\varepsilon,g,h,0}\times I$ is just a thickened 2-manifold with h boundary components and if $\varepsilon=o$, this is a 3-disk with $2g+h-1$ oriented disjoint 1-handles attached. The double of this manifold is precisely $S^3\#(S^2\times S^1)_1\#\cdots\#(S^2\times S^1)_{2g+h-1}$. For $\varepsilon=n$, the manifold $\partial M'$ is just the double of $M^*_{n,g,h,0}\times I$ which can be described in normal form by $N\#(S^2\times S^1)_1\#\cdots\#(S^2\times S^1)_{g+h-2}$. Note that $N=\partial(M^*_{n,0,1,0}\times I)$, $M^*_{n,0,1,0}$ is the projective plane with one hole, that is, the Moebius band.

Consider now the annulus $S^1\times I$ where $S^1\times I$ represents $(o, 0, 1, 1)$. The inverse image of $s\times I$, $s\in S$, is a projective plane. The manifold $P^2\times S^1$ then by Lemma 3 is precisely $M_{o,0,1,1}$.

To obtain $M_{o,g,h,t}$ we just take $M^*_{o,g,h,t}$ which can be regarded as $M^*_{o,g,h,0}$ with t 2-disk holes removed from its interior. Take a small arc I in $M^*_{o,g,h,0}$ which lies entirely within the interior of $M^*_{o,g,h,0}$ except for both endpoints which meet a component of the boundary. The inverse image of I under the orbit map is an invariant 2-sphere. The union of the arc with an arc on the boundary of $M_{o,g,h,0}$ forms a circle which is the boundary of a small 2-disk. The inverse image of this 2-disk is an invariant 3-cell with the aforementioned invariant 2-sphere as its boundary. The fixed point set is just the complement of the arc of the boundary of the disk together with its endpoints. Now do the same thing to the "weighted" annulus $S^1\times I=M^*_{o,0,1,1}$. Form the connected sum of $M^*_{o,g,h,0}$ with $M^*_{o,0,1,1}$ by using an orientation reversing homeomorphism of the arcs. This defines an "equivariant" connected sum of $M_{o,g,h,0}$ with $M_{o,0,1,1}$ to yield $M_{o,g,h,1}$. But this is nothing more than the "equivariant" connected sum of $M_{o,g,h,0}\#P^2\times S^1$. We form $M_{o,g,h,t}$ by iterating this procedure t times.

To form $M_{n,g,h,t}$ we do the same thing as above. We start with $M_{n,g,h,0}$ which is $N\#(S^2\times S^1)_1\#\cdots\#(S^2\times S^1)_{g+h-2}$. We form the "equivariant" connected sum with $M_{o,0,1,1}$. This yields $N\#(S^2\times S^1)_1\#\cdots\#(S^2\times S^1)_{g+h-2}\#P^2\times S^1$ whose normal form is

$$(S^2\times S^1)_1\,\#\cdots\#\,(S^2\times S^1)_{g+h-1}\,\#\,P^2\times S^1.$$

Iterating this procedure t times produces the manifold $M_{n,g,h,t}$.

(iii) If M is homeomorphic to $M_{\varepsilon,g,h,t}$ then the number of topologically distinct actions of the circle on M with fixed points follows directly from (ii) by just counting the possibilities. We leave the details to the reader.

REMARK 1. The descriptions of $M_{o,g,h,0}$ could also have been built up from $M_{o,0,1,0}$ and $M_{o,1,1,0}$ by forming "equivariant" connected sums.

REMARK 2. We append here two more types of actions that have global cross sections but which do not fall under the case treated in Theorem 1. They will be of use in the general case.

a. Suppose that $E=F=SE=\varnothing$. Then the action is a principal fibering on the 3-manifold M. If M is not compact this fibering is trivial and the action is just the product action of S^1 on $M^*\times S^1$, where S^1 operates trivially on the first factor. On the other hand, if the manifold is compact then the principal fibering may not be trivial and the action has no cross section. In fact, the set of all such principal fiberings are classified by the elements of $H^2(M^*;Z)$. In case M^* is orientable, then the action corresponding to the principal fibering determined by a positive integer n is equivalent to the action determined by the principal fibering $-n$. If M^* is nonorientable the 2 distinct principal fiberings represent distinct actions.

b. Suppose that $E=F=\varnothing$ and that $SE\neq\varnothing$. If M is noncompact then the action has a cross section as before. The action can be described geometrically by taking $M^*\times S^1$ and letting S^1 operate trivially on the first factor. Then on $SE^*\times S^1$ collapse each orbit by the action of $Z_2\subset S^1$.

This same description of the action will work when $E=F=\varnothing$, $SE\neq\varnothing$ and M is compact. The point is that once again the action will admit a global cross section.

Lemma 4. *Let G act on M such that C denotes a compact connected component of SE. Let A^* denote an annular neighborhood of C^* in M^* with $C^*\times[0,1]=A^*$ and C^* identified with $C^*\times 0$. If $h\colon C^*\times 1\to C$ is a cross section to the orbit map, then h can be extended to all of A^* to yield a cross section to the action over A^*.*

Proof. In §1, we saw that the action over A^* has a cross section. Thus, $\pi^{-1}(A^*)=A^*\times S^1$ with $a^*\times e^{i\pi}$ identified with $a^*\times e^{i0}$, for each $a^*\in C^*\times 0$. If we form $A^*\times S^1$ with the circle acting trivially on A^* and let h be a cross section of this trivial fibering over $C^*\times 1$, then the obstruction to extending this cross section is given by an element of $H^2(A^*, C^*\times 1; Z)=0$. Hence it may be extended to $A^*\times S^1$. Now by using the identification described above we may use the extension over $A^*\times S^1$ to define the extension over $\pi^{-1}(A^*)$.

c. The lemma above enables us to extend Lemma 2 of §2 to also cover the case $E=F=\varnothing$, $SE\neq\varnothing$, and M compact. The action on $M=M_{\varepsilon,g,0,t}$ is just the trivial action of S^1 on the second factor of $M'=M^*_{\varepsilon,g,0,t}\times S^1$ but with identifications on the boundary of M' by the action of $Z_2\subset S^1$. This manifold M can never be written as the connected sum of two other manifolds with nontrivial fundamental groups (§9).

5. **The exceptional orbit invariants.** In the fourth section we have catalogued all compact connected 3-manifolds which admit effective actions of the circle with fixed points but without exceptional orbits. The solutions were given in terms of a quadruple (ε, g, h, t). This determined a 3-manifold and all the distinct actions on the 3-manifold. The extension of this analysis to the case where exceptional orbits are present necessitates the introduction of more (numerical) invariants. Namely, to each exceptional orbit (and there are at most a finite number of them when the manifold is compact) we will associate a pair of relatively prime integers

(μ, ν) with $0<\nu<\mu$ (and $0<\nu\leqq\mu/2$ if $\varepsilon=n$). Together with the quadruple they will completely determine the action as well as the manifold's specific prime decomposition. Furthermore, each presentation of an action will also determine all other topologically inequivalent actions on the same manifold.

The pair of relatively prime integers (μ, ν) to be associated with each exceptional orbit is just the pair which determines the linear action of the stability group on the slice at an exceptional orbit when the action is differentiable. If the action is only topological then the action is conjugate, in the group of orientation preserving homeomorphisms of the disk, to a specific linear action which is determined by a pair (μ, ν). Seifert considers certain invariants associated with each singular fiber in his singular fiber spaces [18]. A direct comparison will show that the invariants defined here and those of Seifert are identical although the definition here is in terms of transformation groups.

Take the solid torus $D^2\times S^1$ and represent $x\in D^2\times S^1$ by $(\rho e^{i\theta}, e^{i\psi})$ where $0\leqq\rho\leqq 1$, $0\leqq\theta\leqq 2\pi$, $0\leqq\psi\leqq 2\pi$. Define an action

$$T_z\colon D^2\times S^1\to D^2\times S^1 \tag{5.1}$$

by

$$z\times(\rho e^{i\theta}, e^{i\psi})\to(z^\nu\rho e^{i\theta}, z^\mu e^{i\psi})$$

where z ranges over the complex numbers of norm, 1, $0<\nu<\mu$, and ν and μ are relatively prime integers. This is the *standard linear action determined* by (μ, ν).

Observe that the projection map $D^2\times S^1\to S^1$ is a 2-disk bundle projection. The fiber is the disk with structural group Z_μ. The action of the structural group Z_μ on the fiber is the linear action of the rotations generated by $z=e^{2\pi i/\mu}$, that is, for fixed ψ_0,

$$e^{2\pi i/\mu}\times(\rho e^{i\theta}, e^{i\psi_0})\to(e^{2\pi i\nu/\mu}\rho e^{i\theta}, e^{i\psi_0}).$$

Now suppose x is a point in a 3-manifold on an exceptional orbit. The slice S_x at x can be chosen to be a small 2-disk. The stability group $G_x\approx Z_\mu$ operates effectively on S_x and the saturation of S_x, $G(S_x)$ is a *topological* 2-disk bundle over the exceptional orbit $G(x)$ with structural group Z_μ. This fact follows from general facts concerning a slice. See the Remark on page 107 of [1]. Now the action of the stability group on the 2-disk slice is conjugate to a linear action by a well-known theorem [4] or [8]. Assume the solid torus to be oriented. All the orbits are naturally oriented by the group. We can select an orientation of the slice at x so that its intersection with the exceptional orbit is $+1$. Then, in particular, the action of $G_x\approx Z_\mu$ on the slice S_x is conjugate under an *orientation preserving homeomorphism* to a specific linear rotation generated by $e^{i\theta}\to e^{i\theta}e^{2\pi i\nu/\mu}$ where $0<\nu<\mu$, and ν and μ are relatively prime integers. Thus there is an orientation preserving bundle isomorphism of the standard 2-disk bundle over the circle onto the 2-disk bundle neighborhood of $G(x)$. Furthermore, the naturally induced linear action of the circle on $D^2\times S^1$ induced by the rotation

$$T_{e^{i2\pi/\mu}}\colon(\rho e^{i\theta}, 1)\to(e^{i2\pi\nu/\mu}\rho e^{i\theta}, 1)$$

on the slice $(\rho e^{i\theta}, 1)$ is carried over by the bundle isomorphism to the given action on $G(S_x)$. The only choices involved have been the selection of a point x on the exceptional orbit and the choice of slice at x. It is easily checked that neither of these choices affects the ability of finding an equivariant homeomorphism. We will call the pair (μ, ν) the *oriented orbit invariant* associated with the exceptional orbit through x.

We have proved the following lemma:

LEMMA 5. *For each exceptional orbit with oriented orbit invariant* (μ, ν) *there exists an equivariant orientation preserving homeomorphism of* $D^2 \times S^1$ *with a standard linear* S^1*-action determined by* (μ, ν) *onto a neighborhood of the exceptional orbit.*

Notice that if we put $\alpha = \mu$ and $\beta = \nu^{-1} \bmod \alpha$ then we get a pair (α, β) which we shall call the *oriented Seifert invariant associated with the orbit through* x. Clearly (α, β) and (μ, ν) mutually determine one another.

If we reverse the orientation of the circle group, which would occur if we considered the reverse action of the circle, then the orientation of the slice must also be reversed in order to preserve the global orientation of the solid torus neighborhood. Thus the oriented orbit invariant through x would necessarily be the same.

On the other hand, if we reversed the orientation of the solid torus, without changing the action of S^1, then the induced action on the slice would be the inverse of the action on the oppositely oriented disk. *Thus the oppositely oriented orbit invariant would be* $(\mu, \mu - \nu)$ *and the oppositely oriented Seifert invariant would be* $(\alpha, \alpha - \beta)$.

If no orientation on the solid torus was specified, which would occur if the manifold is nonorientable, then since no orientation is preferred we would have to allow orientation reversing homeomorphisms in defining an equivariant homeomorphism. Thus our structural group of the disk bundle must be enlarged to include orthogonal transformations of the disk in the linear case and any homeomorphism in the topological case. Thus the *unoriented orbit invariant* (μ, ν) would satisfy $0 < \nu \leqq \mu/2$ and the *unoriented Seifert orbit invariant* would also be such that $0 < \beta \leqq \alpha/2$. (Reduce ν, of an oriented (μ, ν), on the interval $(-\mu/2, \mu/2) \bmod \mu$ and take the absolute value.) The unoriented orbit invariants and the unoriented Seifert orbit invariants mutually determine each other (but now, $\beta\nu \equiv \pm 1 \bmod \mu$ instead of merely $\beta\nu \equiv 1 \bmod \mu$) and also the equivariant homeomorphism type of the unoriented solid torus. In fact, we should point out that unless we use unoriented orbit invariants, when the orbit space M^* of M is nonorientable, our orbit invariants would not be well determined. We shall now prove a general fact which will be necessary for the classification theorem and which also justifies the use of unoriented invariants when the orbit space is nonorientable.

LEMMA 6. *Let C be an arc or a simple closed curve in the interior of* M^* *which meets only principal orbits except, perhaps, the initial point* (*and the terminal point* *if C is a circle). Let ϕ_t^* be an isotopy of M^* onto itself which moves the initial point of C along the curve and which is fixed outside of a tubular neighborhood of C. Then this isotopy may be equivariantly lifted to an isotopy ϕ_t of M onto itself which leaves M fixed outside of the inverse image of the tubular neighborhood of C.*

Proof. If we take an arc or a simple closed curve in M^* which meets only principal orbits then the manifold over a tubular neighborhood of the curve is a principal S^1-bundle *with a cross section.* Hence, if we isotop M^* along the curve leaving M^* fixed outside of the tubular neighborhood we may equivariantly lift this isotopy to M. If, on the other hand, the initial point of the arc or the initial and terminal point of the simple closed curve meets an exceptional orbit, we may still lift the isotopy to M. This can be effected by boring out a small invariant tubular neighborhood of the exceptional orbit and moving the rest of the manifold along the curve and sewing back in, at each stage of the isotopy, the previously removed invariant tubular neighborhood. This completes the proof of the lemma.

If the manifold M^* is nonorientable and x^* is an element of E^* or a principal orbit, then there exists a simple closed curve which meets, except for x^*, only principal orbits and which reverses the local orientation at x^*. One can find an isotopy of M^* which moves along the curve and is fixed outside a small tubular neighborhood. One can then lift this isotopy to an equivariant isotopy which reverses the (local) orientation of the tubular neighborhood of $G(x)$. The orientation of the orbits is preserved and so the orientation of the slice must be reversed. Thus unless we reduce ν to the absolute value on the interval $-\mu/2$ to $\mu/2$ we would not get a well-defined invariant since no local orientation of a tubular neighborhood of a singular orbit is preferred.

We need now Seifert's more geometric description of the Seifert invariant (α, β) before we can begin our classification. We adhere essentially to the terminology of Seifert [18] here.

Let $x \in M$ be a point on an exceptional orbit and S_x a slice at x, which we can assume is a 2-cell. Let m denote the boundary of S_x. Let H denote a principal orbit issuing out of a point on m. The orbits are all oriented by the circle group and consequently the slice is oriented if we specify an orientation for $G(S_x)$. The orientation of S_x induces an orientation of m. Let B_1 be any curve on $G(m)$ which is homologous in $G(S_x)$ to $G(x)$. If B is any other such curve then $B_1 \sim B+sm$, for any integer m. The curve H satisfies the homology relation $H \sim \mu_1 B_1 + \nu_1 m$, where ν_1 is relatively prime to μ_1. Clearly, as H intersects m μ times and B and H have the same orientation mod the curve m, $\mu_1 = \mu$. We may choose B and s such that $\mu s + \nu_1$ is reduced to ν mod μ on the interval $[0, \mu]$. We have the homology relation

$$H \sim \mu B + \nu m, \tag{5.2}$$

where (μ, ν) are precisely the orbit invariants previously described.

Since G operates freely on $G(m)$, there are cross sections to the trivial principal fibering $G(m) \to G(m)/G$. If Q and Q_1 are two cross sections they are related by the homology relation

$$Q_1 \sim \pm Q + sH, \tag{5.3}$$

for any integers s. The curves Q_1, H form a conjugate pair of curves on $G(m)$ and so we may consider

$$m \sim \alpha Q_1 + \beta_1 H. \tag{5.4}$$

As H intersects m μ times it is clear that $|\alpha| = \mu$. Now substituting (5.3) into (5.4) for a choice s so that $\alpha s + \beta_1$ is an integer β on the interval $[0, \mu]$ we obtain

$$m \sim \alpha Q + \beta H \qquad (\text{or } m \sim \alpha(-Q) + \beta H). \tag{5.5}$$

We may write B in terms of Q and H,

$$B \sim \rho Q + \sigma H \qquad (\text{or } B \sim -\rho(-Q) + \sigma H). \tag{5.6}$$

Solving (5.5) and (5.6) for H we obtain

$$H(\alpha\sigma - \rho\beta) = \alpha B - \rho m \qquad (\text{or } H(\alpha\sigma + \rho B) = \alpha B + \rho m). \tag{5.7}$$

The determinant $\alpha\sigma - \rho\beta$ (or $\alpha\sigma + \rho\beta$) is ± 1.

Let us suppose $+1$. Then we have in lieu of (5.2)

$$\alpha = \mu, \ -\rho \equiv \nu(\mu) \qquad (\text{or } \alpha = \mu, \ \rho \equiv \nu(\mu)).$$

Therefore, $\nu\beta \equiv 1(\alpha)$. If the determinant is -1, then we have in lieu of (5.2)

$$\alpha = -\mu, \ \rho \equiv -\nu(\mu) \quad (\text{or } \alpha = -\mu, \ \rho \equiv -\nu(\mu))$$

and hence

$$-\nu\beta \equiv -1(\alpha).$$

Observe that G_x operates on S_x preserving the orientation of S_x. Therefore, the orientation of S_x and S_x/G_x determine one another. Since $\pi(m) = \pi(Q)$, any choice of orientation for m and Q determines the sign of α.

Hence the Seifert invariants can be chosen so that the coefficient of Q in (5.5) is positive and β is uniquely determined by $\beta\nu \equiv 1(\mu)$.

An analogous computation may be made in the unoriented case. Also it is easy to check that the computation does not depend upon any of the choices, such as the choice of slice, that were made. We leave these details to the reader and summarize the discussion with the following lemma:

LEMMA 7. *(μ, ν) and (α, β) of* (5.2) *and* (5.5) *are the oriented* (*unoriented*) *orbit invariant and the oriented* (*unoriented*) *Seifert orbit invariant of the exceptional orbit through x.*

6. $E \neq \varnothing$. Recall that in Lemma 3 we classified, in particular, effective actions with $E = \varnothing$, $F \neq \varnothing$ up to equivariant homeomorphism. In each instance we had a

global cross section to the orbit map. In Lemma 3, if M is compact and $M-SE$ is orientable then one can find an orientation reversing homeomorphism equivariant with the reverse action. One can also find an orientation reversing homeomorphism of the cross section which induces an orientation reversing homeomorphism of $M-SE$ equivariant with respect to either action. Combining these two we can find an orientation preserving homeomorphism of $M-SE$ such that a given action is equivariantly equivalent to its reverse action.

The orbit invariants are defined in terms of a given orientation. We shall alter the standard definition of equivalence slightly to accommodate this fact. In view of the remark above this altered form is the same ordinary equivalence used earlier when $E=\varnothing$ and $F\neq\varnothing$. In fact, even when $E\neq\varnothing$ and $M-SE$ is nonorientable, it can be seen that the two notions coincide.

Actions of S^1 on M_1 and M_2 will be called *equivalent in the strict sense* (or strictly equivalent) if there exists a homeomorphism $h\colon M_1\to M_2$ and an automorphism $a\colon S^1\to S^1$, perhaps the trivial one, so that for all $m\in M_1$, $g\in S^1$

$$h(g(m)) = a(g)h(m).$$

Furthermore, if $M_k-(SE)_k$, $k=1, 2$ are orientable, we shall choose orientations and demand that h *preserve* the orientations. We make the similar definition in the differentiable case. To establish a classification theorem for actions on an oriented 3-manifold where $E\neq\varnothing$ in terms of the earlier definition of equivalence would force us to look at Kneser's theorem in terms of symmetric and asymmetric irreducible 3-manifolds and we wish to avoid this complication here. We shall now state the first classification theorem in two versions but we shall amalgamate the proofs into one argument.

THEOREM 2a. *Let S^1 act effectively on the connected 3-manifolds M_1 and M_2 so that M_1^* and M_2^* are nonorientable. If the M_k are compact assume that either the fixed point sets or the special exceptional orbits are nonempty. Then the two actions are strictly equivalent if and only if*

1. *there exists a homeomorphism $M_1^*-E_1^*$ onto $M_2^*-E_2^*$ so that F_1^* goes homeomorphically onto F_2^*,*

2. *the set of unoriented Seifert invariants $\{(\alpha_{1i}, \beta_{1i})\}_{i\in I}$ associated with the first action on M_1 is in one to one correspondence with the unoriented Seifert invariants $\{(\alpha_{2j}, \beta_{2j})\}_{j\in J}$ associated with the second action.*

THEOREM 2b. *Let S^1 act effectively on the connected 3-manifold M. If M is compact assume that $F\cup SE\neq\varnothing$. Then two such actions where the orbit spaces are orientable are strictly equivalent, if and only if,*

1. *there exists an orientation preserving homeomorphism of $M_1^*-E_1^*$ onto $M_2^*-E_2^*$ so that F_1^* goes homeomorphically onto F_2^* and where the orientations of the orbit spaces are induced from a choice of orientation of $M-((SE)_1\cup(SE)_2)$,*

2. *the set of oriented Seifert invariants $(\alpha_{1i}, \beta_{1i})_{i\in I}$ associated with the first action*

on M is in one to one correspondence with the oriented Seifert invariants $\{(\alpha_{2j}, \beta_{2j})\}_{j\in J}$ *associated with the second action.*

Proof. In both a and b if the actions are equivalent it is clear that both conditions 1 and 2 must hold. We turn now to the "if" part.

For convenience of handling both arguments simultaneously take identical copies of M of Theorem 2b and call them M_1 and M_2. Assume that the first action is given on M_1 and the second on M_2.

Conditions 1 and 2 together imply that M_1^* and M_2^* are homeomorphic. Denote this homeomorphism by Γ. For Theorem 2b condition 1 guarantees that Γ is orientation preserving where the orientations are induced from $M-(SE_1 \cup SE_2)$. Furthermore it can be constructed so as to preserve the orbit structures on the boundaries (they could be empty if M_i are noncompact) of M_1^* and M_2^*.

Suppose $x_{1i}^* \in E_1^*$ and $\Gamma(x_{1i}^*) \in E_2^*$. By Lemma 6, we may find an equivariant isotopy of M_2 which is fixed outside of a small invariant tubular neighborhood of $\pi_2^{-1}(\Gamma(x_{1i}^*))$ and which moves the exceptional orbit $\pi_2^{-1}(\Gamma(x_{1i}^*))$ a small amount so that its projection misses $\Gamma(E_1^*)$ and the former E_2^*. Thus as the sets are closed and discrete in M_1^* and M_2^* we may as well assume that $\Gamma(E_1^*) \cap E_2^* = \varnothing$.

Let $f\colon E_1^* \to E_2^*$ be an isomorphism. That is, f maps E_1^* one to one onto E_2^* so that the invariant Seifert pair $(\alpha_{1i}, \beta_{1i})$ associated with $x_{1i}^* \in E_1^*$ is identical with the invariant pair $(\alpha_{2j}, \beta_{2j})$ associated with $f(x_{1i}^*) = x_{2j}^* \in E_2^*$. Reindex the set J by I, that is, put $x_{2j}^* \in E_2^*$ equal to x_{2i}^* and $(\alpha_{2j}, \beta_{2j}) = (\alpha_{2i}, \beta_{2i})$ if $x_{2j}^* = f(x_{1i}^*)$. The restriction of Γ to E_1^* does not agree with f and so we wish to deform (isotop) Γ so that it will agree with f.

Surround each member of E_k^*, $k=1, 2$ by small disjoint disks $D_{x_{ki}^*}$, and call the boundary of each of the disks $Q_{x_{ki}^*}$. Set

$$D_k^* = \bigcup_{x_{ki}^* \in E_k^*} D_{x_{ki}^*},$$

$$Q_{x_k^*} = \bigcup_{x_{ki}^* \in E_k^*} Q_{x_{ki}^*}.$$

Put

$$M_k' = (M_k - \pi_k^{-1}(D_k^*)) \cup \pi_k^{-1}(Q_k^*).$$

That is M_k' is just M_k with the interior of tubular neighborhoods of the exceptional orbits discarded. Now, by Lemma 6 again, we can actually find an isotopy $\Gamma_t\colon M_1^* \to M_2^*$, $0 \leqq t \leqq 1$, such that $\Gamma_1(M_1'^*) = M_2'^*$ with $\Gamma_1(Q_{x_{1i}^*}) = Q_{x_{2i}^*}$ for all $i \in I$.

For $x_{ki}^* \in E_k^*$ choose points $x_{ki} \in E_k$ so that $\pi_k(x_{ki}) = x_{ki}^*$. Choose slices $S_{x_{ki}}$ at each x_{ki} and oriented curves Q_{ki} so that

$$m_{ki} \sim \alpha_{ki} Q_{ki} + \beta_{ki} H.$$

We can assume that $\pi_k(S_{x_{ki}}) = D_{x_{ki}^*}$, and $\pi_k(Q_{ki}) = Q_{x_{ki}^*}$. Take homeomorphisms h_{ki}, (orientation preserving if necessary) so that for each (k, i), $k=1, 2$, $i \in I$,

$$h_{ki}\colon Q_{x_{ki}^*} \to Q_{ki}.$$

The actions on M_k' have cross sections and we may extend the given partial cross section $\bigcup_i h_{ki}$ to a global cross section, orientation preserving if necessary,

$$h_k\colon M_k'^* \to M_k'.$$

(If M_k is not compact just use [19, 34.4]. If M_k is compact we have assumed $F \cup SE \neq \varnothing$. Hence if we delete an open tubular neighborhood of a component of $F \cup SE$ we may extend to M_k' minus the deleted neighborhood. Now use Lemma 2 to extend this cross section over the deleted tubular neighborhood if the component is in F or use Lemma 4 if $F = \varnothing$.)

Take the homeomorphism

$$h_2\Gamma_1 h_1^{-1}\colon h_1(M_1'^*) \to h_2(M_2'^*).$$

This extends to an equivariant (orientation preserving if M_k^* is orientable) homeomorphism of M_1' onto M_2'. Note that the curves m_{1i}, Q_{1i} and H in M_1 determine the curves

$$h_2\Gamma_1 h_1^{-1}(m_{1i}) \sim m_{2i}, \qquad h_2\Gamma_1 h_1^{-1}(Q_{1i}) = Q_{2i}.$$

Now extend this homeomorphism to an equivariant homeomorphism (and orientation preserving if necessary) from all of M_1 onto all of M_2 by the fact that there is only one way, up to equivalence, to sew in a solid torus once the curves m, Q, H and the pair (α, β) (and the orientations) have been specified. This completes the proof of both of the theorems.

Theorem 2 has an extension to an effective action of the circle on a 3-manifold. The important point in the proof of Theorem 2 was that after specifying the curves Q on the boundaries of invariant neighborhoods of the orbits of E we could then find a cross section to the orbit map outside of these invariant neighborhoods and which agree with the Q's on the boundaries. This is generally no longer true when M is compact and $SE \cup F = \varnothing$.

Consider now M compact such that $F \cup SE = \varnothing$. The orbit space M^* is a compact 2-manifold without boundary. There is a cross section on M' but we must sew in the deleted tori in a very specific way. Hence if we specify cross sections to $\pi^{-1}(Q^*)$ to be the Q_i, then there is a one-one correspondence between $H^2(M', \pi^{-1}(Q^*); Z)$ and the principal bundles on M' which extend to the given actions on $\pi^{-1}(D_i^*)$. This group is isomorphic to Z or Z_2 depending on whether M^* is orientable or not. *This yields an integer b or an integer reduced modulo 2 which in addition to the orbit space and the set of Seifert invariants completely determine the action on M.*

In the special case that at least one of the unoriented orbit invariants is (2, 1) and M^* is nonorientable *the two distinct values of b yield the same action.* A very detailed account of this fact in terms of Seifert singular fiberings can be found in [18, Satz 8]. For the situation at hand the argument is roughly as follows. Specify cross sections Q_i near each exceptional orbit except for the one assumed to have unoriented orbit invariant (2, 1). Let N denote the tubular neighborhood of this exceptional orbit. We may extend this cross section over (M^*-N^*) (the disks for which Q_i^* are the boundaries). Denote the intersection of the cross section with the toral boundary of N by Q. By the previous obstruction argument and a slight variant of Lemma 6 any other cross section Q' can be deformed to satisfy $Q' \sim Q+H$ or $Q' \sim Q$ on the boundary of N, (cf. [18, Hilfsatz VII]). This corresponds to both values of b. Now consider the solid torus $D^2 \times S^1$ with linear action whose unoriented orbit invariant is (2, 1). By matching $-Q$ to $(e^{i\psi_0}, e^{i\theta})$ or $-Q'$ to $(ze^{i\psi_0}, ze^{i\theta_0})$ we can match the orbits on the boundary of N with those on $D^2 \times S^1$.

It is now easily seen that the orbit map of each action of G with $F \cup SE = \varnothing$ on a compact manifold gives rise to a singular fibering in the sense of Seifert [18] of type (O, o) or (N, n, I). Furthermore, each of the above classes of Seifert fiberings do correspond to orbit maps with $F \cup SE = \varnothing$. The other four classes of Seifert fiberings do *not* correspond to actions. Seifert classified each of the singular fiberings up to "fiber" preserving homeomorphisms. It is done in terms of the same set of invariants which determine an oriented or unoriented action with $F \cup SE = \varnothing$ on a manifold M. Furthermore an examination of the arguments in [18] show that the reconstruction of M by the orbit data or the Seifert singular fiber data yields exactly the same manifold and the same decomposition ("fiber" or orbit) map. Thus we have:

(6.1) *The classification up to strict equivalence of compact M with $SE \cup F = \varnothing$ coincides with Seifert's singular fiber classification.*

Combining the preceding discussion with Theorems 2a and 2b we have the following corollaries.

COROLLARY 2a. *The set of all strictly inequivalent effective actions of the circle on compact connected* 3*-manifolds with nonorientable orbit space is in one to one correspondence with the set of unordered tuples*

$$(b; (n, g, h, t); (\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots, (\alpha_n, \beta_n)),$$

where b is either 0 *or* 1 *modulo* 2 ($b=0$ *if* $h+t>0$, *or if some* $\alpha_i=2$).

COROLLARY 2b. *The set of all strictly inequivalent effective actions of the circle on compact connected* 3*-manifolds with oriented orbit space is in one to one correspondence with the set of unordered tuples*

$$(b; (o, g, h, t); (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)),$$

where b denotes the integer obtained from the cohomology class described above and o denotes a specific orientation on the orbit space.

If $h+t>0$, i.e. $F \cup SE \neq \varnothing$, then $b=0$, of course.

In both a and b we allow quadruples $(\varepsilon, g, 0, t)$, which were not allowed in Theorem 1, to account for the absence of any fixed point in M^* when $F=\varnothing$. Of course (α_i, β_i) are relatively prime and $0<\beta_i \leqq \alpha_i/2$ in (a) and $0<\beta_i<\alpha_i$ in (b).

This completes the list of all possible actions of the circle on 3-manifolds. The next two sections identify explicitly the manifolds which admit actions with fixed points and completely classify this type of action up to strict equivalence.

7. **Actions with fixed points on lens spaces.** For each (μ, ν) we shall construct an action on a lens space homeomorphic to $L(\mu, \nu)$. By using equivariant connected sums the action

$$\{(\varepsilon, g, h, t); (\mu_1, \nu_1), (\mu_2, \nu_2), \ldots, (\mu_n, \nu_n)\}$$

will be constructed on the space

$$M_{\varepsilon,g,h,t} \# L(\mu_1, \nu_1) \# \cdots \# L(\mu_n, \nu_n).$$

This makes precise how the invariants determine the manifold on which the action is taking place.

Recall that a lens space can be defined as the union of two solid tori sewn together by a homeomorphism on their boundaries. In particular let V_1 be one oriented solid torus, T_1 be its boundary. The resulting space is completely determined if we specify how the curve m_2 on T_2 is mapped onto T_1. This is determined by the homology relation

$$m_2 \sim \mu B_1 + \nu m_1 \tag{7.1}$$

where B_1 is an oriented curve of V_1 homologous to the center circle, m_1 is an oriented curve which is homologous to zero in V_1 and μ and ν are relatively prime. Notice that the oriented B_1 may be replaced by a similarly oriented B_1' where

$$B_1 \sim B_1' + x m_1.$$

Let us suppose that we have a lens space $L(\mu, \nu)$ where $0<\nu<\mu$, μ and ν are relatively prime and is determined by the homology relation (7.1). We shall define an action of S^1 on V_1 so that we have a single exceptional orbit corresponding to the center circle and all other orbits are principal. Let G operate on V_1 with oriented orbit invariant (μ, ν). We may as well assume that $V_1 = D^2 \times S^1$ such that $D^2 \times 1$, an oriented 2-disk, coincides with a slice S_x to the exceptional orbit $G(0 \times 1)$. Define an action of G on $V_2 = D^2 \times S^1$ by $z(\rho e^{i\psi}, e^{it}) = (z\rho e^{i\psi}, e^{it})$ where $0 \leqq \rho \leqq 1$, $0 \leqq \psi \leqq 2\pi$, $0 \leqq t \leqq 2\pi$, $|z| = 1$. Here the center circle $(0, e^{it})$ is left fixed and $m_2 = (e^{i\psi}, 1)$ is an orbit. Now map $H_2 = m_2$ onto H_1 and $Q_2 = (1, e^{it})$ onto $-Q_1$, where Q_1 is a cross section associated with the action on V_1. Note that we have matched V_2 to V_1 by an orientation reversing homeomorphism of T_2 onto T_1. In fact,

$$m_1 \sim -\alpha Q_2 + \beta H_2, \qquad B_1 \sim \rho Q_2 + \sigma H_2.$$

The matrix has determinant -1 since it must represent an orientation reversing homeomorphism. Thus one can check that $\mu = \alpha$ and $\beta\nu \equiv 1(\mu)$, (cf. [18, p. 154]) and the action is well defined on $L(\mu, \nu)$. It has exactly one exceptional orbit, the center circle of V_1, with oriented orbit invariant (μ, ν) and oriented Seifert invariant (α, β) and exactly one circle of fixed points, the center circle of V_2. This action is strictly equivalent to $(0; (o, 0, 1, 0); (\mu, \nu))$.

We remark that any compact manifold M, with finite fundamental group, which admits an effective action of the circle with fixed points must be a lens space and the action must be of the type just described. This is an easy consequence of lifting the action to the universal covering space of M, (see [5] and §2), and Corollary 2b. It also follows from Theorem 4.

It is well known that $L(\mu, \nu)$ and $L(\mu', \nu')$ are homeomorphic if and only if $\mu=\mu'$, $\nu\nu' \equiv \pm 1(\mu)$ or $\nu \pm \nu' \equiv 0(\mu)$ and $(\mu, \nu)=(\mu', \nu')=1$.

THEOREM 3. *If S^1 acts effectively and with fixed points on an oriented lens space $L(\mu, \nu)$ then there exists exactly one exceptional orbit. Furthermore the topologically inequivalent actions correspond to pairs (α, β) which satisfy*

1. $\alpha=\mu$, *and*
2. $\nu\beta \equiv \pm 1(\mu)$ *or* $\nu \pm \beta \equiv 0(\mu)$.

It is easily checked that there are exactly four topologically inequivalent actions on the oriented $L(\mu, \nu)$ unless $\nu^2 \equiv \pm 1(\mu)$ and then there are exactly two. The proof of this theorem is an immediate corollary of the preceding remark and the earlier construction and also will follow independently from Theorem 4.

8. **Equivariant connected sums.** Let M_1^* be a closed 2-disk with one point in the interior corresponding to an exceptional orbit with oriented orbit invariant (μ, ν), and with boundary the image of the fixed point set. We may as well assume that M_1^* is homeomorphic by means of an orientation preserving homeomorphism to the unit 2-disk in the plane with the image of the exceptional orbit equal to the origin. Let Q_1^* be the circle of radius 1/2. The set of points A_2 whose modulus is $\geqq 1/2$ can be lifted to a cross section over this set. The solid torus $V_2=\pi^{-1}(A_2)$ has center circle corresponding to the lifted image of the outer boundary. Now $\pi^{-1}(M_1^*-A_2)=V_1$ is a solid torus with exceptional orbit equal to $\pi^{-1}(0)$. The two solid tori are sewn together along their boundaries in such a way that

$$m \sim \alpha Q_1+\beta H_1$$

where H_1 corresponds to an orbit (and hence is a bounding curve on the boundary of V_2) and Q_1 is the lifted image of Q^*, perhaps adjusted. Hence M_1 is an oriented lens space $L(\mu, \nu)$ with oriented Seifert invariant (α, β) so that $\alpha=\mu$ and $\nu\beta \equiv 1(\alpha)$.

If we remove from the disk M_1^* the set of all points whose x-coordinates are less than $-3/4$, then we have removed in M_1 the interior of an invariant 3-cell. Call the deleted manifold M_1' and the deleted orbit space $M_1'^*$. Let I_1 denote the set of points of $M_1'^*$ whose x-coordinates are equal to $-3/4$.

Let M_2^* be a 2-manifold with boundary. We remove part of an oriented 2-cell from M_2^* by taking an oriented 2-cell whose boundary meets a boundary component of the 2-manifold in precisely an arc and with the rest of the 2-cell lying entirely in the interior of M_2^*. Remove all of the 2-cell except for the boundary points of the disk I_2 which are not interior points of the boundary of M_2^*. Call the deleted 2-manifold $M_2'^*$. Sew together $M_1'^*$ to $M_2'^*$ along I_1 and I_2 by an orientation reversing homeomorphism. This results in a manifold homeomorphic to M_2^*. If the manifold M_2^* is oriented then we shall assume that the orientation of the deleted 2-cell is that induced by the orientation of M_2^*. Then when we form $M_1'^* \# M_2'^*$ as described the new manifold is homeomorphic to M_2^* under an orientation preserving homeomorphism.

Now suppose that M_2^* is the orbit space of an effective action with fixed points on M_2. Assume also that the cell removed from M_2^* met only principal orbits except for those points meeting the boundary and there they meet fixed points. Take a cross section h_2 of the action on $M_2 - E$. Lift this 2-cell by h_2. Similarly, lift the deleted 2-cell by a cross section h_1 to $M_1 - E$. Now paste the *deleted* cross sections together exactly as we did on the orbit spaces. This gives an *equivariant* connected sum $M_1 \# M_2 = M_3$ and an action of S^1 on M_3 with fixed point sets and orbit spaces homeomorphic to the given action on M_2. But the set of exceptional orbits has been increased by the addition of one orbit that corresponds to $\pi_1^{-1}(0)$ in M_1, and of course the oriented orbit invariant of this orbit is (μ, ν).

Let $G = S^1$ act effectively and with fixed points on the connected 3-manifold M. Let M^* denote the orbit space and orient $M - SE$ by an orientation of M^* *if M^* is orientable*. Let M_1 be the unique 3-manifold which admits an effective action, with no exceptional orbits, of S^1 and whose orbit space is homeomorphic to M^* so that the fixed point sets correspond under the homeomorphism. Assume that the homeomorphism is orientation preserving if M^* is oriented. For each $x_i^* \in E^* \subset M^*$, $i \in I \subset Z^+$, let (μ_i, ν_i) be the pair associated with the orbit invariant of x_i^*. We shall find it convenient to use the notation $L'(\mu_i, \nu_i)$ to mean the oriented lens space $L(\mu_i, \nu_i)$ with the circle action described in §7 which has exactly one exceptional orbit of oriented type (μ_i, ν_i). We have shown

THEOREM 4. *M is equivalent in the strict sense to*

$$M_1 \# L'(\mu_1, \nu_1) \# L'(\mu_2, \nu_2) \# \cdots \# L'(\mu_i, \nu_i), \ldots, i \in I.$$

COROLLARY. *If M is compact then M is equivalent in the strict sense to*

$$M_{\varepsilon,g,h,t} \# L'(\mu_1, \nu_1) \# \cdots \# L'(\mu_n, \nu_n).$$

From the corollary it is easy to write down all strict inequivalent actions on a compact manifold. For example, suppose for convenience $\mu_i \neq \mu_j$ for $i \neq j$, and $\nu_i^2 \not\equiv \pm 1 (\mu_i)$. In Theorem 1 (iii) we have given a table for the number of inequivalent actions for $M_{\varepsilon,g,h,t}$. For each i, there exist four inequivalent actions if M^* is orientable and two actions if M^* is not orientable. So on M there exist

$$4^n \text{ (number of inequivalent actions on } M_{\varepsilon,g,h,t})$$

oriented actions on M if M^* is orientable and

$$2^n \text{ (number of inequivalent actions on } M_{\varepsilon,g,h,t})$$

if M^* is nonorientable.

Let G act effectively on M, compact, with action given by $(b; (\varepsilon, g, h, t); (\mu_1, \nu_1), \ldots, (\mu_n, \nu_n))$. Here we are assuming a general action as given by Corollaries 2a and 2b. Let H denote a principal orbit. Perform a surgery on an invariant tubular neighborhood of H. The solid torus removed is just $S_x \times H$ where S_x is a slice at x and the interior of a nicely imbedded disk. The solid torus $D^2 \times S^1$ that replaces $S_x \times H$ has an action of the circle on the first factor and trivial on the second factor. The principal orbits of the boundary of $S_x \times H$ are matched with the principal orbits of $D^2 \times S^1$. The new manifold M' now admits an action of the circle identical to the previous action outside of the surgery and on the replaced solid torus $D^2 \times S^1$ we have the standard action with fixed points.

COROLLARY. *With the notation of the preceding paragraph M' is strictly equivalent to*

$$M_{\varepsilon,g,h+1,t} \# L'(\mu_1, \nu_1) \# \cdots \# L'(\mu_n, \nu_n).$$

9. M, **compact,** $F = \varnothing$. To classify actions of the circle *without* fixed points on compact 3-manifolds in view of Corollaries 2a and 2b, which list all possible strictly inequivalent actions of the circle on 3-manifolds, it is necessary to identify, in some way, each of the 3-manifolds which admit such an action. The last corollary in §8, unfortunately, is not sufficient to solve this problem but it does suggest various approaches to a solution.

Recently, P. Orlik, E. Vogt and H. Zieschang [14] and independently F. Waldhausen [20] have shown that most of the Seifert singular fiberings are topologically distinct. In other words, under certain mild restrictions two Seifert singular fiber spaces are homeomorphic if and only if they are equivalent under a "fiber" preserving homeomorphism. Because of (6.1) we are able to state their result in terms of actions of the circle.

THEOREM 5 (ORLIK, VOGT, ZIESCHANG AND WALDHAUSEN). *Suppose M, compact, admits actions of type*

$$(b; (\varepsilon, g, 0, 0); (\mu_1, \nu_1), \ldots, (\mu_n, \nu_n)) \text{ and}$$
$$(b'; (\varepsilon', g', 0, 0); (\mu'_1, \nu'_1), \ldots, (\mu'_{n'}, \nu'_{n'})),$$

where all permissible values are allowed provided in the special cases,

$$\varepsilon = o \begin{cases} g \text{ or } g' = 0 \text{ implies } n > 3 \text{ or respectively } n' > 3, \\ g \text{ or } g' = 1 \text{ implies } n \geqq 1 \text{ or respectively } n' \geqq 1; \end{cases}$$

$$\varepsilon = n \begin{cases} g \text{ or } g' = 2 \text{ implies } n \geqq 1 \text{ or respectively } n' \geqq 1, \\ g \text{ or } g' = 1 \text{ implies } n > 1 \text{ or respectively } n' > 1. \end{cases}$$

Then, $b=b'$, $(\varepsilon, g, 0, 0)=(\varepsilon', g', 0, 0)$, $n=n'$, *and* $((\mu_1, \nu_1), \ldots, (\mu_n, \nu_n))$ *is a permutation of* $((\mu_1', \nu_1'), \ldots, (\mu_{n'}', \nu_{n'}'))$. *That is, the actions are strictly equivalent.*

The theorem is to be interpreted to mean that M has a fixed orientation if orientable and then the orbit invariants are oriented with respect to this fixed orientation. Seifert has shown [18] that the fundamental group of M can be presented in terms of the invariants which classify the Seifert fibering up to "fiber" preserving homeomorphism. The proof of the theorem in [14] consists in showing that each of the invariants are topological invariants. We refer the reader to [14].

The next proposition shows that manifolds which admit actions with fixed points usually do not admit actions without fixed points.

PROPOSITION 1. *Let* M *be a connected compact* 3-*manifold, and such that* $\pi_1(M)$ *is isomorphic to the nontrivial free product of* G_1 *and* G_2. *If* M *admits an action of the circle, then the action must have fixed points.*

Proof. By Hilfsatz VIII and Satz 22 of [18], M cannot be a Seifert fibering of type (O, o) or (N, n, I). One can present the fundamental groups of manifolds which admit actions with $h=0$, $t>0$. The presentations are analogous to those for $h=0$, $t=0$ and consequently we may apply Hilfsatz VIII and deduce that if M admitted an action where $h=0$, $t>0$, then $\pi_1(M)$ is isomorphic to the free product of two copies of Z_2. An examination of the possible fundamental groups shows that this is impossible.

COROLLARY. *If* $\pi_1(M)$ *is not cyclic and* $M \neq P^2 \times S^1$, *compact, admits an action without fixed points then* M *admits no action with fixed points.*

PROPOSITION 2. *A necessary condition that* M, *compact, admits an action with* $(g>0, n>1, h=0)$, $(g=0, n>3, h=0, t=0)$, *or* $(g=0, n>1, h=0, t>0)$ *is that the principal orbit* H *generates the center and that the center is infinite cyclic. Furthermore the manifold admits no action with fixed points.*

Proof. By generalizing an argument of Brody [3], P. Orlik showed in his thesis, University of Michigan, 1966, that a Seifert singular fibering of type (O, o) or (N, n, I) had an infinite cyclic center generated by a principal orbit H provided that the conditions of the proposition hold. The proof consists in observing that the group generated by H, (H), is in the center. Then, by factoring out the normal subgroup generated by H one obtains a group with no center. Seifert showed that H has infinite order unless $\pi_1(M)$ has finite order. This can occur only when $g=0$, $n \leqq 3$. This yields the first part of the proposition provided that $t=0$. Now using the presentation of the fundamental group for M, alluded to in Proposition 1, when $h=0$, $t>0$, we may apply a similar argument and also obtain the analogous result for this case.

The only compact manifolds which admit actions of the circle with fixed points and have nontrivial centers for their fundamental groups are $L(\mu, \nu)$, N, $S^2 \times S^1$,

$P^2 \times S^1$. None of these manifolds admit actions under the minor restrictions of the proposition. This completes the sketch of the proof of the proposition.

Note the proposition says that if a compact *orientable* manifold admits an action with enough exceptional orbits (at least 2 if $g>0$ and at least 4 if $g=0$) and without fixed points, then this manifold admits no other inequivalent action of the circle. The proposition, Theorems 4 and 5 yield under minor restrictions the complete classification, up to strict equivalence, of actions on compact connected oriented 3-manifolds.

We suspect that Theorem 5 (as well as the preceding remark in the nonorientable case) can be extended to cover actions where SE is not necessarily empty. Consider G acting on M, compact, such that $F=\varnothing$ and $SE\neq\varnothing$. Let $Z_2\subset G=S^1$. Then $M/Z_2=M'$, is a 3-manifold with a toral boundary component for each connected component of SE. The group G/Z_2, which is isomorphic to the circle, now operates on M' such that on the boundary it operates freely. The induced action, of course, has $F=SE=\varnothing$. In [14] a more general theorem than Theorem 5 is proved so that the analogue of Theorem 5 also holds in this case of a 3-manifold with boundary. Furthermore, the invariants of the action on M' are completely determined by the action on M. What one needs to do now is to study the epimorphism $\pi_1(M)\rightarrow\pi_1(M')$ so that under reasonable conditions $\pi_1(M_1')\neq\pi_1(M_2')$ will imply $\pi_1(M_1)\neq\pi_1(M_2)$, where M_1 and M_2 denote two manifolds admitting actions such that $F=\varnothing$, $SE\neq\varnothing$.

10. **Concluding remarks.** To prove Theorem 6 it is only necessary to observe that we have shown that each topological action is strictly topologically equivalent to a standard action. Each standard action, of course, can be chosen to be differentiable. If a given action is differentiable to begin with then all our constructions can be done in the differentiable category provided that we use an equivariant form of straightening the angle.

We have not said much about open 3-manifolds other than Theorems 2a, 2b, 4 and 6. However we shall state two propositions without proof to show that open 3-manifolds which admit actions are often very tractable. The proofs use the techniques of [16]. Let us suppose that G acts on the open connected 3-manifold M such that each component of SE is compact and E^* has at most a finite number of points.

PROPOSITION 3. *M is an open dense subspace of a compact* 3-*manifold* (*without boundary*) *if M^* is an open subspace of a compact* 2-*manifold* (*with boundary if M^* has boundary and without boundary if M^* lacks a boundary*). *Moreover, if such a compactification exists, then one can find a unique "minimal" compactification and an explicit extension of the action on the compactification.*

If M is open and connected and G acts such that $F\neq\varnothing$, then, using [5], we saw in §2 that the action of G can be lifted to the universal covering space $\tilde{M}$ of M.

PROPOSITION 4. *The lifted action on $\tilde{M}$ is equivariantly homeomorphic to the linear action, with fixed points, of S^1 on the 3-sphere with a closed totally disconnected set of fixed points removed.*

The fundamental group $\pi_1(M, x_0)$, $x_0 \in F$, now acts on $\tilde{M}/G$ so that

$$(\tilde{M}/G)/\pi_1(M, x_0) = M/G.$$

The stability group of $\pi_1(M, x_0)$ at $\tilde{x}^*$ is precisely $G_x/G_{\tilde{x}}$. And so $\pi_1(M, x_0)$ acts freely on $\tilde{F}^*$ and almost free everywhere else. That is, if $x \in E \cup SE$, then the stability group at $\tilde{x}^*$ is precisely isomorphic to Z_μ where μ is the first integer in (μ_i, ν_i), the orbit invariant associated with x if $x \in E$, and Z_2 if $x \in SE$. In the latter case the action of Z_2 must be an orientation reversing involution of the disk, otherwise all are orientation preserving. The entire analysis of the earlier sections could be developed exclusively from this point of view.

Another application of this fact is that every tame 2-sphere in M which bounds a contractible set in M must bound a 3-disk in M since the entire contractible set can be lifted to the universal covering space $\tilde{M}$. Hence if $F \neq \varnothing$, M, compact or not, is irreducible.

We also point out that we could have discussed cohomology 3-manifolds over Z instead of locally Euclidean 3-manifolds.

PROPOSITION 5. *Let G be a compact connected group $\neq e$ which acts effectively on a connected separable metric cohomology 3-manifold M over Z. Then the group is a Lie group, the cohomology manifold is locally Euclidean and the action is equivalent to a differentiable action.*

Proof. That all the groups are Lie groups is known [17], (or [2] if M is locally Euclidean). To see that M is locally Euclidean follows without much difficulty from [17, Theorem 1], and the fact that the circle is a subgroup of G.

If 2-or 3-dimensional orbits occur then these have been classified by Mostert in [13]. (Actually several cases have been omitted in [13], as pointed out by R. W. Richardson, Jr., but these can be readily filled in.) The actions in [13] are all equivalent to differentiable ones. If the maximal dimension of any orbit is 1-dimensional then $G = S^1$. A complete list has been given here and all of these are equivalent to differentiable actions by Theorem 6.

To generalize the results of this paper to differentiable actions of the circle on higher-dimensional manifolds with such completeness is impossible. Some parts do generalize but the classifications are much more complicated. For example, suppose that M is a compact connected differentiable n-manifold, $n > 4$, with G acting differentiably such that the dimension of each component of the fixed point set is $(n-2)$-dimensional. Then M/G is an $(n-1)$-manifold with nonempty boundary F/G. The principal bundle over $M/G - F/G$ is classified by $H^2(M/G - F/G; Z)$. The actions corresponding to the 0-element of this group are given by taking $(M/G \times I^2)$ and letting G act orthogonally on the 2-disk I^2 and trivially on the first factor. The action on M is then obtained by restricting the action to $\partial(M/G\times I^2)$. Thus all such inequivalent actions are in one-one correspondence with $(n-1)$-manifolds with boundary such that the double of their Cartesian product with I is diffeomorphic to M. For each $(n-1)$-manifold with boundary, N, we may find an *infinite* number of distinct N_i such that $\partial(N_i\times I^2)$ is diffeomorphic to $\partial(N\times I^2)$. Thus all such actions are differentiably and even topologically inequivalent.

For topological n-manifolds, $n>3$, the situation is still much worse, even when the action is as above except for differentiability. Then the orbit space $N=M/G$ need only be a cohomology $(n-1)$-manifold with boundary such that $\partial(N\times I^2)$ is homeomorphic to M. For *each topological action* of M with the properties as above one can find an *uncountable* number of N's and correspondingly an *uncountable* number of distinct topological actions of G on the same M with all the nice properties demanded above except for differentiability. Thus, only in dimension 3, are there few enough distinct actions to make a reasonable classification possible.

Added in proof. Since acceptance of this paper for publication, P. Orlik and I have completed the topological classification of the S^1 actions on closed 3-manifolds without any restrictions such as those imposed in §9. This will appear, under joint authorship, as *Actions of SO*(2) *on* 3-*manifolds*, in the Proceedings of the Conference on Transformation Groups, to be published by Springer.

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