

# SEMIRECURSIVE SETS AND POSITIVE REDUCIBILITY

BY

CARL G. JOCKUSCH, JR.<sup>(1)</sup>

**1. Introduction.** In this paper the notions of “semirecursive set” and “positive reducibility” introduced in [8] are studied and applied to problems in recursive function theory. It will be shown that every r.e. set with regressive complement is semirecursive and that every r.e. truth-table degree contains an r.e. semirecursive set. It will follow that there are hypersimple sets  $A$  with  $A \times A \leq_m A$  and that there are sets which are truth-table complete but not  $p$ -complete. It will be shown that every degree contains a semirecursive set but that the degrees of *immune* semirecursive sets are precisely the nonrecursive degrees which are r.e. in  $\mathbf{0}'$ . From the latter result it follows at once that every nonrecursive degree which is r.e. in  $\mathbf{0}'$  contains a hyperimmune set, for immune semirecursive sets are hyperimmune. Finally it will be shown that there is a simple semirecursive set with a nonregressive complement. As McLaughlin has pointed out, the construction also shows that the intersection of an r.e. set with a regressive set need not be regressive.

**2. Notation.** We will refer freely to the standard reducibilities, such as many-one and Turing reducibilities, which are defined in [13] and [14]. In general, if  $R$  is a reducibility and  $A$  and  $B$  are sets of integers, we let  $A \leq_R B$  mean that  $A$  is  $R$ -reducible to  $B$ . An  $R$ -degree is the collection of all sets which are  $R$ -equivalent to some fixed set, and a *degree* is a Turing degree.  $\mathbf{0}'$  is the degree of the halting problem. A set  $A$  is  $R$ -complete if  $A$  is recursively enumerable (r.e.) and each r.e. set is  $R$ -reducible to  $A$ . We let  $\varphi_j$  denote the  $j$ 'th partial recursive function in some standard enumeration of all partial recursive functions, and we let  $W_j$  denote the domain of  $\varphi_j$ .

If  $x_1, x_2, \dots, x_n$  are distinct integers and  $x = 2^{x_1} + 2^{x_2} + \dots + 2^{x_n}$ , then  $E_x$  denotes  $\{x_1, x_2, \dots, x_n\}$ , and  $E_0$  denotes  $\{0\}$ . Thus  $x$  encodes effectively the non-empty finite set  $E_x$ . We let  $\langle x, y \rangle$  denote the image of the ordered pair of  $x$  and  $y$  under a fixed effective coding of all ordered pairs of integers onto the integers. We define, for sets  $A, B$ ,

$$A \times B = \{\langle x, y \rangle : x \in A \ \& \ y \in B\}, \quad A \text{ join } B = \{2n : n \in A\} \cup \{2n+1 : n \in B\}.$$

If  $A$  is a set,  $\bar{A}$  is  $N - A$ , where  $N$  is the set of all natural numbers.  $A$  is said to be *coimmune* if  $\bar{A}$  is immune, and similarly for other properties of sets.

---

Received by the editors February 13, 1967.

<sup>(1)</sup> The first four sections of this paper are contained in the author's doctoral dissertation at M.I.T. Also, this research was supported in part by National Science Foundation Grant GP 4361 at M.I.T.

### 3. Some existence theorems for semirecursive sets.

DEFINITION 3.1. A set of natural numbers  $A$  is *semirecursive* if there is a recursive function  $f$  of two variables such that for every  $x$  and  $y$

- (i)  $f(x, y) = x$  or  $f(x, y) = y$  and
- (ii)  $(x \in A \text{ or } y \in A) \Rightarrow f(x, y) \in A$ .

If a recursive function  $f$  has properties (i) and (ii) for a set  $A$ ,  $f$  is called a *selector function for  $A$* .

Trivially, every recursive set is semirecursive, and any set  $A$  is semirecursive iff  $\bar{A}$  is semirecursive.

To show that every r.e. nonrecursive degree contains a hypersimple semirecursive set it is sufficient to show that the hypersimple sets constructed by Dekker [3, Theorem 1] are semirecursive. This is accomplished by Theorem 3.2.

THEOREM 3.2. *If a set  $A$  is recursively enumerable and  $\bar{A}$  is regressive, then  $A$  is semirecursive.*

**Proof.** Suppose that  $A$  is r.e. and  $\bar{A}$  is regressive. The theorem is immediate if  $\bar{A}$  is finite, so we may assume that  $\bar{A}$  is infinite. Then by definition of regressiveness [4, p. 80], there is a 1-1 enumeration  $\{a_i\}$  of  $\bar{A}$  and a partial recursive function  $\psi$  such that  $\psi(a_0) = a_0$  and  $\psi(a_{i+1}) = a_i$  for each  $i \geq 0$ .

We now give informal instructions for computing a selector function  $f$  for  $A$ . Fix an effective enumeration of  $A$ . Given  $x$  and  $y$ , simultaneously enumerate  $A$  and  $\{\psi^n(x) : n \geq 0\}$  and  $\{\psi^n(y) : n \geq 0\}$ . ( $\psi^n$  denotes the  $n$ th iterate of the partial function  $\psi$ .) Stop the procedure the first time any of the following occurs:

- (i)  $x$  is found in  $A$ .
- (ii)  $y$  is found in  $A$ .
- (iii)  $x$  is found in  $\{\psi^n(y) : n \geq 0\}$ .
- (iv)  $y$  is found in  $\{\psi^n(x) : n \geq 0\}$ .

If event (i) or (iv) stops the procedure, set  $f(x, y) = x$ .

If event (ii) or (iii) stops the procedure, set  $f(x, y) = y$ .

$f$  is a partial recursive function. Also  $f$  is everywhere defined, because if for some  $x$  and  $y$  the above procedure never stops, then  $x \in \bar{A}$  and  $y \in \bar{A}$ . Then it is clear from the definition of regressiveness that event (iii) or (iv) must occur. Thus  $f$  is recursive.

Now suppose  $f(x, y) \notin A$ . It must be shown that  $x \notin A$  and  $y \notin A$ . Since  $f(x, y) \notin A$ ,  $f(x, y)$  was computed via event (iii) or event (iv). Suppose  $f(x, y)$  was computed with event (iii). Thus  $f(x, y) = y$ , so  $y \notin A$ . Since  $y \notin A$  and event (iii) occurred, it follows from the definition of regressiveness that  $x \notin A$ . A similar argument shows that neither  $x$  nor  $y$  is in  $A$  if  $f(x, y)$  was computed via event (iv).

COROLLARY 3.3. *Every r.e. but not recursive degree contains a hypersimple semirecursive set.*

**Proof.** In [7, Theorem 3] it is shown that each such degree contains a hypersimple set with retraceable complement.

We now turn to a theorem which will imply that every truth-table degree contains a semirecursive set. First we define positive reducibility.

DEFINITION 3.4. If  $A$  and  $B$  are sets, then  $A$  is *positively reducible to  $B$*  (in symbols,  $A \leq_p B$ ) just in case there is a recursive function  $f$  such that, for all  $x$ ,  $x \in A \Leftrightarrow (\exists y)[y \in E_{f(x)} \ \& \ E_y \subset B]$ .

We digress to give an intuitive explanation and motivation of the above definition. Recall that  $A$  is *tt-reducible to  $B$*  just in case, given any  $x$ , one can effectively compute a finite list of numbers, say  $x_1, x_2, \dots, x_n$ , and a Boolean combination of statements of the form  $x_i \in B$ , such that  $x$  belongs to  $A$  iff the Boolean combination holds for the  $x_i$ . Of course, the Boolean combination can be represented as a formula of the propositional calculus in which the statement letters have the form " $y \in B$ " for numbers  $y$ . In the definition of positive reducibility, it is required that this representing formula have the form of a disjunction of conjunctions of statement letters. One could equivalently require that the representing formula be *positive*, i.e. use only conjunction and disjunction as logical connectives, since every positive formula is easily seen to be equivalent to a disjunction of conjunctions of statement letters. Our terminology does not agree with that of [9] since our positive truth tables are essentially the *e*-tables of [9].

PROPOSITION 3.5. (i)  $A \leq_p B, B \leq_p C \Rightarrow A \leq_p C$ .

(ii)  $A \leq_p B, B \text{ r.e.} \Rightarrow A \text{ r.e.}$

The straightforward proof is omitted. Part (ii) of the above proposition shows that  $p$  and *tt*-reducibility are distinct, for it implies that r.e. sets cannot in general be  $p$ -reduced to their complements. However, it is not so trivial to show that  $p$ -reducibility and *tt*-reducibility differ on the r.e. nonrecursive sets. Indeed, practically all *tt*-reductions between r.e. sets in the literature are actually positive reductions. However, the following theorem on semirecursive sets will lead to the result that there are sets which are *tt*-complete but not  $p$ -complete.

THEOREM 3.6. *For any set  $A$  there is a semirecursive set  $B$  such that  $B \leq_p A$  and  $A \leq_{tt} B$ .*

**Proof.** (The construction to be used is due to McLaughlin and Martin and was used by them to prove part (i) of Corollary 3.7 below.)

Let  $A$  be given. To avoid trivial cases, assume that  $A$  is nonempty and that  $\bar{A}$  is infinite. Define a real number  $r_A$  by  $r_A = \sum_{n \in A} 2^{-n}$ . For each integer  $x$ , define a rational number  $r_x$  by  $r_x = \sum_{n \in E_x} 2^{-n}$ . Define  $B = \{x : r_x \leq r_A\}$ .  $B$  is the desired set.  $B$  is semirecursive with the following selector function:

$$\begin{aligned} f(x, y) &= x && \text{if } r_x \leq r_y, \\ &= y && \text{if } r_y < r_x. \end{aligned}$$

Now define a recursive function  $h$  by  $h(x) =$  the largest member of  $E_x$ .

To show that  $B \leq_p A$ , it will be more than sufficient to show that  $x \in B$  iff there

exists a  $y$  such that  $E_y \subset \{0, 1, \dots, h(x)\}$ ,  $r_x \leq r_y$ , and  $E_y \subset A$ . Suppose  $x \in B$ . Let  $E_y = A \cap \{0, 1, \dots, h(x)\}$ . Then, since  $A$  is co-infinite, it follows from the fact that  $r_x \leq r_A$  and an elementary property of binary expansions that  $r_x \leq r_y$ , so the desired  $y$  exists.

Conversely, suppose that such a  $y$  exists. Since  $E_y \subset A$ ,  $r_y \leq r_A$ . Since  $r_x \leq r_y$ , it follows that  $r_x \leq r_A$  and so  $x \in B$ .

The straightforward but more tedious demonstration that  $A \leq_{tt} B$  is omitted.

COROLLARY 3.7.

- (i) (McLaughlin, Martin) *There exist continuously many semirecursive sets.*
- (ii) *Every r.e. tt-degree contains an r.e. semirecursive set.*

#### 4. Characterizations and simple properties of semirecursive sets.

THEOREM 4.1. *The following statements are equivalent.*

- (i)  *$A$  is semirecursive;*
- (ii)  *$A \times \bar{A}$  and  $\bar{A} \times A$  are recursively separable, i.e. there exists a recursive set  $C$  such that  $A \times \bar{A} \subset C$  and  $\bar{A} \times A \subset \bar{C}$ .*
- (iii) (McLaughlin, Appel (unpublished))  *$A$  is an initial segment of some recursive linear ordering of the natural numbers.*
- (iv) *There is a recursive function  $h$  such that  $h(x) \in E_x \cap A$  whenever  $E_x \cap A$  is nonempty.*

**Proof.** It is trivial that (i) and (ii) are equivalent and that (iv) implies (i). Thus it will suffice to show that (i) implies (iii) and that (iii) implies (iv).

First assume that  $A$  is semirecursive with a selector function  $f$ . To prove (iii) we must define a recursive linear ordering  $\leq_0$  of  $N$  such that whenever any number  $y$  is in  $A$ , then all  $x$  with  $x \leq_0 y$  are also in  $A$ . The ordering  $\leq_0$  will be defined by  $x \leq_0 y \Leftrightarrow g(x) \leq g(y)$ , (in the natural ordering of the rationals) where  $g$  is a function mapping the integers 1-1 into the rationals. The function  $g$  to be defined will be recursive when the rationals are effectively coded onto the integers, so  $\leq_0$  will be a recursive linear ordering.

We now define  $g(n)$  by induction on  $n$ .

Let  $g(0) = 0$ .

Now assume that  $n \geq 0$  and  $g(0), g(1), \dots, g(n)$  are all defined. Let  $x_0, x_1, \dots, x_n$  be the integers from 0 to  $n$  indexed such that  $g(x_0) < g(x_1) < \dots < g(x_n)$ .

Case 1.  $f(n+1, x_0) = n+1$ . Then define  $g(n+1) = g(x_0) - 1$ .

Case 2.  $f(n+1, x_n) = x_n$  and the first case does not apply. Then define  $g(n+1) = g(x_n) + 1$ .

Case 3. Neither Case 1 nor Case 2 applies. Then let  $j$  be the largest number  $i$  such that  $f(n+1, x_i) = x_i$ . Then define  $g(n+1) = (g(x_j) + g(x_j + 1))/2$ .

Note that  $j$  exists and is less than  $n$  because neither Case 1 nor Case 2 applies.

This completes the definition of  $g$  and hence of the recursive linear ordering  $\leq_0$ . It is easy to verify by induction on  $\max\{x, y\}$  that  $y \in A, x \leq_0 y \Rightarrow x \in A$ .

This completes the proof of (iii).

Now assume that (iii) holds and let  $\leq_0$  be a recursive linear ordering of  $N$  such that  $A$  is an initial segment of  $\leq_0$ . We define the recursive function  $h$  by letting  $h(x)$  be the least member with respect to the ordering  $\leq_0$  of the set  $E_x$ . Then, since  $A$  is an initial segment of  $\leq_0$ ,  $E_x \cap A \neq \emptyset \Rightarrow h(x) \in E_x \cap A$ .

Thus (iv) is proved. It should be noted that (iv) can also be proved directly from (i) without difficulty by defining  $h(x)$  by induction on the cardinality of  $E_x$ .

The following theorem, which gives some simple properties of semirecursive sets, will yield many results on reducibilities when it is combined with the results in §3.

**THEOREM 4.2.** *Let  $A$  be semirecursive and let  $B$  be any set. Then,*

- (i)  $A \times A \leq_m A, \bar{A} \times \bar{A} \leq_m \bar{A}$ .
- (ii)  $B \leq_p A \Rightarrow B \leq_m A$ .
- (iii)  $B \leq_p A \Rightarrow B$  semirecursive.
- (iv) *The positive degree of  $A$  consists of a single  $m$ -degree.*
- (v) *If  $A$  is immune, then  $A$  is hyperimmune.*
- (vi)  $A \leq_p \bar{A} \Rightarrow A$  recursive.

**Proof.** Throughout the proof, let  $f$  be a selector function for  $A$ .

- (i) We have, since  $f$  is a selector function for  $A$ ,

$$\langle x, y \rangle \in \bar{A} \times \bar{A} \Leftrightarrow f(x, y) \in \bar{A}.$$

Therefore,  $\bar{A} \times \bar{A} \leq_m \bar{A}$ . Since  $\bar{A}$  is semirecursive, we symmetrically have  $A \times A \leq_m A$ .

(ii) Suppose that  $B \leq_p A$  and that  $g$  is a recursive function such that  $x \in B \Leftrightarrow (\exists u)[u \in E_{g(x)} \ \& \ E_u \subset A]$ . Let  $h$  be a recursive function such that  $E_w \cap A \neq \emptyset \Leftrightarrow h(w) \in A$ .  $h$  exists since the function  $h$  guaranteed to exist by (iv) of Theorem 4.1 may be assumed to have the property that  $h(x) \in E_x$  for all  $x$ . Let  $h'$  be a recursive function such that  $E_u \subset A \Leftrightarrow h'(u) \in A$ .  $h'$  is seen to exist by applying the preceding argument to  $\bar{A}$ . Now,

$$\begin{aligned} x \in B &\Leftrightarrow (\exists u)[u \in E_{g(x)} \ \& \ h'(u) \in A], \\ &\Leftrightarrow E_w \cap A \neq \emptyset, \quad \text{where } E_w = \{h'(u) : u \in E_{g(x)}\}, \\ &\Leftrightarrow h(w) \in A. \end{aligned}$$

Given  $x$ , one may effectively compute  $h(w)$ , so  $B \leq_m A$ .

(iii) Assume  $B \leq_p A$ . Then  $B \leq_m A$  by (ii). Let  $g$  be a recursive function such that  $x \in B \Leftrightarrow g(x) \in A$ .

Define  $f'$ :

$$\begin{aligned} f'(x, y) &= x \quad \text{if } f(g(x), g(y)) = g(x), \\ &= y \quad \text{if } f(g(x), g(y)) \neq g(x). \end{aligned}$$

Then  $f'$  is a selector function for  $B$ , so  $B$  is semirecursive.

(iv) Assume  $B \equiv_p A$ . To show:  $B \equiv_m A$ . By (ii),  $B \leq_m A$ . By (iii)  $B$  is semirecursive. Hence by (ii), applied with  $B$  and  $A$  interchanged,  $A \leq_m B$ .

(v) Assume that  $A$  is infinite and not hyperimmune. Then there is a recursive

function  $k$  such that the sets  $E_{k(x)}$  are pairwise disjoint and all intersect  $A$ . Since  $A$  is semirecursive, there is a recursive function  $h$  such that  $E_x \cap A \neq \emptyset \Rightarrow h(x) \in E_x \cap A$ .

Then since, for each  $x$ ,  $hk(x) \in E_{k(x)} \cap A$ , the function  $hk$  is a 1-1 recursive function with range a subset of  $A$ , so  $A$  is not immune.

(vi) Suppose  $A \leq_p \bar{A}$ . Then  $A \leq_m \bar{A}$ , and there is a function  $g$  such that  $x \in A \Leftrightarrow g(x) \in \bar{A}$ . Hence,  $x \in A \Leftrightarrow f(x, g(x)) = x$ . Therefore,  $A$  is recursive.

**COROLLARY 4.3.**

- (i) *There exist hypersimple sets  $A$  such that  $A \times A \leq_m A$ .*
- (ii) *No  $p$ -complete set is semirecursive.*
- (iii) *There exists a set which is  $tt$ -complete but not  $p$ -complete.*
- (iv) *Each  $tt$ -degree contains two  $p$ -degrees which are incomparable and which each consist of a single  $m$ -degree.*

**Proof.**

(i) is immediate from Corollary 3.3 and (i) of the theorem. (i) answers a question raised by Young [19]. It is shown in [8, Corollary 5.3] that there does not exist a simple, nonhypersimple set  $A$  such that  $A \times A \leq_m A$ .

(ii) Assume that  $B$  is  $p$ -complete. Let  $A$  be any set which is simple but not hypersimple. Then  $A$  is not semirecursive by (v) of the theorem. Since  $B$  is  $p$ -complete,  $A \leq_p B$ , so  $B$  is not semirecursive by (iii) of the theorem.

(iii) By Corollary 3.7 there exists a  $tt$ -complete semirecursive set, and this set is not  $p$ -complete by (ii).

(iv) In the recursive  $tt$ -degree, the  $p$ -degrees of  $N$  and  $\emptyset$  are incomparable and each consist of a single  $m$ -degree. In any nonrecursive  $tt$ -degree, the  $p$ -degrees of  $A$  and  $\bar{A}$ , where  $A$  is a semirecursive set in the  $tt$ -degree, have the desired properties.

Yates [18, Theorem 2] has shown that every r.e. nonrecursive degree contains a simple set which is not hypersimple. It follows from this result and Theorem 4.2 that every r.e. nonrecursive degree contains an r.e. set which is not semirecursive and hence contains at least two r.e.  $p$ -degrees.

We have seen that immune semirecursive sets are hyperimmune. The following result, due to Martin, will show that such sets are never hyperhyperimmune.

**THEOREM 4.4 (MARTIN).** *If  $A$  is an infinite semirecursive set, then  $A$  has an infinite co-r.e. retraceable subset.*

**Proof.** Let  $A$  be infinite and semirecursive. We may assume that  $A$  is immune, since otherwise the conclusion is immediate. Suppose that  $A$  is an initial segment of a recursive linear ordering  $\leq_0$  of  $N$ . Define  $B = \{x : (\forall y)[x \leq y \Rightarrow x \leq_0 y]\}$ . We claim that  $B$  is the desired infinite co-r.e. retraceable subset of  $A$ .

$B$  is clearly co-r.e. To show that  $B \subset A$ , assume that  $x \in B$ . Let  $y$  be any member of  $A$  which is greater than  $x$ . Then  $x \leq_0 y$ , since  $x \in B$ . Therefore  $x \in A$ , since  $A$  is an initial segment of  $\leq_0$ .

To show that  $B$  is infinite, assume otherwise and let  $u$  be any member of  $A$

which is larger than every member of  $B$ . Define  $C = \{x : x \geq u \ \& \ x \leq_0 u\}$ . Since  $u \in A$ ,  $C \subset A$ . Since  $C$  is r.e. and  $A$  is immune,  $C$  is finite. Let  $x_0$  be the largest member of  $C$ . ( $C$  is nonempty since  $u \in C$  and hence  $C$  has a largest member.) Then for no  $y > x_0$  is it the case that  $y \leq_0 u$ , so for each  $y > x_0$  we have  $x_0 \leq_0 u \leq_0 y$ . Therefore  $x_0$  is in  $B$ . But every member of  $C$  is larger than any member of  $B$ . This contradiction shows that  $B$  is infinite.

It remains to show that  $B$  is retraceable. Observe that if any number  $y$  is in  $B$  and  $x$  is less than  $y$ , then  $x \in B \Leftrightarrow (\forall u)[x \leq u \leq y \Rightarrow x \leq_0 u]$ . It follows easily that  $B$  is retraceable.

**COROLLARY 4.5 (MARTIN).** *No semirecursive set is hyperhyperimmune.*

**Proof.** Suppose that  $A$  is an infinite semirecursive set. Then by the theorem,  $A$  has an infinite co-r.e. retraceable subset  $B$ . By Theorem 2 of [17],  $B$  is retraced by a finite-one partial recursive function. (Actually, the retracing function for  $B$  suggested by the proof of Theorem 4.4 is finite-one.) It now follows from the proof of Theorem 6 of [17] that  $A$  is not hyperhyperimmune, i.e. there is a recursive function  $f$  such that the sets  $W_{f(x)}$  are pairwise disjoint, finite, and all intersect  $A$ .

**COROLLARY 4.6.** *If  $A$  is r.e. and semirecursive and  $B$  is a maximal set, then  $B$  is not reducible to  $A$  by bounded truth tables.*

**Proof.** In [8, Theorem 3.3] it is shown that if  $B$  is a maximal set and  $B \leq_{btt} A$ , where  $A$  is r.e., then  $B \leq_p A$ . Hence, if  $A$  is r.e. and semirecursive and  $B$  is maximal, and  $B \leq_{btt} A$ , then  $B$  must also be semirecursive. But  $B$  cannot be semirecursive by Corollary 4.5.

**5. The degrees of immune semirecursive sets.** Although we have seen that every degree contains a semirecursive set, we shall see that the degrees of immune semirecursive sets are drastically limited.

**THEOREM 5.1.** *Assume that  $A$  is immune and  $\bar{A} \times \bar{A} \leq_m \bar{A}$ .*

- (i)  *$A$  is r.e. in  $\mathbf{0}'$ .*
- (ii) *If  $A$  is either effectively immune or retraceable,  $A$  is co-r.e.*

**Proof.** Assume that  $A$  is immune and  $\bar{A} \times \bar{A} \leq_m \bar{A}$ . Let  $g$  be a recursive function such that  $(x \in A \text{ or } y \in A) \Leftrightarrow g(x, y) \in A$ .

To prove part (i) we claim that for any  $x$ ,  $x \in A \Leftrightarrow \{g(x, y) : y \in N\}$  is finite. Once the claim is established, it will follow that  $A$  is in  $\Sigma_2$  in the arithmetical hierarchy and hence r.e. in  $\mathbf{0}'$ , for we will have  $x \in A \Leftrightarrow (\exists u)(\forall y)[g(x, y) \leq u]$ .

To prove the claim, first assume that  $x \in A$ . Then  $\{g(x, y) : y \in N\}$  is an r.e. subset of  $A$  and therefore finite.

Assume now that  $x \notin A$  and, contrary to the claim,  $\{g(x, y) : y \in N\}$  is a finite set. Then, for all  $z$ ,  $z \in A \Leftrightarrow g(x, z) \in A \cap \{g(x, y) : y \in N\}$ .

Thus  $A$  is recursive, contrary to assumption. This completes the proof of part (i).

To prove part (ii), assume now that  $A$  is effectively immune. This assumption

means [12, p. 33] that there is a recursive function  $h$  such that whenever  $W_x \subset A$ , the cardinality of  $W_x$  is at most  $h(x)$ . It follows that there is a recursive function  $h'$  such that  $x \in A \Rightarrow \{g(x, y) : y \in N\}$  has at most  $h'(x)$  members.

Hence, by the proof of part (i),  $x \in \bar{A} \Leftrightarrow$  there are more than  $h'(x)$  numbers of the form  $g(x, y)$ . Therefore,  $\bar{A}$  is r.e.

Finally, assume that  $A$  is retraceable. Let  $\psi$  be a partial recursive retracing function for  $A$ , and let  $a_0$  be the least member of  $A$ . Define  $B$  by

$$B = \{x : (\exists n)(\exists y)[g(x, y) > x \ \& \ \psi^n g(x, y) = a_0 \\ \& \ x \notin \{\psi g(x, y), \psi^2 g(x, y), \dots, \psi^n g(x, y)\}]\}.$$

$B$  is clearly r.e.

To show that  $B \subset \bar{A}$ , assume that some number  $x$  were in  $B \cap A$ . Let  $n$  and  $y$  be such that

$$g(x, y) > x \ \& \ \psi^n g(x, y) = a_0 \ \& \ x \notin \{\psi g(x, y), \psi^2 g(x, y), \dots, \psi^n g(x, y)\}.$$

Since  $x \in A$ ,  $g(x, y) \in A$ . Since  $\psi^n g(x, y) = a_0$ , every member of  $A$  which is less than  $g(x, y)$  is in  $\{\psi g(x, y), \dots, \psi^n g(x, y)\}$ . In particular,  $x$  belongs to this set, but this contradicts the assumption on  $n$  and  $y$ . Hence  $B \subset \bar{A}$ .

To show that  $\bar{A} \subset B$ , assume that  $x \in \bar{A}$ . Then the set  $C$  is infinite, where  $C = \{g(x, y) : y \in A\}$ . For if  $C$  were finite, the equivalence  $y \in A \Leftrightarrow g(x, y) \in C$  which holds because  $x \in \bar{A}$  and  $C \subset A$ , would show that  $A$  is recursive.

Since  $C$  is infinite, there is a number  $y \in A$  such that  $g(x, y) > x$ . Since  $y \in A$ ,  $g(x, y) \in A$ , so there is a number  $n$  with  $\psi^n g(x, y) = a_0$ . Also, since  $g(x, y) \in A$  and  $x \in \bar{A}$ ,  $x \notin \{\psi g(x, y), \psi^2 g(x, y), \dots, \psi^n g(x, y)\}$ . Thus this  $n$  and this  $y$  show that  $x \in B$ . Therefore  $\bar{A} = B$ , so  $\bar{A}$  is r.e.

Of course, Theorem 5.1 implies that every immune semirecursive set is r.e. in  $\mathbf{0}'$  and that every immune and coimmune semirecursive set is recursive in  $\mathbf{0}'$ . We now turn to the other half of the problem of classifying the degrees of immune and coimmune semirecursive sets.

**THEOREM 5.2.** *Each nonrecursive degree which is recursive in  $\mathbf{0}'$  contains a semi-recursive set which is both immune and coimmune.*

**Proof.** As in the proof of Theorem 3.6, we define

$$r_x = \sum_{n \in E_x} 2^{-n} \quad \text{and} \quad r_A = \sum_{n \in A} 2^{-n}$$

for numbers  $x$  and sets  $A$ . Recall that it was shown in that proof that  $A$  and  $\{x : r_x \leq r_A\}$  are sets of the same degree. For the purposes of this proof, let us call a set  $A$  *strongly non-r.e.* if  $\{x : r_x \leq r_A\}$  is neither r.e. nor co-r.e.

The theorem is an obvious consequence of the following three lemmas.

**LEMMA 5.3.** *Every nonrecursive degree contains a set  $A$  such that  $A$  is strongly non-r.e.*



LEMMA 5.4. *If  $A$  is recursive in  $\mathbf{0}'$ , there exists a recursive function  $f$  such that  $\lim r_{f(n)} = r_A$ .*

LEMMA 5.5. *If  $A$  is strongly non-r.e. and  $f$  is as in Lemma 5.4 and  $B = \{n : r_{f(n)} \leq r_A\}$  then  $B$  is immune, coimmune, semirecursive, and has the same degree as  $A$ .*

To prove Lemma 5.3 we note that it is sufficient to prove that every r.e. non-recursive degree contains a strongly non-r.e. set, for any set belonging to a non-r.e. degree is strongly non-r.e. Let  $B$  be a given r.e. nonrecursive set. By Corollary 2 on page 67 of [15],  $B$  is the disjoint union of r.e. sets  $C$  and  $D$  of incomparable degree. Define  $A$  to be  $C$  join  $\bar{D}$ .

Lemma 5.3 will follow once it is established that  $A$  is a strongly non-r.e. set and has the same degree as  $B$ . The degree of  $B$  is the least upper bound of the degrees of  $C$  and  $D$ , because  $C$  and  $D$  are r.e. and disjoint. Since  $A$  is the join of  $C$  and  $\bar{D}$ , the degree of  $A$  is also the least upper bound of the degrees of  $C$  and  $D$ , so  $A$  and  $B$  have the same degree.

Now suppose that  $A$  fails to be strongly non-r.e., so that  $\{x : r_x \leq r_A\}$  is either r.e. or co-r.e. Let us consider the case where this set is r.e. To get a contradiction in this case it is sufficient to show that  $D \leq_T C$ . Let  $n$  be given. We give a method for determining, recursively in  $C$ , whether  $n$  is in  $D$ . We assume inductively that we already know for each number  $m < n$  whether  $m$  is in  $D$ .

Let  $P(x)$  be the predicate  $(\forall y)_{\leq 2n} [(y \in E_x \leftrightarrow y \in C \text{ join } \bar{D}) \ \& \ 2n+1 \in E_x \ \& \ r_x \leq r_A]$ .

We claim that  $n \in \bar{D} \leftrightarrow (\exists x)P(x)$ . If  $n \in \bar{D}$ , then  $2n+1 \in A$ , so if we let

$$E_x = (C \text{ join } \bar{D}) \cap \{0, 1, \dots, 2n+1\},$$

then  $P(x)$  holds. Conversely, suppose that  $x$  is such that  $P(x)$  holds. Then the two real numbers  $r_x$  and  $r_A$  have identical binary expansion through the  $2n$ th place and  $r_x$  has a 1 in the  $(2n+1)$ th place. Since  $r_x \leq r_A$  and  $A$  is coinfinite, it follows that  $r_A$  has a 1 in the  $(2n+1)$ th place, so that  $n \in \bar{D}$ .

Now to see whether  $n$  is in  $D$ , we simultaneously enumerate  $D$  and look for an  $x$  such that  $P(x)$  holds. If  $n$  appears in the enumeration of  $D$ , we are done. If  $n \notin D$ , there will be an  $x$  such that  $P(x)$  holds, and it is clear from the definition of  $P(x)$  that we will be able to determine that  $P(x)$  holds using only the ability to enumerate  $\{x : r_x \leq r_A\}$ , knowledge of the membership of  $C$ , and knowledge of the membership of  $D$  for numbers  $m$  with  $m < n$ . Hence, from this knowledge we will be able to decide whether  $n \in D$ . We have shown that  $D \leq_T C$  and thus contradicted the supposition that  $\{x : r_x \leq r_A\}$  is r.e.

If we assume that  $\{x : r_x \leq r_A\}$  is co-r.e. we can show in the same way that  $C \leq_T D$  and reach a contradiction. This completes the proof of Lemma 5.3.

Lemma 5.4 is merely a restatement of the well-known fact that the characteristic function of any set recursive in  $\mathbf{0}'$  is the limit of a two-argument recursive function.

To prove Lemma 5.5, assume that  $A$  is strongly non-r.e. and  $\lim r_{f(n)} = r_A$ . Let  $B = \{n : r_{f(n)} \leq r_A\}$ . It follows from the definition of  $B$  that  $B \leq_m \{x : r_x \leq r_A\}$ , and so

$B$  is semirecursive and  $B \leq_T A$ , since  $\{x : r_x \leq r_A\}$  is a semirecursive set of the same degree as  $A$ .

$B$  is infinite. For if we assume that  $B$  is finite, it follows that the equivalence  $r_x \geq r_A \Leftrightarrow (\exists n)[n \notin B \ \& \ r_x \geq r_{f(n)}]$  shows that  $\{x : r_x \geq r_A\}$  is r.e., contrary to assumption. Similarly,  $\bar{B}$  is infinite.

Now assume that  $B$  is not immune, and let  $C$  be an infinite r.e. subset of  $B$ . Then  $r_x \leq r_A \Leftrightarrow (\exists n)[n \in C \ \& \ r_x \leq r_{f(n)}]$ . This contradicts the assumption that  $A$  is strongly non-r.e. and shows that  $B$  is immune. A similar proof shows that  $\bar{B}$  is immune.

It remains to show that  $A \leq_T B$ , so it is sufficient to show that  $\{x : r_x \leq r_A\}$  is recursive in  $B$ . Since  $B$  is infinite and coinfinite, we have the relationships:

$$r_x \leq r_A \Leftrightarrow (\exists n)[n \in B \ \& \ r_x \leq r_{f(n)}],$$

$$r_x \geq r_A \Leftrightarrow (\exists n)[n \notin B \ \& \ r_x \geq r_{f(n)}].$$

Thus  $\{x : r_x \leq r_A\}$  is both r.e. in  $B$  and co-r.e. in  $B$ , whence it is recursive in  $B$ .

**COROLLARY 5.6.** *Every nonrecursive degree which is recursive in  $\mathbf{0}'$  contains a set which is both hyperimmune and cohyperimmune.*

Corollary 5.6 strengthens a result of [11]. Corollary 5.6 can be relativized to state that any degree which is recursive in the jump of any strictly lower degree contains a set which is both hyperimmune and cohyperimmune.

**THEOREM 5.7.** *Every nonrecursive degree which is r.e. in  $\mathbf{0}'$  contains an immune semirecursive set.*

**Proof.** The proof is similar to the proof of Theorem 5.2, except that no counterpart to Lemma 5.3 is needed, and the counterpart to Lemma 5.4 will require proof.

**LEMMA 5.8.** *If  $A$  is r.e. in  $\mathbf{0}'$ , then there is a recursive function  $f$  such that  $\liminf r_{f(n)} = r_A$ .*

**LEMMA 5.9.** *If  $A$  is strongly non-r.e. (cf. proof of Theorem 5.2) and  $f$  is as in Lemma 5.8 and  $B = \{n : r_{f(n)} \leq r_A\}$ , then  $B$  is an immune semirecursive set of the same degree as  $A$ .*

We first prove the theorem from the lemmas. Every r.e. nonrecursive degree contains an immune semirecursive set by Corollary 3.3 (or Theorem 5.2). Every non-r.e. degree contains a strongly non-r.e. set, and therefore Lemmas 5.8 and 5.9 imply that every non-r.e. degree which is r.e. in  $\mathbf{0}'$  contains an immune semirecursive set.

We first discuss the proof of Lemma 5.9. The proof of Lemma 5.5 will suffice for Lemma 5.9 if we can show that if  $A$  is strongly non-r.e. and  $r_A = \liminf r_{f(n)}$ , then the sequence  $r_{f(n)}$  has a subsequence converging to  $r_A$  from below and a subsequence converging to  $r_A$  from above. Suppose that there were no subsequence

of  $r_{f(n)}$  converging to  $r_A$  from above. Then there would exist a rational number  $r$  such that  $r > r_A$  and  $(\forall n)[r_{f(n)} > r_A \Rightarrow r_{f(n)} > r]$ .

Since  $r_A$  is a limit point of  $r_{f(n)}$ , there must be a subsequence converging to  $r_A$  from below, so we have  $r_x \leq r_A \Leftrightarrow (\exists n)[r_x \leq r_{f(n)} < r]$ . This contradicts the assumption that  $A$  is strongly non-r.e. Similarly, one may show that  $r_{f(n)}$  has a subsequence converging to  $r_A$  from below.

To prove Lemma 5.8 it is sufficient to find a recursive function  $f$  having the following two properties:

(I)  $n \in A \Leftrightarrow (n \in E_{f(s)}$  for all but finitely many  $s$ ).

(II) For each  $k$ , there is an  $s$  such that  $(\forall n)_{\leq k}[n \in A \Leftrightarrow n \in E_{f(s)}]$ .

We obtain such a function  $f$  in several steps. First, since  $A$  is r.e. in  $\mathbf{0}'$ , there exists a recursive three-place predicate  $R$  such that  $x \in A \Leftrightarrow (\exists u)(\forall v)R(x, u, v)$ .

Define a recursive two-place function  $h$  by

$$\begin{aligned} h(s, x) &= (\mu u)_{\leq s}(\forall v)_{\leq s}R(x, u, v) && \text{if } (\exists u)_{\leq s}(\forall v)_{\leq s}R(x, u, v), \\ &= s+2 && \text{otherwise.} \end{aligned}$$

Define the recursive function  $g$  by:  $E_{g(s)} = \{x : x \leq s \ \& \ h(x, s) = h(x, s+1)\}$ . (For the purposes of this proof we let  $E_0$  denote the empty set rather than  $\{0\}$ .) Since  $x$  is in  $A$  iff  $\lim_s h(s, x)$  exists and is finite, the members of  $A$  are precisely those integers  $n$  which belong to  $E_{g(s)}$  for all but finitely many  $s$ . Thus  $g$  satisfies property (I) mentioned above for  $f$ , but there is no reason to think that it has property (II). However, we obtain the desired  $f$  by "shrinking" the sets  $E_{g(s)}$ . Define

$$E_{f(s)} = E_{g(s)} - E_{k(s)}$$

where  $k$  is a recursive function defined by

$$\begin{aligned} E_{k(s)} = \{n : (\exists t)_{< s}(\exists n_1)_{< t}[n_1 \notin E_{g(t)} \ \& \ (\exists n_2)_{\leq n_1}[s = \mu z[z > t \ \& \ n_2 \notin E_{g(z)}]] \\ \ \& \ (\exists w)[t \leq w \leq s \ \& \ n \notin E_{g(w)}]]\}. \end{aligned}$$

First we must show that  $n \in A \Leftrightarrow (n \in E_{f(s)}$  for all but finitely many  $s$ ). The arrow to the left in the above claim is immediate because  $E_{f(s)} \subset E_{g(s)}$  for all  $s$ , and  $g$  has property (I). To prove the converse implication, it is sufficient, since  $g$  has property (I), to show that if  $n \in A$ , then there are only finitely many  $s$  such that  $n \in E_{k(s)}$ . Suppose an element  $n$  of  $A$  has been fixed. Then there is a number  $w_0$  such that  $n \in E_{g(w)}$  for every  $w \geq w_0$ . Now

$$\begin{aligned} n \in E_{k(s)} \Rightarrow (\exists t)_{< w_0}(\exists n_1)_{< w_0}[n_1 \notin E_{g(t)} \ \& \ (\exists n_2)_{< w_0}[s = \mu z[z \geq t \ \& \ n_2 \notin E_{g(z)}]] \\ \ \& \ (\exists w)_{< w_0}[t \leq w \leq s \ \& \ n \notin E_{g(w)}]]. \end{aligned}$$

Here the bounding on  $w$  is permissible by the assumption on  $w_0$ . The other parameters may be bounded by  $w_0$  because in the definition of  $E_{k(s)}$ ,  $w$  bounds  $t$ ,  $t$  bounds  $n_1$ , and  $n_1$  bounds  $n_2$ . But then there are only finitely many numbers  $s$  of the form  $\mu z[z \geq t \ \& \ n_2 \notin E_{g(z)}]$  when  $n_2$  and  $t$  are restricted to range over numbers less than  $w_0$ . Hence  $n \in E_{k(s)}$  for only finitely many  $s$ .

Now it must be shown that for every  $k$  there is an  $s$  such that

$$(\forall n)_{\leq k} [n \in A \Leftrightarrow n \in E_{f(s)}].$$

Let  $k$  be fixed, and let  $q_0$  be so large that  $n \in E_{f(q)}$  for every  $n \leq k$  such that  $n \in A$  and every  $q \geq q_0$ .  $q$  exists since  $f$  satisfies property (I). If every number less than or equal to  $k$  is in  $A$ , we may take  $s = q_0$ , and we will be done. Otherwise, let  $n_1$  be the largest member of  $\bar{A}$  which is less than or equal to  $k$ . Let  $t$  be a number larger than both  $q_0$  and  $n_1$  such that  $n_1 \notin E_{g(t)}$ .

Define a finite partial function  $p$  for numbers  $n_2$  which are  $\leq n_1$  and not in  $A$  by  $p(n_2) = \mu z [z > t \ \& \ n_2 \notin E_{g(z)}]$ .

Now fix a particular number  $n_2$  on which  $p$  attains its maximum and let  $s = p(n_2)$ . We claim that for each  $n \leq k$ ,  $n \in A \Leftrightarrow n \in E_{g(s)}$ . If  $n \in A$  and  $n \leq k$ , then since  $s \geq q_0$ ,  $n \in E_{f(s)}$ .

Now assume that  $n \leq k$  &  $n \notin A$ . Let  $w = p(n)$ . Then

$$\begin{aligned} t < s \ \& \ n_1 < t \ \& \ n_1 \notin E_{g(t)} \ \& \ n_2 \leq n_1 \ \& \ s \\ &= \mu z [z > t \ \& \ n_2 \notin E_{g(z)}] \ \& \ t \leq w \leq s \ \& \ n \notin E_{g(w)}. \end{aligned}$$

Therefore,  $n \in E_{k(s)}$ , so  $n \notin E_{f(s)}$ . This concludes the proof of Lemma 5.8 and of the theorem.

**COROLLARY 5.9.** *Every nonrecursive degree which is r.e. in  $\mathbf{0}'$  contains a hyperimmune set.*

Corollary 5.9 generalizes a result of [11]. Corollary 5.9 also holds in a relativized form suggested by Martin: every degree which is r.e. in the jump of any strictly lower degree contains a hyperimmune set. Corollary 5.9 cannot be improved with regard to the arithmetical hierarchy, for it is shown in [11] that there are nonrecursive degrees below  $\mathbf{0}''$  which contain no hyperimmune set.

**6. Regressiveness and semirecursiveness.** We saw in §3 that each r.e. nonrecursive degree contains an immune semirecursive regressive set. However, the following result shows, in view of §5, that not all immune semirecursive sets are regressive.

**PROPOSITION 6.1.** *If  $A$  is semirecursive, immune, and coimmune, then neither  $A$  nor  $\bar{A}$  is regressive.*

**Proof.** By Proposition 3.2 of [2], the complement of an immune regressive set is never hyperimmune.

We mention without proof a result which implies Proposition 6.1. Every regressive semirecursive set is the difference of two r.e. sets. We do not know whether every regressive semirecursive set is either r.e. or co-r.e.

It is easy to check that a regressive set is r.e. in each of its infinite subsets, just as a retracable set is recursive in each of its infinite subsets. Immune semirecursive sets share this property of regressive sets.

**THEOREM 6.2.** *If  $A$  is immune and semirecursive, then  $A$  is r.e. in each of its infinite subsets.*

**Proof.** Suppose that  $A$  is immune and semirecursive, and let  $f$  be a selector function for  $A$ . Let  $B$  be any infinite subset of  $A$ . Define

$$C = \{x : (\exists y)[y \in B \ \& \ f(x, y) = x]\}.$$

$C$  is certainly r.e. in  $B$ , so it is sufficient to show that  $C = A$ . Since  $f$  is a selector function and  $B \subset A$ ,  $C \subset A$ . Hence if  $C \neq A$ , there must be some number  $x_0$  in  $A - C$ . Define  $D = \{y : f(x_0, y) = y\}$ .

$D$  is an r.e. subset of  $A$ . Since  $x_0 \notin C$ , we have further that  $(\forall y)[y \in B \Rightarrow f(x_0, y) = y]$ . Hence  $B \subset D$ , so  $D$  is infinite. This contradicts the assumption that  $A$  is immune.

**COROLLARY 6.3.** *If  $A$  and  $B$  are simple sets belonging to different degrees,  $A$  join  $B$  is not semirecursive. Hence, the join of r.e. semirecursive sets need not be semirecursive.*

**Proof.** Assume that  $A$  and  $B$  are simple sets and that  $A$  join  $B$  is semirecursive. Let  $C$  be the complement of  $A$  join  $B$ . Then  $C$  is an immune semirecursive set and hence r.e. in each of its infinite subsets. Since  $\bar{C}$  is r.e., it follows that  $C$  is recursive in each of its infinite subsets. Hence  $A$  join  $B \leq_T A$ ,  $A$  join  $B \leq_T B$ .

The above corollary was inspired by proposition P17(d) of [5], which was pointed out to the author by T. G. McLaughlin, who has obtained P17(d) independently.

**THEOREM 6.4.** *If  $A$  is strongly effectively immune and semirecursive, then  $A$  is regressive.*

**Proof.** Assume that  $A$  is strongly effectively immune. This means [12, p. 33] that  $A$  is infinite and there is a recursive function  $f$  such that every member of  $W_x$  is less than  $f(x)$  whenever  $W_x$  is contained in  $A$ .

We assume also that  $A$  is semirecursive, so that  $A$  is an initial segment of some recursive linear ordering  $\leq_0$  of  $N$ . Say that  $y$  is an  $o$ -predecessor of  $x$  if  $y \leq_0 x$ . Any  $x$  in  $A$  has only finitely many  $o$ -predecessors, because the set of its  $o$ -predecessors is an r.e. subset of the immune set  $A$ . Thus the restriction of  $\leq_0$  to  $A$  is an ordering in which every element has only finitely many predecessors and hence has order type  $\omega$ . Thus there is an enumeration  $\{a_i\}$  of  $A$  such that  $a_0 <_0 a_1 <_0 a_2 \dots$ .

If  $x$  is given, it is possible to compute effectively an r.e. index for the set of  $o$ -predecessors of  $x$ . Thus, since  $A$  is strongly effectively immune, there is a recursive function  $g$  such that, for each  $x$  in  $A$ , every  $o$ -predecessor of  $x$  is less than  $g(x)$ . It follows that there is a recursive function  $h$  such that  $h(a_0) = a_0$  and  $h(a_{i+1}) = a_i$  for all  $i \geq 0$ , so that  $A$  is regressed by the function  $h$ .

We now show that Theorem 6.4 cannot be extended to all effectively immune sets.

**THEOREM 6.5.** *There exist r.e. sets  $A, B$  such that  $A \leq_1 B$ ,  $\bar{B}$  is retraceable, and  $\bar{A}$  is effectively immune but not regressive.*

**Proof.** We employ a priority argument in a movable markers format. The movable markers will be used to spoil each partial recursive function as a regressing function for  $\bar{A}$ . To facilitate this spoiling we will use the following terminology:

$\varphi_j(x)$  is *strongly convergent* if there is a number  $p$  such that  $\varphi_j^p(x)$  and  $\varphi_j^{p+1}(x)$  are defined and equal. Clearly, if  $S$  is the set of all  $\langle j, x \rangle$  such that  $\varphi_j(x)$  is strongly convergent, then  $S$  is r.e. We fix an effective enumeration of  $S$  and say that  $\varphi_j(x)$  is *strongly convergent in at most  $n$  steps* if  $\langle j, x \rangle$  is among the first  $n$  numbers listed in our effective enumeration of  $S$ .

As in [2], we define

$$\hat{\varphi}_j(x) = \{y : (\exists p)[p \geq 0 \ \& \ \varphi_j^p(x) = y]\}.$$

We say that the numbers  $a, a'$  *spoil*  $\varphi_j$  if any of the following conditions hold:

- (i)  $a \in \bar{A}$  and  $\varphi_j(a)$  is not strongly convergent.
- (ii)  $a \in \bar{A}, a' \in \bar{A}$  and  $a \notin \hat{\varphi}_j(a')$  and  $a' \notin \hat{\varphi}_j(a)$ .
- (iii)  $a' \in \bar{A}$  and  $a \in (A \cap \hat{\varphi}_j(a'))$ .

Clearly,  $\varphi_j$  cannot regress  $\bar{A}$  if there exist numbers  $a, a'$  which spoil  $\varphi_j$ .

The construction will use movable markers  $\Lambda_j$  which will be associated with various integers as the construction proceeds. Each marker will achieve a final resting place. If  $a_{2j}$  and  $a_{2j+1}$  are the respective final resting places of  $\Lambda_{2j}$  and  $\Lambda_{2j+1}$ , the numbers  $a_{2j}, a_{2j+1}$  will spoil  $\varphi_j$ . Hence  $\bar{A}$  will not be regressive.

As the construction proceeds, a 1-1 recursive function  $f$  will be defined.  $B$  is defined to be the deficiency set of  $f$ :

$$B = \{x : (\exists y)[y > x \ \& \ f(y) < f(x)]\}.$$

Then, according to [7, Theorem 3],  $B$  is r.e. and  $\bar{B}$  is retraceable.

Also, a 1-1 recursive function  $g$  will be defined as the construction proceeds.  $A$  will be defined as  $g^{-1}(B)$  so that it is automatically true that  $A \leq_1 B$ . We let  $f_n$  and  $g_n$  denote the finite subfunctions of  $f$  and  $g$  respectively which have been defined by the end of the  $n$ th stage. We also define:

$$B^n = \{x : (\exists y)[y > x \ \& \ f_n(y) < f_n(x)]\} \quad \text{and} \quad A^n = g_n^{-1}(B^n).$$

*Stage 0.* Associate  $\Lambda_0$  with 0.

*Stage 3n. ( $n > 0$ ).* Stage  $3n$  is devoted to extending the definitions of  $f$  and  $g$  and to spoiling regressiveness of  $\bar{A}$ . Let  $y$  be the least integer which is greater than every member of the domain of  $g_{3n-1} \cup f_{3n-1}$  and does not bear a marker. Associate  $\Lambda_n$  with  $y$ . For each  $x \leq y$  such that  $g(x)$  has not yet been defined, let  $g(x) = 2x + 1$ .

Let  $u$  be the least integer such that  $f(u)$  has not yet been defined. For each  $x \leq u$  such that  $f(x)$  is as yet undefined, set  $f(x) = 3x$ .

Let  $a_0, a_1, \dots, a_n$  be the present positions of  $\Lambda_0, \Lambda_1, \dots, \Lambda_n$  respectively.

Let  $j$  be the least number such that  $\varphi_j(a_{2j})$  is strongly convergent in at most  $n$  steps and  $g(a_{2j+1}) = 2a_{2j+1} + 1$ . (If  $j$  does not exist, go to stage  $3n + 1$ .)

Let  $z$  be the least number which is greater than every member of  $\hat{\varphi}_j(a_{2j}) \cup \{a_n\}$ . Move the marker  $\Lambda_{2j+1}$  to  $z$  and define  $g(z)$  to be  $2a_{2j}$ . Also, move all markers  $\Lambda_k$ , for  $2j+1 < k \leq n$  to integers greater than  $z$  without disturbing their relative order and without associating any two markers with the same integer.

Stage  $3n$  is arranged so that  $a_{2j}, z$  will spoil  $\varphi_j$  by clause (ii) in the definition of spoiling unless the marker  $\Lambda_{2j}$  moves at some later stage of the construction or  $a_{2j} \in \hat{\varphi}_j(z)$ . In this latter case, it will be possible to spoil  $\varphi_j$  under clause (iii) in stage  $3m + 1$  for some  $m \geq n$ .

*Stage  $3n + 1$ .* Let  $a_0, a_1, \dots, a_n$  be the present positions of  $\Lambda_0, \Lambda_1, \dots, \Lambda_n$  respectively. Let  $j$  be the least number such that  $\varphi_j(a_{2j+1})$  is strongly convergent in at most  $n$  steps,  $a_{2j} \in \hat{\varphi}_j(a_{2j+1})$ ,  $a_{2j} \notin A^{3n}$ , and  $g(a_{2j+1}) = 2a_{2j}$ . (If  $j$  does not exist, go to stage  $3n + 2$ .)

To spoil  $\varphi_j$ , we want to put  $a_{2j}$  into  $A$  without putting  $a_{2j+1}$  into  $A$ . This is possible, since we have:  $g(a_{2j}) = 2a_{2j} + 1$ , and therefore  $fg(a_{2j}) = 6a_{2j} + 3$  and  $g(a_{2j+1}) = 2a_{2j}$ , and therefore  $fg(a_{2j+1}) = 6a_{2j}$ .

Let  $y$  be the least number such that  $y > 2a_n + 1$  and  $y$  is not in the domain of  $f_{3n}$ , and define  $f(y) = 6a_{2j} + 1$ . This definition of  $f(y)$  has the desired effect, so that  $a_{2j}, a_{2j+1}$  will spoil  $\varphi_j$  unless  $\Lambda_{2j}$  is caused to move at some later stage of the construction. This definition of  $f(y)$ , moreover, puts each number  $a_k$ , for  $2j+1 < k \leq n$ , into  $A$  and hence may undo earlier attempts to spoil  $\varphi_k$  for  $k > j$ . Hence we move all markers  $\Lambda_k$ , for  $2j+1 < k \leq n$ , to numbers greater than  $y$ .

*Stage  $3n + 2$ .* This stage is devoted to making  $\bar{A}$  effectively immune and employs a slight modification of the method of Post [13]. Let  $a_0, a_1, \dots, a_n$  be the present marker positions. Let  $W_j^n$  be the subset of  $W_j$  obtained by performing  $n$  steps in the enumeration of  $W_j$ .

Let  $j$  be the least number such that  $W_j^n$  does not intersect  $A^{3n+1}$ , but  $W_j^n$  does contain some number  $x > a_{2j+1}$ . (If  $j$  does not exist, go to the next stage.) Let  $y$  be the least number such that  $y > x$ ,  $y > a_n$ , and  $f(2y + 1)$  is as yet undefined. We desire to put all numbers  $z$  such that  $a_{2j+1} < z \leq y$  into  $A$ , since we will then have insured that  $W_j \cap A \neq \emptyset$ . To this end, let  $u = 2y + 1$ , and define  $f(u) = 6a_{2j} + 2$ .

Since for all  $z$  such that  $a_{2j+1} < z \leq y$  we will have  $fg(z) > 6a_{2j} + 2$ , this definition has the desired effect.

Move all markers  $\Lambda_k$  for  $2j+1 < k \leq n$  to integers greater than  $y$ .

This completes the description of the construction. Each marker moves only finitely often, since  $\Lambda_{2j}$  and  $\Lambda_{2j+1}$  move only to allow us to attempt to spoil some  $\varphi_k$  for  $k \leq j$  or to allow us to make some  $W_k$ , for  $k < j$ , intersect  $A$ . Hence each function  $\varphi_j$  is eventually spoiled, so  $\bar{A}$  is not regressive. To see that  $\bar{A}$  is effectively immune, one should first verify that if  $\Lambda_j$  is associated with a number  $a_j$  at some stage of the construction, then each  $x < a_j$  such that  $x \in \bar{A}$  also bears a marker at

that stage of the construction. Now, it follows from stage  $3n+2$ , that if  $W_j \subset \bar{A}$ , then

$$W_j \subset \{a_0, a_1, \dots, a_{2j}\}$$

where, for each  $i$ ,  $a_i$  is the  $i$ th largest member of  $\bar{A}$ . Hence if  $W_j \subset \bar{A}$ , the cardinality of  $W_j$  is at most  $2j+1$ , and  $\bar{A}$  is effectively immune.

**COROLLARY 6.6.** *There exist effectively immune semirecursive sets which are not regressive.*

**COROLLARY 6.7 (MARTIN [10]).** *There exist co-r.e. sets which are effectively immune but not strongly effectively immune.*

We can use Theorem 6.5 to answer a question of Dekker and Myhill [6, p. 121] concerning the recursive equivalence types of co-r.e. sets. Let the notation  $C \leq D$  mean that the RET of the set  $C$  is below that of  $D$  in the standard ordering of the RET's.

**COROLLARY 6.8.** *There exist co-r.e. sets  $C, D$  such that  $C \leq_1 D$  although it is not the case that  $C \leq D$ .*

**Proof.** Let  $C$  and  $D$  be the respective complements of the sets  $A$  and  $B$  constructed in Theorem 6.5. Now  $C$  and  $D$  are obviously co-r.e. sets with  $C \leq_1 D$ . If it were also the case that  $C \leq D$ , it is easy to see that the regressiveness of  $D$  would entail the regressiveness of  $C$ . Hence it is not true that  $C \leq D$ .

**COROLLARY 6.9 (MARTIN AND R. W. ROBINSON, UNPUBLISHED).** *There exist r.e. sets  $A, B$  such that  $A \leq_1 B$  although  $A$  cannot be 1-reduced to  $B$  with any function having a recursive range.*

**Proof.** Let  $A$  and  $B$  be the sets constructed in Theorem 6.5. If  $A$  could be 1-reduced to  $B$  via a function with recursive range, it would follow that  $A \leq B$ , which violates the proof of Corollary 6.8.

Although we have deduced Corollary 6.9 from Corollary 6.8, T. G. McLaughlin has pointed out that Corollary 6.8 can easily be deduced from Corollary 6.9. He has also observed that Theorem 6.5 implies the result of Appel [1] that the intersection of two regressive sets need not be regressive.

**COROLLARY 6.10 (MCLAUGHLIN, UNPUBLISHED).** *The intersection of an r.e. set with a co-r.e. retraceable set need not be regressive.*

**Proof.** Let  $A, B$  be as in Theorem 6.5 and suppose  $A = g^{-1}(B)$ , where  $g$  is a 1-1 recursive function. Let  $C = (\text{range } g) \cap \bar{B}$ .  $C$  is clearly the intersection of an r.e. set and a retraceable set.  $C$  is not regressive because  $g$  gives a recursive equivalence between  $C$  and the nonregressive set  $\bar{A}$ .

Actually, as McLaughlin has pointed out, Appel's proof that the intersection of two regressive sets need not be regressive shows that the intersection of a co-r.e.



regressive set with a co-r.e. retraceable set need not be regressive, so Corollary 6.10 complements Appel's result.

## REFERENCES

1. K. I. Appel, *There exist two regressive sets whose intersection is not regressive*, J. Symbolic Logic **32** (1967), 322–324.
2. K. I. Appel and T. G. McLaughlin, *On properties of regressive sets*, Trans. Amer. Math. Soc. **155** (1965), 83–93.
3. J. C. E. Dekker, *A theorem on hypersimple sets*, Proc. Amer. Math. Soc. **5** (1954), 791–796.
4. ———, *Infinite series of isols*, Proc. Sympos. Pure Math., Vol. 5, Amer. Math. Soc. Providence, R. I., 1962, pp. 77–96.
5. ———, *The minimum of two regressive isols*, Math. Z. **83** (1964), 345–366.
6. J. C. E. Dekker and J. Myhill, *Recursive equivalence types*, Univ. California Publ. Math. **3** (1960), 73, 74, 121.
7. ———, *Retraceable sets*, Canad. J. Math. **10** (1958), 357–373.
8. C. G. Jockusch, Jr., *Reducibilities in recursive function theory*, Doctoral Dissertation, Massachusetts Institute of Technology, Cambridge, Mass., 1966.
9. A. H. Lachlan, *Some notions of reducibility and productiveness*, Z. Math. Logik Grundlagen Math. **11** (1965), 17–44.
10. D. A. Martin, *Completeness, the recursion theorem, and effectively simple sets*, Proc. Amer. Math. Soc. **17** (1966), 838–842.
11. D. A. Martin and W. Miller, *On the degrees of hyper-immune sets*, Z. Math. Logik Grundlagen Math. (to appear).
12. T. G. McLaughlin, *On a class of complete simple sets*, Canad. Math. Bull. **8** (1965), 33–37.
13. E. L. Post, *Recursively enumerable sets of integers and their decision problems*, Bull. Amer. Math. Soc. **50** (1944), 284–316.
14. H. Rogers, Jr., *Recursive functions and effective computability*, McGraw-Hill, New York, 1967.
15. G. E. Sacks, *Degrees of unsolvability*, Ann. of Math. Studies No. 55, Princeton Univ. Press, Princeton, N. J., 1963.
16. R. M. Smullyan, *Effectively simple sets*, Proc. Amer. Math. Soc. **15** (1964), 893–895.
17. C. E. M. Yates, *Recursively enumerable sets and retracing functions*, Z. Math. Logik Grundlagen Math. **8** (1962), 331–345.
18. ———, *Three theorems on the degrees of recursively enumerable sets*, Duke Math. J. **32** (1965), 461–468.
19. P. R. Young, *A note on pseudo-creative sets and cylinders*, Pacific J. Math. **14** (1964), 749–753.

NORTHEASTERN UNIVERSITY,  
BOSTON, MASSACHUSETTS