

CLOSURE THEOREMS FOR SOME DISCRETE SUBGROUPS OF R^k

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Introduction. The study of lattice-point translates of an L^2 function was begun in [2] where it was required that the set of these functions form an orthogonal set. In this paper, the study is continued, but the orthogonality condition is dropped.

Thus, given a function K in L^2 on R^k , we should like to know when its lattice-point translates are dense in the largest subspace of L^2 for which this is possible, i.e. the subspace of L^2 functions F such that the support of \hat{F} , the Fourier transform of F , is contained in the support of \hat{K} . The problem is solved in Theorem 1, and the solution involves a geometric and measure-theoretic condition on S , the support of \hat{K} . Some more or less immediate corollaries follow which clarify the situation. It is the subgroup of lattice-points and certain linear images of it to which the title of the paper refers in this context. In the second theorem however, we consider, for the first and last occasion in this paper, a closure result for certain nondiscrete subgroups to which the already established methods are applicable.

In the second section, results analogous to the preceding are established for the case L^p , $1 < p < \infty$. They are, as to be expected, less precise than for the L^2 case. In the last section, the main theorem is established in the setting of locally compact, abelian groups.

In the matter of notation, m, n, \dots will stand for lattice-points. Thus, $m = (m_1, m_2, \dots, m_k)$ where each m_j is an integer. $L^p(\hat{S})$ will denote the space of L^p functions F such that the support of \hat{F} , written as $\text{supp } \hat{F}$, is in the set S . The set S is normally associated with an L^2 function K such that $\text{supp } \hat{K} = S$. As usual, χ_S is the characteristic function of S . $S_{2\pi}$ will denote the hypercube of side 2π , center at the origin, and with sides parallel to the coordinate axes. It is thus identified with the k torus. $H^*(K)$ signifies the L^2 closure of the linear span of the functions $K(x+m)$. $H_p^*(K)$ has an analogous meaning for L^p closure.

1. **Density of translates.** For the statement of our first theorem, a class of measurable sets which will serve as supports for acceptable functions \hat{K} must be identified. We say that S is of special form if, for almost every x ,

$$(1) \quad \sum_n \chi_S(x + 2\pi n) \leq 1.$$

The sum is taken over all lattice-points n . By integration of the left side of (1) over

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$S_{2\pi}$, it is seen that $|S|$, the measure of S , does not exceed $(2\pi)^k$. It also follows from (1) that, given n a lattice-point other than the origin, for almost every x

$$\chi_S(x)\chi_S(x+2\pi n) = 0.$$

In particular, $S_{2\pi}$ is a set of special form, and all such sets arise in the following way. Let E_n be a class of measurable and mutually disjoint sets, each contained in $S_{2\pi}$. The set $S = \bigcup_n (E_n + 2\pi n)$ is then of special form.

In the theory of cardinal series [2], sets of special form played a distinctive role in that they were only ones for which the corresponding spaces $H^*(K)$ were translation invariant. Similarly, in the present situation, these sets play a decisive role.

THEOREM 1. *Let S be the support of \hat{K} . Then $H^*(K) = L^2(\hat{S})$ if and only if S is of special form.*

First let S be a set of special form. Given F in $L^2(\hat{S})$, we may write $F = F_1 + F_2$, where F_1 is in $H^*(K)$, and F_2 is in the orthogonal complement. Since clearly, $\text{supp } \hat{F}_1 \subset S$, then also $\text{supp } \hat{F}_2 \subset S$. Also, for every lattice point m ,

$$0 = \int F_2(x)\bar{K}(x+m) dx = \int \hat{F}_2(x)\tilde{K}(x)e^{-i(m,x)} dx.$$

Here \tilde{K} denotes the complex conjugate of \hat{K} . The last integral can be written as

$$\int_{S_{2\pi}} \left\{ \sum_n \hat{F}_2(x+2\pi n)\tilde{K}(x+2\pi n) \right\} e^{-i(m,x)} dx.$$

Thus, the periodic and integrable function $\sum_n \hat{F}_2(x+2\pi n)\tilde{K}(x+2\pi n)$ has Fourier coefficients all equal to 0, and so is itself 0. But for each x , at most one of the terms in the sum $\sum_n \hat{F}_2(x+2\pi n)\tilde{K}(x+2\pi n)$ is nonzero. Thus, $\hat{F}_2(x)\tilde{K}(x) = 0$, and $F_2(x) = 0$ because $\text{supp } \hat{F}_2 \subset S$.

Now assume that S is not of special form. We wish to prove that $H^*(K) \neq L^2(\hat{S})$. There exists a nonzero lattice point N such that $\chi_S(x)\chi_S(x+2\pi N) \neq 0$ almost everywhere. For convenience we may assume there exists a set T_1 of positive measure in $S_{2\pi}$ such that $\chi_S(x)\chi_S(x+2\pi N) = 1$ on T_1 . Let $T = T_1 \cup (T_1 + 2\pi N)$, a subset of S . Now if $H^*(K) = L^2(\hat{S})$, then $H^*(K_T) = L^2(\hat{T})$, where $\hat{K}_T = \chi_T \hat{K}$. This follows easily from the fact that $(K - K_T)$ is orthogonal to $L^2(\hat{T})$. We now construct a function F in $L^2(\hat{T})$ which is not in $H^*(K_T)$. Let $\hat{F}(x) = \hat{K}_T(x+2\pi N)$ for x in T_1 ; let $\hat{F}(x) = -\hat{K}_T(x-2\pi N)$ for x in $T_1 + 2\pi N$; let \hat{F} be 0 elsewhere. Certainly F is in $L^2(\hat{T})$ and $\|F\|_2 > 0$. Let $\sum c_m K(x+m)$ be any finite sum, and designate $\sum c_m \exp i(m, x)$ by $P(x)$. Then

$$\begin{aligned} \|F(x) - \sum c_m K_T(x+m)\|_2^2 &= \|\hat{F} - \hat{K}_T P\|_2^2 \\ &= \|\hat{F}\|_2^2 + \|\hat{K}_T P\|_2^2 + \int_T \hat{F}(x)\tilde{K}_T(x)\bar{P}(x) dx \\ &\quad + \int_T \bar{F}(x)\hat{K}_T(x)P(x) dx. \end{aligned}$$

Here \tilde{F} denotes the complex conjugate of \hat{F} . To show that F is not in $H^*(K_T)$, it is sufficient to prove that the last two terms on the right are 0, i.e. then no finite sum $\sum c_m K_T(x+m)$ can approximate F arbitrarily closely. But

$$\int_T \hat{F}(x) \tilde{K}_T(x) \bar{P}(x) dx = \int_{T_1} \{\hat{F}(x) \tilde{K}_T(x) + \hat{F}(x+2\pi N) \tilde{K}_T(x+2\pi N)\} \bar{P}(x) dx.$$

It is easily checked from the definition of \hat{F} that the bracketed term in the second integral is 0.

COROLLARY 1. $H^*(K)$ is translation invariant if and only if $S = \text{supp } \hat{K}$ is a set of special form.

If S is contained in a set of special form, $H^*(K) = L^2(\hat{S})$. The latter is clearly translation invariant.

If $H^*(K)$ is translation invariant, it contains, in particular, $K(x+t)$ for every translate t . Thus there exist constants c_m such that

$$\left\| \hat{K}(x) \left\{ e^{i(x,t)} - \sum c_m e^{i(m,x)} \right\} \right\|_2 = \left\| K(x+t) - \sum c_m K(x+m) \right\|_2 < \varepsilon$$

for every $\varepsilon > 0$. Hence on S there is a sequence $P(x)$ of trigonometric polynomials approaching $\exp i(x, t)$, at least in measure. If S is not a set of special form, there exists a nonzero lattice point N such that S and $S+2\pi N$ intersect in a set T of positive measure. But P is periodic in each variable of period 2π . If t is chosen so that $\exp i(2\pi N, t) \neq 1$, a contradiction is obtained.

Let us reexamine the integral which must be small in order that F be in $H^*(K)$. It can be written as

$$\int_S \left| \hat{F}(x) - \hat{K}(x) \sum_m c_m e^{i(m,x)} \right|^2 dx = \int_S \left| \frac{\hat{F}(x)}{\hat{K}(x)} - \sum_m c_m e^{i(m,x)} \right|^2 |\hat{K}(x)|^2 dx.$$

The function \hat{F}/\hat{K} occurring on the right need not be integrable, but it is measurable and finite almost everywhere on S . In this form, it is clear that the trigonometric polynomial $\sum_m c_m \exp i(m, x)$ must be close to \hat{F}/\hat{K} in some weak sense: say a sequence of trigonometric polynomials approaches \hat{F}/\hat{K} in measure. This is clearly a necessary condition if S has finite measure. This being so, it is then clear that \hat{F}/\hat{K} is "periodic on S ": i.e. if x and $x+2\pi m$ are both in S , then \hat{F}/\hat{K} takes the same value at both points. The essence of Theorem 1 was that there are no such points if S is of special form and hence no conditions for F to satisfy other than that it be in $L^2(\hat{S})$. Under certain weaker conditions, the assumption of periodicity for \hat{F}/\hat{K} —which we take to include the fact that F is in $L^2(\hat{S})$ —is sufficient to insure that F be in $H^*(K)$.

COROLLARY 2. Let $\sum_n \chi_S(x+2\pi n)$ be in L^∞ where $S = \text{supp } \hat{K}$. Then F is in $H^*(K)$ if and only if \hat{F}/\hat{K} is periodic on S .

In view of the above discussion, it is sufficient to assume that \hat{F}/\hat{K} is periodic on S and to show that F is in $H^*(K)$. Let

$$\left\| \sum_n \chi_S(x+2\pi n) \right\|_\infty = M < \infty.$$

Consider, for each x in S , the lattice-point translates (with factor 2π) of x which are also in S . There exist, at most, M of these. Among them, there exists one, say y , such that $|\hat{K}(y)| \geq |\hat{K}(y+2\pi n)|$ for all n , since no more than M values of n will lead to nonzero values of \hat{K} . If a set of measure zero is disregarded, this is so for any x of S . It may happen that more than one such point y corresponding to a given x satisfies this inequality. If so, the sets $S_{2\pi} + 2\pi n$ may be indexed by the positive integers, and the y in that translate of $S_{2\pi}$ of least index will correspond to x . In this way, we obtain a measurable subset T of S such that for x in T , $|\hat{K}(x)| \geq |\hat{K}(x+2\pi n)|$ for all n , and such that for any x of S , there exists precisely one y of T such that $y-x=2\pi n$ for some n . It is clear that T is a set of special form.

Let S_1 be the difference set, $S \sim T$. For each x of S_1 , there exist at most M values of n such that $x+2\pi n$ is in S . But for one such value, $x+2\pi n$ is in T . Hence, there exist at most $M-1$ values of n such that $x+2\pi n$ is in S_1 so that $\|\sum_n \chi_{S_1}(x+2\pi n)\|_\infty \leq M-1$. Now we construct a subset T_1 of S_1 as above. The process is continued, producing sets $T_j \subset S_j$, $j=1, 2, \dots, M-1$. The sets T_j are of special form, $S_{j+1} = S_j \sim T_j$, and $T_{M-1} = S_{M-1}$. Also, if T_0 is taken as T , we have

$$S = \bigcup_{j=0}^{M-1} T_j$$

where the T_j 's are mutually disjoint.

Let F belong to $L^2(\hat{S})$ such that \hat{F}/\hat{K} is periodic on S . We temporarily consider the restrictions of both \hat{F} and \hat{K} to the set T . By Theorem 1, there exists a trigonometric polynomial P such that

$$\int_T \left| \frac{\hat{F}(x)}{\hat{K}(x)} - P(x) \right|^2 |\hat{K}(x)|^2 dx < \varepsilon.$$

With the same polynomial P , we have

$$\int_{T_j} \left| \frac{\hat{F}(x)}{\hat{K}(x)} - P(x) \right|^2 |\hat{K}(x)|^2 dx < \varepsilon, \quad j = 1, 2, \dots, M-1.$$

That this is so follows from these facts. For each x of T_j , there exists a point $x+2\pi n$ of T . The functional values of $\hat{F}/\hat{K} - P$ are the same for the two points because of the periodicity of \hat{F}/\hat{K} . Moreover, $|\hat{K}(x+2\pi n)| \geq |\hat{K}(x)|$ by the construction of T . Adding the results gives

$$\int_S |\hat{F}(x) - P(x)\hat{K}(x)|^2 dx < \varepsilon.$$

We now consider the effect on K of a similarity transformation in R^k . For $\alpha > 0$, define $K_\alpha(x) = K(\alpha x)$. Then $\hat{K}_\alpha(x) = \hat{K}(x/\alpha)/\alpha^k$. If $S(\alpha)$ is the support of \hat{K}_α and S the support of \hat{K} , then $\chi_{S(\alpha)} = \chi_S(x/\alpha)$. Define

$$\psi_\alpha(x) = \sum_n \chi_{S(\alpha)}(x + 2\pi n)$$

which is the function occurring in (1) with $\alpha = 1$. The space $L^2(\hat{S}(\alpha))$ consists of the functions F_α where $F_\alpha(x) = F(\alpha x)$ and F is in $L^2(\hat{S})$. An immediate consequence of Theorem 1 is the following.

COROLLARY 3. *Let $S = \text{support } \hat{K}$. The closure of the linear span of the functions $\{K(x + \alpha m)\}$ is $L^2(\hat{S})$ if and only if $\|\psi_\alpha\|_\infty \leq 1$.*

A natural question which arises in this connection is concerned with the existence of an $\alpha > 0$ corresponding to a set S such that $\|\psi_\alpha\|_\infty \leq 1$. In other words, even though S is not of special form, it might be so with respect to some α . However, it is easy to show the existence of an S of arbitrarily small measure such that $\|\psi_\alpha\|_\infty > 1$ for every $\alpha > 0$. We do this now in dimension one. A consequence of Corollary 3 is the following. *There exist sets S of R^1 of arbitrarily small measure such that if support $\hat{K} = S$, then for no discrete subgroup G of R^1 is $\{K(x + g), g \text{ in } G\}$ dense in $L^2(\hat{S})$.* It is, in fact, sufficient to show the existence of such a set S of finite measure since then, for every $\alpha > 0$, the set αS has the same property. The condition to be attained can be restated in more convenient form. For any $\alpha > 0$, we want $\sum_n \chi_S(x + 2\pi\alpha n) > 1$ on a set of positive measure. In this form, it is clear that we need consider only large values of α since halving α only adds to the set of lattice points. Let S be a union of intervals E_ν where $E_0 = (0, a)$, $2\pi < a < 3\pi$, and $E_\nu = (2\pi(\nu + 1), 2\pi(\nu + 1 + \epsilon_\nu))$, $\nu = 1, 2, \dots$, ϵ_ν is chosen so that $0 < \epsilon_\nu < 1$ and $\sum \epsilon_\nu < \infty$. Let $j \leq \alpha < j + 1$ for a large integer j . Since $2\pi(j + 1) - 2\pi\alpha \leq 2\pi < a$, we may let

$$2\pi(j + 1) - 2\pi\alpha < x < \min(a; 2\pi(j + 1) - 2\pi\alpha + \epsilon_j).$$

Since $0 < x < a$, then $\chi_S(x) = 1$. It is also clear that $x + 2\pi\alpha$ is in E_j so that $\chi_S(x + 2\pi\alpha) = 1$.

Every discrete subgroup of R^k has j generators where $j \leq k$. If $j = k$, then there exists an invertible linear transformation carrying the generators into unit orthogonal vectors. This leads to theorems of the type of Theorem 1.

Although our primary interest is discrete subgroups of R^k , the techniques used above can be applied in other situations. We end this section with a sample of such. Let x^j and m^j denote an arbitrary point and a lattice point respectively of R^j , $j \leq k$. Thus (x^j, x^{k-j}) denotes a point of R^k . Let S denote a measurable subset of R^k such that

$$(2) \quad \sum_{n^j} \chi_S(x^j + 2\pi n^j, x^{k-j}) \leq 1$$

for almost every $x=(x^j, x^{-j+k})$. For example, if $j=1$ and $k=2$, S might be included in a vertical strip. Let $H_j^*(K)$ denote the L^2 closure of the functions $K(x^j+m^j, x^{k-j}+r^{k-j})$ where m^{k-j} ranges over the lattice points of R^{k-j} and r^{k-j} ranges over the points of R^{k-j} . S , as usual, denotes the support of \hat{K} .

THEOREM 2. *Let $1 \leq j < k$. Then $H_j^*(K) = L^2(\hat{S})$ if and only if (2) is satisfied.*

Let (2) be satisfied, and let F be any function of $L^2(\hat{S})$. Then $F = F_1 + F_2$ where F_1 is in $H_j^*(K)$ and F_2 is orthogonal to it. Since the supports of \hat{F} and of \hat{F}_1 are in S , the same is true of \hat{F}_2 . The orthogonality condition can be expressed as

$$\int \hat{F}_2(x)\tilde{K}(x) \exp \{-i(m^j, x^j) - i(r^{k-j}, x^{k-j})\} dx = 0.$$

Now let

$$G^*(x^j, x^{k-j}) = \sum_{n^j} \hat{F}_2(x^j + 2\pi n^j, x^{k-j})\tilde{K}(x^j + 2\pi n^j, x^{k-j}).$$

This is periodic in x^j and integrable over $S_{2\pi}$ (in dimension j) for almost every x^{k-j} . The orthogonality condition becomes

$$\int_{R^{k-j}} \exp \{-i(r^{k-j}, x^{k-j})\} dx^{k-j} \int_{S_{2\pi}} G^*(x^j, x^{k-j}) \exp \{-i(m^j, x^j)\} dx^j = 0$$

for every (m^j, r^{k-j}) . The inner integral is integrable over R^{k-j} and hence is 0 for every lattice point m^j and almost every x^{k-j} . Thus $G^*(x^j, x^{k-j}) = 0$. But at most one term in the sum defining G^* is nonzero, and so

$$\hat{F}_2(x^j, x^{k-j})\tilde{K}(x^j, x^{k-j}) \equiv 0.$$

Since the support of \hat{F}_2 is in S , $\hat{F}_2 \equiv 0$.

To prove the converse, we need only mimic the proof of the corresponding part of Theorem 1 or, alternatively, make use of that periodicity of \hat{F}/\tilde{K} which is implicit in the given condition.

2. The case L^p , $1 < p < \infty$. In this section, we extend the ideas of Theorem 1 to L^p spaces with p different from 1 and ∞ . We shall require a hypothesis that is implicit in the L^2 case: that the characteristic function χ_S of the support of \hat{K} be a multiplier of class M_p^p (cf. [5] for notation) and that it lead to a projection onto the space $L^p(\hat{S})$. The latter is meant to imply that $L^p(\hat{S})$ is an invariant subspace for the operator. Thus, if F belongs to $L^p(\hat{S})$, then "multiplication" of the Fourier transform of F by χ_S is the identity operator. In the cases we consider, this operator will be realized as convolution with an ordinary function F_S so that if F is in $L^p(\hat{S})$, then $F * F_S = F$. It is to be noted that if $1 < p \leq 2$, then the assumption that χ_S is in M_p^p is sufficient to insure that a projection is obtained (cf. [7]). The two hypotheses can be combined by stating that χ_S gives a bounded, linear projection onto $L^p(\hat{S})$.

THEOREM 3. Let χ_S define a bounded, linear projection of L^p onto $L^p(\hat{S})$. Let S denote the support of \hat{K} .

(i) Let $2 \leq p < \infty$, and let K be in $L^2(\hat{S})$. If S is compact and of special form, then $H_p^*(K) = L^p(\hat{S})$.

(ii) Let $1 < p \leq 2$, and let $H^*(K) = L^p(\hat{S})$. Then S is of special form.

For the proof of (i), it is enough to show that $L^2(\hat{S})$ is dense in $L^p(\hat{S})$. For, if this is the case, then $H^*(K) = L^2(\hat{S})$. Let F be in $L^p(\hat{S})$ and G in $L^2(\hat{S})$ such that $\|F - G\|_p < \epsilon$. Let P be a linear combination of the functions $K(x+m)$ such that $\|G - P\|_2 < \epsilon$. Then

$$\|F - P\|_p \leq \|F - G\|_p + \|G - P\|_p.$$

But, since S is compact, $\|G - P\|_p \leq C\|G - P\|_2$ [3].

To show that $L^2(\hat{S})$ is dense in $L^p(\hat{S})$, we let F be in $L^p(\hat{S})$. Let ϕ be a smooth function such that $\|F - \phi\|_p$ is small. Application of the multiplier χ_S to both functions, F and ϕ , is realized by convolution with F_S , the function whose Fourier transform is χ_S . F_S is, of course, in L^2 , but it is also in L^q , $1/p + 1/q = 1$. This can be seen as follows. By duality, χ_S belongs to M_q^* . A large hypercube T will contain S . The function F_T , whose Fourier transform is χ_T , belongs to L^q (cf. [3]). Since $\chi_T \chi_S = \chi_S$, we have that F_S itself is in L^q . The same argument shows that F_S is in all L^r spaces, $q \leq r \leq p$. By hypothesis, $F_S * F = F$ so that

$$\|F - (F_S * \phi)\|_p = \|F_S * (F - \phi)\|_p \leq C\|F - \phi\|_p.$$

The function $F_S * \phi$ is in $L^2(\hat{S})$ if ϕ is only in L^2 .

Implicit in the hypothesis of part (ii) is the fact that K is in $L^p(\hat{S})$. If, under the given conditions, S is not of special form, then there exists a set T of positive measure and a nonzero lattice-point m such that

$$T_1 = T \cup (T + 2\pi m) \subset S.$$

We may take T_1 to be compact and then construct a smooth function ϕ —say $\hat{\phi}$ has compact support and ϕ is in L^p —such that $\hat{\phi}/\hat{K}$ is not periodic on T_1 . In fact, if it is periodic, it may be changed slightly to lose this periodicity. Now the function ψ with Fourier transform $\chi_S \hat{\phi}$ is in $L^p(\hat{S})$, but $\hat{\psi}/\hat{K}$ is not periodic on S . By virtue of the Hausdorff-Young theorem and arguments analogous to those of §1, it is clear that the periodicity on S of $\hat{\psi}/\hat{K}$ is a necessary condition for ψ to be in $H_p^*(K)$. This contradiction with the hypothesis that $H_p^*(K) = L^p(\hat{S})$ shows that S is of special form. We remark that, in this case, since S is of special form, then it has finite measure. Hence, K , being in L^p , is also in L^2 ; and by Theorem 1, $H^*(K) = L^2(\hat{S})$, (cf. [3]).

Examples of the situation described in Theorem 3 (ii) are provided by certain cardinal series (cf. [2]). We shall consider only the simplest situation: the support of \hat{K} is $S_{2\pi}$, and $|\hat{K}| = 1$ here. This implies that the functions $K(x+m)$ are

orthogonal. A smoothness condition is required: \hat{K} is in M_p^p . Then K is in $L^p(\hat{S}_{2\pi})$, since the characteristic function of $S_{2\pi}$ has a Fourier transform in L^p .

THEOREM 4. *Let $|\hat{K}|$ equal the characteristic function of $S_{2\pi}$, and let \hat{K} be in M_p^p , $1 < p \leq 2$. Then $H_p^*(K) = L^p(\hat{S}_{2\pi})$.*

By duality, \hat{K} is in M_q^q with $1/p + 1/q = 1$. For the same reason as above K is in L^q . Let F be in $L^p(\hat{S})$ with $S = S_{2\pi}$, and let G be the function whose Fourier transform is $\hat{F}\hat{K}$. Thus, G is in $L^p(\hat{S})$. Under these circumstances,

$$\sum_m |G(m)|^p < \infty$$

(cf. [6]). For the same reason $\sum_m |K(x+m)|^q$ is uniformly bounded. Thus the series $\sum_m G(-m)K(x+m)$ converges pointwise by Hölder's inequality. All the functions considered are in L^2 (cf. [3]), and the series converges in the L^2 norm to a constant multiple of F , which is thus also the pointwise limit. The L^p convergence will follow if we can show that

$$\|F\|_p^p \leq C \sum_m |G(m)|^p.$$

But this is the same as asking that $\|F\|_p^p \leq C \|G\|_p^p$ (cf. [6]). The latter inequality follows from the fact that $1/\hat{K}$ is in M_p^p , i.e. $1/\hat{K} = \tilde{K}$ on $S_{2\pi}$.

3. Group-theoretic background. The notation and basic notions for general groups will be taken from [8]. In particular, “+” will denote the group operation in all cases. Let G be a locally compact abelian group which is also Hausdorff. Let H be a discrete (so closed) subgroup of G such that G/H is compact. Let Γ be the dual group of G . We wish this to be σ -compact; and it will be so if G is first countable (cf. [4, p. 397]). Let Λ be the annihilator group of H . Λ is a closed subgroup of Γ and the dual group of G/H . Since G/H is compact, then Λ is discrete. Together with the fact that Γ is σ -compact, this implies that Λ is countable. Also, H is the annihilator group of Λ , and so Γ/Λ is the dual group of H . Our assumptions are perhaps stronger than they have to be, but they will lead quite directly to the result we want.

Let m_G be the Haar measure of G with the other Haar measures of the above groups designated accordingly. If f is a function of $L^1(\Gamma)$, then

$$\int_{\Lambda} f(\gamma + \lambda) dm_{\Lambda}(\lambda)$$

exists finitely for almost every γ of Γ [1]. Since it is invariant under changes of γ by elements of Λ , it may be considered as a function on Γ/Λ . The formula

$$(3) \quad \int_{\Gamma} f(\gamma) dm_{\Gamma}(\gamma) = \int_{\Gamma/\Lambda} dm_{\Gamma/\Lambda}(\omega) \int_{\Lambda} f(\omega + \lambda) dm_{\Lambda}(\lambda)$$

is valid after adjustments in the constants of the Haar measures involved and will be essential in our proof.

As a generalization of (1), the notion of a set of special form is introduced. We say that S , a measurable subset of Γ , is of special form with respect to Λ if

$$(4) \quad \int_{\Lambda} \chi_S(\gamma + \lambda) dm_{\Lambda}(\lambda) \leq 1 \quad \text{for almost every } \gamma \text{ of } \Gamma.$$

In this condition, it is important that ordinary discrete measure be used on Λ , i.e. each point has measure 1. It is also to be noted that γ may be interpreted as a point of Γ/Λ , i.e. the condition is invariant under changes of γ by elements of Γ . It happens that, in this case, to say that the exceptional set is of measure 0 in Γ is equivalent to saying that the corresponding set of Γ/Λ is of measure 0 with respect to Haar measure in Γ/Λ . Since this fact will be used elsewhere, we include a proof.

Let π denote the natural homomorphism from Γ to Γ/Λ . Let Q denote a measurable subset of Γ . Then $\pi(Q)$ is a measurable subset of Γ/Λ .

LEMMA. *Let $\pi^{-1}(\pi(Q)) = Q$. Then Q is of measure 0 in Γ if and only if $\pi(Q)$ is of measure 0 in Γ/Λ .*

Let ω be a point of Γ/Λ . We shall also write $\Lambda + \omega$ to denote a coset in Γ/Λ as well as a subset of Γ . In this case ω denotes any representative point in the coset. Now let $Q(\omega) = Q \cap (\Lambda + \omega)$, and assign to this set its "transplanted" measure (cf. [1]), i.e. $Q(\omega) - \omega$ is a subset of Λ and, as such, $m_{\Lambda}(Q(\omega) - \omega)$ exists and is, in fact, the number of points of $Q(\omega)$. We have [1]

$$m_{\Gamma}(Q) = \int_{\Gamma/\Lambda} m_{\Lambda}(Q(\omega) - \omega) dm_{\Gamma/\Lambda}(\omega)$$

and

$$m_{\Lambda}(Q(\omega) - \omega) = \sum_{\lambda_n \text{ in } \Lambda} \chi_{Q(\omega) - \omega}(\lambda_n)$$

where, as usual, χ denotes a characteristic function. Combining these equalities gives

$$m_{\Gamma}(Q) = \sum_{\lambda_n \text{ in } \Lambda} \int_{\Gamma/\Lambda} \chi_{Q(\omega) - \omega}(\lambda_n) dm_{\Gamma/\Lambda}(\omega).$$

Now we wish to assert that

$$(5) \quad \chi_{Q(\omega) - \omega}(\lambda_n) = \chi_{\pi(Q)}(\omega).$$

In fact, if (5) holds, then it follows from the preceding formula that

$$m_{\Gamma}(Q) = \sum_{\lambda_n \text{ in } \Lambda} m_{\Gamma/\Lambda}(\pi(Q))$$

so that the lemma is then proved. To establish (5) we argue as follows. To say that λ_n belongs to $Q(\omega) - \omega$ is equivalent to saying that ω is in $(Q - \lambda_n) \cap (\Lambda + \omega - \lambda_n)$. But $\Lambda - \lambda_n = \Lambda$, and $Q - \lambda_n = Q$ for every element λ_n of Λ . The latter fact follows since $\pi^{-1}(\pi(Q)) = Q$. Thus λ_n belongs to $Q(\omega) - \omega$ if and only if ω belongs to

$Q(\omega)$. But the latter is equivalent to saying that ω belongs to Q , i.e. each representative of the coset does. For such ω , and no others, the corresponding element of Γ/Λ belongs to $\pi(Q)$. This proves (5) and the lemma.

Let K be in $L^2(G)$ and $\text{supp } \tilde{K} = S$. $H^*(K)$ denotes the closure of the linear span of the functions $K(x+h)$, h in H ; and $L^2(\hat{S})$ has its previous meaning. In the following theorem, which is a general form of Theorem 1, it is understood that all the assumptions made in the first paragraph of this section are to hold.

THEOREM 5. $H^*(K) = L^2(\hat{S})$ if and only if S is of special form with respect to Λ .

Let S be a set of special form with respect to Λ , i.e. (4) is satisfied. Let F be in $L^2(\hat{S})$. Then $F = F_1 + F_2$ where F_1 is in $H^*(K)$ and F_2 is orthogonal to it. As before, $\text{supp } \hat{F}_j \subset S$, $j = 1, 2$. By use of Parseval's theorem, the orthogonality condition can be written as

$$0 = \int_G F_2(x)\bar{K}(x+h) dm_G(x) = \int_\Gamma \hat{F}_2(\gamma)\tilde{K}(\gamma)(h, \gamma) dm_\Gamma(\gamma), \quad h \text{ in } H.$$

Since $\hat{F}_2(\gamma)\tilde{K}(\gamma)(h, \gamma)_c$, where the subscript c denotes complex conjugate, is in $L^1(\Gamma)$, we have by (3) that

$$0 = \int_{\Gamma/\Lambda} dm_{\Gamma/\Lambda}(\omega) \int_\Lambda \hat{F}_2(\gamma + \lambda)\tilde{K}(\gamma + \lambda)(h, \gamma + \lambda)_c dm_\Lambda(\lambda)$$

where, as before, the inner integral may be considered as a function of ω in Γ/Λ rather than as a function of γ in Γ . Thus

$$0 = \int_{\Gamma/\Lambda} (h, \omega)_c dm_{\Gamma/\Lambda}(\omega) \int_\Lambda \hat{F}_2(\gamma + \lambda)\tilde{K}(\gamma + \lambda) dm_\Lambda(\lambda).$$

The last equality follows from the fact that Λ is the annihilator of H , i.e. $(h, \gamma + \lambda) = (h, \gamma)$. The function $\int_\Lambda \hat{F}_2(\gamma + \lambda)\tilde{K}(\gamma + \lambda) dm_\Lambda(\lambda)$ is in $L^1(\Gamma/\Lambda)$. Since H is the dual group of Γ/Λ , it follows from the uniqueness theorem that it is the zero function, i.e. it is 0 for almost every ω of Γ/Λ . But, by the lemma, this is equivalent to saying that it is 0 for almost every γ of Γ . Hence, using the series form of this last integral, which is legitimate since Λ is countable, and the fact that $\text{supp } \tilde{K} = S$ we obtain

$$\sum_{\lambda_n \text{ in } \Lambda} \hat{F}_2(\gamma + \lambda_n)\tilde{K}(\gamma + \lambda_n)\chi_S(\gamma + \lambda_n) = 0$$

for almost every γ . It is clear from (4) that at most one term in this sum is nonzero. Hence, $\hat{F}_2(\gamma)\tilde{K}(\gamma) = 0$ for almost every γ . Finally, since $\text{supp } \hat{F}_2 \subset S$, \hat{F}_2 is itself 0 almost everywhere.

Now let $H^*(K) = L^2(\hat{S})$. If (4) is not satisfied, there exists, in view of the countability of Λ , a fixed λ in Λ such that

$$\chi_S(\gamma) + \chi_S(\gamma + \lambda) = 2$$

for γ in a subset of S of positive measure. \hat{F} may be constructed as in the proof of Theorem 1 such that F is in $L^2(\hat{S})$ but such that \hat{F}/\hat{K} is not periodic on S in the above sense. Hence, F is not in $H^*(K)$, and this contradiction shows that (4) is satisfied.

The most important example of this last theorem seems to be our original case, that in which $G=R^k$. However, it does have some nontrivial things to say if G is taken to be either the torus or the set of lattice points in R^k .

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