FREE Σ-STRUCTURES

BY

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Introduction. The concepts of free semigroups, free groups, free lattices, and so on, play a central role in algebra. These concepts were unified by G. Birkhoff [1] who showed that given any class of algebras, defined by a set Σ of identities, free algebras can be defined and their existence can be proved. The construction is, roughly speaking, the following: we start with a set X and from the elements of X we generate an algebra as freely as possible, that is, elements will be identified only if they have to be identified because of the identities in Σ. The free algebras, thus constructed, have the property that every algebra satisfying Σ is a homomorphic image of a free algebra.

If we consider classes of algebras defined by axiom systems more involved than identities, the situation is not so simple. The case of universal axiom systems Σ shows that free algebras need not exist, but if they do, they can be constructed using Birkhoff's method.

Arbitrary first order axiom systems raise new problems. If, by introducing new operations, they can be reduced to universal axiom systems (as in case of groups), then there is no new problem. But even in this case the question arises whether such axiom systems can be handled directly. However, it can be shown that most first order axiom systems cannot be reduced to universal ones if we want the reduction to preserve the important algebraic properties of the class. Thus arises the necessity of introducing some new concept of free algebras over arbitrary first order axiom systems.

The first task is to find a proper definition of subalgebras and homomorphisms. If an operation is replaced by an axiom requiring the existence of an element, we immediately see that our subalgebras must be closed under the formation of "inverses" which are guaranteed by the axioms and that a homomorphism should have the substitution property for these elements as well.

In this paper a solution to these problems is proposed. For an arbitrary first order axiom system Σ we define the concepts of Σ-subalgebras and Σ-homomorphisms. In terms of these concepts we define free Σ-algebras.

Σ-subalgebras and Σ-homomorphisms are not related to each other the way subalgebras and homomorphisms are. The injection of a Σ-subalgebra of a Σ-algebra

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into the Σ-algebra need not be a Σ-homomorphism. Also, a map of a Σ-generating set can, in general, be extended to many Σ-homomorphisms. Thus free Σ-algebras cannot be handled using the adjoint functor theorem. Therefore it is somewhat surprising that we still get the Uniqueness Theorem (Theorem 2 of §3) for free Σ-algebras.

The basic idea of the paper is the use of multi-valued functions, called Σ-polynomials, which are constructed from Σ. These will turn out to be uniformly bounded if free Σ-algebras exist. Then these are used to define covering families of finite sets on certain algebras, and the problems considered are localized on a suitable finite set.

The basic properties of Σ-subalgebras and Σ-homomorphisms are given in §§1 and 2. §3 contains the Uniqueness Theorem. In §4 it is proved that for a given Σ either there exists a positive integer n such that the free Σ-algebra on k generators exists if and only if $k<n$, or *all* free Σ-algebras exist. In §5 necessary and sufficient conditions are given for the existence of free Σ-algebras and some partial results are given on the problem: when can one construct the free Σ-algebras using inverse limits. The last section (§6) investigates when free Σ-algebras can be constructed as free algebras over a richer type.

Most of the results of this paper were announced in [4]. The present approach is somewhat more general, since we will formulate the results for structures rather than algebras. This framework is much more natural for the whole theory; also, it allows us to give some very natural examples (e.g., lattices as partially ordered sets). Since relations are admitted, this theory subsumes the theory of algebras with a scheme of operators (Higgins [5]) in a more natural way. As a result of this greater generality one definition and three proofs become longer but otherwise there is no essential change.

NOTATIONS. A *type* τ is defined as a pair $\langle \tau_0, \tau_1\rangle$, where $\tau_0=\langle n_0, \ldots, n_\gamma, \ldots\rangle$, $\gamma<o_0(\tau)$, and $\tau_1=\langle m_0, \ldots, m_\gamma, \ldots\rangle$, $\gamma<o_1(\tau)$ where $o_0(\tau)$ and $o_1(\tau)$ are arbitrary ordinals, and m_γ and n_γ are nonnegative integers. A *structure* $\mathfrak{A}$ of type τ is a triple $\langle A; F, R\rangle$, where $F=\langle f_0, \ldots, f_\gamma, \ldots\rangle$, $\gamma<o_0(\tau)$, $R=\langle r_0, \ldots, r_\gamma, \ldots\rangle$, $\gamma<o_1(\tau)$, where f_γ is an n_γ-ary operation and r_γ is an m_γ-ary relation. All structures will be assumed to be of a fixed type τ, unless otherwise specified. We write $o(\tau)$ for $o_0(\tau)+o_1(\tau)$.

A *homomorphism* φ of the structure $\mathfrak{A}$ into the structure $\mathfrak{B}$ is a map of A into B such that

$$f_\gamma(a_0, \ldots, a_{n_\gamma-1})\varphi = f_\gamma(a_0\varphi, \ldots, a_{n_\gamma-1}\varphi)$$

for all $a_0, \ldots, a_{n_\gamma-1} \in A$ and $\gamma<o_0(\tau)$, and

$$r_\gamma(a_0, \ldots, a_{m_\gamma-1}) \text{ implies } r_\gamma(a_0\varphi, \ldots, a_{m_\gamma-1}\varphi),$$

for all $a_0, \ldots, a_{m_\gamma-1} \in A$ and $\gamma<o_1(\tau)$.

A *polynomial* p over $\mathfrak{A}$ is a function from A^n into A, for some $n<\omega$, which we get from the projections by substituting them into the f_γ, $\gamma<o_0(\tau)$.

$L(\tau)$ is the first order language (with equality) associated with τ in the usual sense. A set Σ of first order sentences is called an *axiom system*. The class of all structures satisfying Σ will be denoted by Σ^*. If $\mathfrak{A} \in \Sigma^*$ then $\mathfrak{A}$ will be called a Σ-structure. In this paper Σ will be kept fixed unless otherwise specified; we assume that every $\Phi \in \Sigma$ is given in prenex normal form.

There are four basic results which will be frequently used.

THEOREM A (THE COMPACTNESS THEOREM). *If every finite subset of a first order axiom system Σ has a model, then Σ has a model.*

THEOREM B (INVERSE LIMIT THEOREM). *The inverse limit of finite nonvoid sets is never void.*

Let $f \in \prod (A_i \mid i \in I)$ and $\mathscr{D}$ a dual prime ideal of the Boolean algebra $\mathfrak{P}(I)$ of all subsets of I. $\prod_{\mathscr{D}} (\mathfrak{A}_i \mid i \in I)$ will denote the prime product and $f^{\vee}$ the equivalence class containing f.

THEOREM C. *Let $\Phi(x_0, \ldots, x_{n-1})$ be a formula free at most in $x_0, \ldots, x_{n-1}$. Then $\Phi(f_0^{\vee}, \ldots, f_{n-1}^{\vee})$ if and only if $\{i \mid \Phi(f_0(i), \ldots, f_{n-1}(i))\} \in \mathscr{D}$.*

THEOREM D. *Let $\mathfrak{A}$ be a relational system with constants and Σ a universal axiom system which contains the relations of $\mathfrak{A}$ and an additional relation R. If R can be defined on every finite subset of A so as to satisfy Σ, then R can be defined on A so as to satisfy Σ.*

REFERENCES. [3] for Theorems A, C, and D, [2] for Theorem B.

1. **Inverses and Σ-subalgebras.** If Σ is universal then every substructure of a Σ-structure satisfies Σ, so there is no problem; Σ-substructure should mean substructure. As an example of an axiom system which is not universal, let us take the axiom system Σ_1 of lattices as partially ordered sets:

$$\Phi_1: (x)(y)(z)(x \leqq x \wedge ((x \leqq y \wedge y \leqq x) \rightarrow x = y) \wedge ((x \leqq y \wedge y \leqq z) \rightarrow x \leqq z)).$$

$$\Phi_2: (x)(y)(\exists z)(u)(x \leqq z \wedge y \leqq z \wedge ((x \leqq u \wedge y \leqq u) \rightarrow z \leqq u)).$$

$$\Phi_3: (x)(y)(\exists z)(u)(z \leqq x \wedge z \leqq y \wedge ((u \leqq x \wedge u \leqq y) \rightarrow u \leqq z)).$$

Then a sublattice of $\langle L; \leqq \rangle$ is not any sub-partially ordered set $\langle H; \leqq \rangle$ satisfying Σ_1, but it has the additional property that if $x, y \in H$, then the z_1 required by Φ_2 and the z_2 required by Φ_3 are also in H.

In this example x and y uniquely determine z in Φ_2 and Φ_3. What should we do if this is not the case? Let us consider the axiom system Σ_2 of complemented lattices with 0 and 1:

$$\begin{aligned}\Phi_1: (x)(y)(z)(x \vee x = x \wedge x \wedge x = x \wedge x \vee y = y \vee x \wedge x \wedge y \\ = y \wedge x \wedge x \vee (y \vee z) = (x \vee y) \vee z \wedge x \wedge (y \wedge z) \\ = (x \wedge y) \wedge z \wedge x \vee (x \wedge y) = x \wedge x \wedge (x \vee y) = x).\end{aligned}$$

$$\Phi_2: (x)(\exists y)(x \vee y = \mathbf{1} \wedge x \wedge y = \mathbf{0}).$$

Set $L=\{0, 1, p_0, p_1, p_2\}$, $p_i \vee p_j=1$ and $p_i \wedge p_j=0$ if $i\neq j$. If $\langle H; \vee, \wedge, 0, 1\rangle$ is a substructure of $\mathfrak{L}$ satisfying Σ_2 and $p_0 \in H$, then, by Φ_2, p_1 or $p_2 \in H$. Thus $\langle\{p_0, p_1, 0, 1\}; \vee, \wedge, 0, 1\rangle$, $\langle\{p_0, p_2, 0, 1\}; \vee, \wedge, 0, 1\rangle$ and $\mathfrak{L}$ are the substructures of $\mathfrak{L}$ satisfying Σ_2 which contain p_0.

Now any good substructure concept should have the property that every subset B is contained in a smallest substructure of that kind. By symmetry, in the above example, the substructure generated by $\{p_0\}$ has to be $\mathfrak{L}$. That is, along with x, the substructure has to contain all y required by Φ_2. This leads us to the concept of Σ-substructure: $\mathfrak{B}$ is a Σ-substructure of $\mathfrak{A}$, if $\mathfrak{B}$ is a substructure, and if $x_1, \ldots, x_n \in B$ and there is a $\Phi \in \Sigma$ with $(x_1)\cdots(x_n)(\exists y)\cdots$, then all $y \in A$ satisfying Φ are in B. To illustrate this, take the following two axioms:

(1) $(\boldsymbol{x})(\exists \boldsymbol{y})\Psi_1(\boldsymbol{x}, \boldsymbol{y})$,

(2) $(\boldsymbol{x})(\exists \boldsymbol{y})(\boldsymbol{u})(\exists \boldsymbol{v})\Psi_2(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{u}, \boldsymbol{v})$,

where Ψ_1 and Ψ_2 do not contain quantifiers.

For (1) this definition means that if $a \in B$, $b \in A$ and $\Psi_1(a, b)$ then $b \in B$. For (2) this means that if $a \in B$, $b \in A$ and

(3) $(\boldsymbol{u})(\exists \boldsymbol{v})\Psi_2(a, b, \boldsymbol{u}, \boldsymbol{v})$ in $\mathfrak{A}$([2]),

then $b \in B$, and furthermore if for $a, c \in B$ and $d \in A$ there exists a $b \in A$ such that (3) holds and $\Psi_2(a, b, c, d)$, then $d \in B$.

A Φ sequence for (1) means a, b with $\Psi_1(a, b)$; a Φ sequence for (2) will be a, b, c, d satisfying (3) and $\Psi_2(a, b, c, d)$.

To give a rigorous definition of Σ-substructures first we must define the concept of inverse.

Definition 1. Let $\Phi \in \Sigma$ be of the following form:

$$(4)\quad \begin{aligned}&(\boldsymbol{x}_0)\cdots(\boldsymbol{x}_{n_0-1})(\exists \boldsymbol{y}_0)(\boldsymbol{x}_{n_0})\cdots(\boldsymbol{x}_{n_1-1})(\exists \boldsymbol{y}_1)(\boldsymbol{x}_{n_1})\cdots(\exists \boldsymbol{y}_k)(\boldsymbol{x}_{n_k})\cdots\\ &\qquad(\boldsymbol{x}_{n-1})\Psi(\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{n_0-1}, \boldsymbol{y}_0, \boldsymbol{x}_{n_0}, \ldots, \boldsymbol{x}_{n_1-1}, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_k, \boldsymbol{x}_{n_k}, \ldots, \boldsymbol{x}_{n-1}),\end{aligned}$$

where $0\leqq n_0 \leqq n_1 \leqq \cdots \leqq n_k \leqq n$; $0=n_0$ means that no universal quantifier precedes $\exists \boldsymbol{y}_0$, $n_0=n_1$ means that there is no universal quantifier between $\exists \boldsymbol{y}_0$ and $\exists \boldsymbol{y}_1$, and so on, $n_k=n$ means that no universal quantifier follows $\exists \boldsymbol{y}_k$; Ψ contains no quantifiers. Set $e(\Phi)=k+1$. The concepts of $\Phi-l$ *inverse* and $\Phi-l$ *sequence* are defined for all $0\leqq l<e(\Phi)$ by induction on l. Let $\mathfrak{A}$ be a Σ-structure, $a_0, a_1, \ldots$ and $b_0, b_1, \ldots \in A$.

(i) b_0 is a $\Phi-0$ inverse of $a_0, \ldots, a_{n_0-1}$ in $\mathfrak{A}$ if

$$\begin{aligned}&(\boldsymbol{x}_{n_0})\cdots(\boldsymbol{x}_{n_1-1})(\exists \boldsymbol{y}_1)\cdots(\exists \boldsymbol{y}_k)(\boldsymbol{x}_{n_k})\cdots(\boldsymbol{x}_{n-1})\\ &\qquad\Psi(a_0, \ldots, a_{n_0-1}, b_0, \boldsymbol{x}_{n_0}, \ldots, \boldsymbol{x}_{n_1-1}, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_k, \boldsymbol{x}_{n_k}, \ldots, \boldsymbol{x}_{n-1})\end{aligned}$$

holds in $\mathfrak{A}$; in this case, $a_0, \ldots, a_{n_0-1}, b_0$ is a $\Phi-0$ sequence;

([2]) Note that (3) is not a formula in our language, since a, b are elements of A; (3), and all such formulas similarly, should read: $\langle a, b\rangle$ satisfies $(\boldsymbol{u})(\exists \boldsymbol{v})\Psi(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{u}, \boldsymbol{v})$ in $\mathfrak{A}$.

(ii) b_l is a $\Phi-l$ inverse of $a_0, \ldots, a_{n_l-1}$ in $\mathfrak{A}$ if there exists a $\Phi-(l-1)$ sequence $a_0, \ldots, a_{n_0-1}, b_0, \ldots, a_{n_l-1}, b_{l-1}$ such that

$$(x_{n_l})\cdots(x_{n_{l+1}-1})(\exists y_{l+1})\cdots(\exists y_k)(x_{n_k})\cdots(x_{n-1})\Psi(a_0, \ldots, a_{n_0-1}, b_0, \ldots, a_{n_{l-1}-1}, b_{l-1}, a_{n_{l-1}}, \ldots, a_{n_l-1}, b_l, x_{n_l}, \ldots, x_{n_{l+1}-1}, y_{l+1}, \ldots, y_k, x_{n_k}, \ldots, x_{n-1})$$

holds in $\mathfrak{A}$. Then, $a_0, \ldots, a_{n_0-1}, b_0, \ldots, a_{n_{l-1}-1}, b_{l-1}, a_{n_{l-1}}, \ldots, a_{n_l-1}, b_l$ is a $\Phi-l$ sequence.

Φ-*inverse* will mean $\Phi-l$ inverse for some $l<e(\Phi)$ and Σ-*inverse* will mean Φ-inverse for some $\Phi\in\Sigma$. A *complete* Φ *sequence* is a $\Phi-(e(\Phi)-1)$ sequence.

REMARK. Note that $e(\Phi)$ is the number of existential quantifiers in the prefix of Φ. Intuitively, a $\Phi-l$ inverse is an element whose existence is guaranteed by $(\exists y_l)$, the lth existential quantifier.

Most proofs of statements on inverses can be carried out only by induction on l as in Definition 1, which is sometimes technically involved. Therefore, with the exceptions of Lemma 1 and Theorem 1 we will work out the proofs only for axioms of the forms (1) and (2) and leave the details of a formal proof to the reader.

The following lemma shows that the two concepts introduced in Definition 1 can be expressed by first order relations.

LEMMA 1. *For every $\Phi\in\Sigma$ and $l<e(\Phi)$ there exists a formula $\Phi^{[l]}(x_0, \ldots, y)$ in $L(\tau)$ free in $x_0, \ldots, y$ such that for a Σ-structure $\mathfrak{A}$ and $a_0, \ldots, b\in A$, b is a $\Phi-l$ inverse of $a_0, \ldots$ if and only if $\Phi^{[l]}(a_0, \ldots, b)$. Furthermore, there exists a formula $\Phi^{(l)}(x_0, \ldots, x_{n_0-1}, y_0, \ldots, x_{n_l-1}, y_l)$ in $L(\tau)$ free in $x_0, \ldots, y_0, \ldots, y_l$ such that if $\mathfrak{A}$ is a Σ-structure and $a_0, \ldots, a_{n_0-1}, b_0, \ldots, a_{n_l-1}, b_l\in A$, then $a_0, \ldots, a_{n_0-1}, b_0, \ldots, a_{n_l-1}, b_l$ is a $\Phi-l$ sequence in $\mathfrak{A}$ if and only if $\Phi^{(l)}(a_0, \ldots, a_{n_0-1}, b_0, \ldots, a_{n_l-1}, b_l)$ in $\mathfrak{A}$.*

Proof. If Φ is of the form (1), then $\Phi^{[0]}(x, y)\equiv\Psi_1(x, y)\equiv\Phi^{(0)}(x, y)$. If Φ is of the form (2), then $\Phi^{[0]}(x, y)\equiv(u)(\exists v)\Psi_2(x, y, u, v)$ and

$$\Phi^{[1]}(x_0, x_1, y) \equiv (\exists z)(\Phi^{[0]}(x_0, z) \wedge \Psi_2(x_0, z, x_1, y)).$$

Furthermore, $\Phi^{(0)}(x, y)=\Phi^{[0]}(x, y)$ and

$$\Phi^{(1)}(x_0, y_0, x_1, y_1) = \Phi^{(0)}(x_0, y_0) \wedge \Psi_2(x_0, y_0, x_1, y_1).$$

The general proof for Φ, as it is given in Definition 1, proceeds by induction on l. For $l=0$,

$$\Phi^{[0]}(x_0, \ldots, x_{n_0-1}, y_0) \equiv (x_{n_0})\cdots(x_{n_1-1})(\exists y_1)(x_{n_1})\cdots(\exists y_k)(x_{n_k})\cdots(x_{n-1})\Psi(x_0, \ldots, x_{n_0-1}, y_0, x_{n_0}, \ldots, y_k, x_{n_k}, \ldots, x_{n-1}),$$

and

$$\Phi^{(0)} \equiv \Phi^{[0]}.$$

Suppose that we have already constructed $\Phi^{(l-1)}$ and $\Phi^{[l-1]}$. Then

$$\Phi^{(l)}(x_0, \ldots, x_{n_0-1}, y_0, \ldots, x_{n_l-1}, y_l) \equiv \Phi^{(l-1)}(x_0, \ldots, x_{n_0-1}, y_0, \ldots, x_{n_{l-1}-1}, y_{l-1}) \wedge (x_{n_l})\cdots(x_{n_{l+1}-1})(\exists y_{l+1})\cdots(x_n)\Psi(x_0, \ldots, y_l, x_{n_l}, \ldots, x_n)$$

and

$$\Phi^{[l]}(x_0, \ldots, x_{n_l-1}, y_l) \equiv (\exists y_0)\cdots(\exists y_{l-1})\Phi^{(l)}(x_0, \ldots, y_0, x_{n_1}, \ldots, x_{n_l-1}, y_l).$$

REMARK. We constructed the $\Phi^{[l]}$ and $\Phi^{(l)}$ from Φ. It is easy to see that Φ can be constructed from $\Phi^{[l]}$ and $\Phi^{(l)}$.

DEFINITION 2. Let $\mathfrak{A}$ be a Σ-structure and let $\mathfrak{B}$ be a substructure of $\mathfrak{A}$. Then $\mathfrak{B}$ is a Σ-*substructure* of $\mathfrak{A}$ if whenever $a_0, \ldots, a_t \in B$, $b \in A$ and b is a Σ-inverse of $a_0, \ldots, a_t$ in $\mathfrak{A}$ then $b \in B$.

The most important property of Σ-substructures is the following:

THEOREM 1. *Let $\mathfrak{A}$ be a Σ-structure and let $\mathfrak{B}$ be a Σ-substructure of $\mathfrak{A}$; let $\Phi \in \Sigma$ and $l < e(\Phi)$. If $a_0, \ldots, b_0, \ldots \in B$ and $\Phi^{(l)}(a_0, \ldots, b_0, \ldots, a_{n_l-1}, b_l)$ in $\mathfrak{A}$, then it also holds in $\mathfrak{B}$.*

Proof. Illustration: Let Φ be of the form (2) and $\Phi^{(0)}(a_0, b_0)$ in $\mathfrak{A}$; that is, $(u)(\exists v)\Psi_2(a_0, b_0, u, v)$ in $\mathfrak{A}$. If $c \in B$ there exists a $d \in A$ with $\Psi_2(a_0, b_0, c, d)$. Now $\Phi^{(0)}(a_0, b_0)$ implies d is a $\Phi - 1$ inverse of a and c, whence $d \in B$. This proves that $\Phi^{(0)}(a_0, b_0)$ in $\mathfrak{B}$. $\Phi^{(1)}(a_0, b_0, a_1, b_1)$ can be handled similarly.

To prove Theorem 1 we first note that every Φ sequence can be extended to a complete Φ sequence, hence it is enough to prove Theorem 1 for complete Φ sequences.

For $e(\Phi) = 0$ Theorem 1 follows from the known (and obvious) theorem that if a universal sentence holds for the structure $\mathfrak{A}$, it holds for the substructure $\mathfrak{B}$ of $\mathfrak{A}$.

Assume that Theorem 1 is proved for all sentences with less than $k+1$ existential quantifiers, let Φ be given as in (4), let $a_0, \ldots, b_0, a_{n_0}, \ldots, b_k$ be a complete Φ sequence in $\mathfrak{A}$ with $a_0, \ldots, b_0, \ldots, b_k \in B$ and let $\tau' = \tau \oplus (n_0 + 1)$ be the type we get from τ by adding the constants $p_0, \ldots, p_{n_0-1}, q$. Let $\mathfrak{A}'$ and $\mathfrak{B}'$ be structures of type τ' which we get from $\mathfrak{A}$ and $\mathfrak{B}$ by interpreting p_i as a_i and q as b_0. Finally, let

$$\Phi' \equiv (x_{n_0})\cdots(x_{n_1-1})(\exists y_1)(x_{n_1})\cdots(\exists y_k)\cdots (x_{n-1})\Psi(p_0, \ldots, p_{n_0-1}, q, x_{n_0}, \ldots, y_k, \ldots, x_{n-1}).$$

Then $e(\Phi') = e(\Phi) - 1$; $a_{n_0}, \ldots, b_k$ is a complete Φ' sequence in $\mathfrak{A}'$ (this follows from the definition of satisfaction), hence by the induction hypothesis, it is a complete Φ' sequence in $\mathfrak{B}'$, which, in turn, implies that $a_0, \ldots, b_0, a_{n_0}, \ldots, b_k$ is a complete Φ sequence in $\mathfrak{B}$. This completes the proof of Theorem 1.

COROLLARY 1. *Under the same conditions as in Theorem 1, if $\Phi^{[l]}(a_0, \ldots, a_{n_l-1}, b)$ in $\mathfrak{A}$ then it also holds in $\mathfrak{B}$.*

Thus in a Σ-substructure $\Phi - 0$ inverses exist, and so Σ is satisfied.

COROLLARY 2. *A Σ-substructure of a Σ-structure is again a Σ-structure.*

It should be emphasized that the converse to Theorem 1 does not hold. A trivial example is the following: let $\langle B; \cdot \rangle$ be a two-element commutative idempotent semigroup, $B=\{0, 1\}$, and $A=\{0, 1, a\}$, $1\cdot a=a\cdot 1=a$, $0\cdot a=a$, $a\cdot 0=1$, $a\cdot a=a$. Let $\Phi\equiv(\exists y_0)(x_0)(y_0x_0=x_0y_0)$. Then there is only one $\Phi-0$ inverse in $\mathfrak{A}$, namely 1, hence $\mathfrak{B}$ is a Φ-substructure. But there are two $\Phi-0$ inverses in $\mathfrak{B}$ (0 and 1).

A few elementary properties of Σ-substructures follow.

LEMMA 2. *Let $\mathfrak{A}$ be a Σ-structure and let $\mathfrak{B}$ be a Σ-substructure of $\mathfrak{A}$. Let $\mathfrak{C}$ be a Σ-substructure of $\mathfrak{B}$. Then $\mathfrak{C}$ is a Σ-substructure of $\mathfrak{A}$.*

REMARK. By Corollary 2 to Theorem 1, $\mathfrak{B}$ is a Σ-structure.

Proof. Let $a_0, \ldots, a_t \in C$ and $b \in A$ and let b be a Σ-inverse of $a_0, \ldots, a_t$. Then $b \in B$, since $\mathfrak{B}$ is a Σ-substructure. By Corollary 1 to Theorem 1, b is a Σ-inverse of $a_0, \ldots, a_t$ in $\mathfrak{B}$ whence $b \in C$ since $\mathfrak{C}$ is a Σ-substructure of $\mathfrak{B}$.

LEMMA 3. *Let $\mathfrak{A}$ be a Σ-structure and $\varnothing \neq H \subseteq A$. Then there exists a smallest Σ-substructure $\mathfrak{B}$ with $H \subseteq B$.*

Proof. Obvious, since the intersection of Σ-substructures is again a Σ-substructure, provided it is not void.

We will set $B=[H]_\Sigma$ and we will say that H *Σ-generates* $\mathfrak{B}$ or H is a *Σ-generating set* of $\mathfrak{B}$.

LEMMA 4. *Let $\mathfrak{A}$ be a Σ-structure, $\varnothing \neq H \subseteq A$. Set $H_0=H$, $\overline{H}_{n-1}=\{a \mid a \in A$ and $a=p(a_0, \ldots, a_{k-1})$, where p is a polynomial and $a_0, \ldots, a_{k-1} \in H_{n-1}\}$, $H_n=\overline{H}_{n-1} \cup \{a \mid a \in A$, and there exist $b_0, \ldots, b_t \in H_{n-1}$ such that a is a Σ-inverse of $b_0, \ldots b_t$ in $\mathfrak{A}\}$. Then*

$$[H]_\Sigma = \bigcup (H_i \mid 0 \leqq i < \omega).$$

Proof. $H_n \subseteq [H]_\Sigma$ can be proved by induction on n, so we get

$$\bigcup (H_i \mid 0 \leqq i < \omega) \subseteq [H]_\Sigma.$$

It is routine to check that $\langle \bigcup (H_i \mid 0 \leqq i < \omega); F, R\rangle$ is a Σ-substructure, so we get equality.

A useful criterion for $a \in [H]_\Sigma$ can be given in terms of Σ-polynomials. Σ-polynomial symbols are a new type of expression defined as follows:

DEFINITION 3. Let n be a positive integer. The set $\boldsymbol{P}_n(\Sigma)$ of *n-ary Σ-polynomial symbols* is defined by rules (i)–(iv) below.

(i) $\boldsymbol{x}_i \in \boldsymbol{P}_n(\Sigma)$, $i=0, \ldots, n-1$;

(ii) if $\boldsymbol{P}_0, \ldots, \boldsymbol{P}_{n_\gamma-1} \in \boldsymbol{P}_n(\Sigma)$, then $\boldsymbol{f}_\gamma(\boldsymbol{P}_0, \ldots, \boldsymbol{P}_{n_\gamma-1}) \in \boldsymbol{P}_n(\Sigma)$;

(iii) if $\Phi \in \Sigma$, $l < e(\Phi)$, n_l universal quantifiers precede the $\exists y_l$ and $\boldsymbol{P}_0, \ldots, \boldsymbol{P}_{n_l-1} \in \boldsymbol{P}_n(\Sigma)$ then $\Phi^{(l)}(\boldsymbol{P}_0, \ldots, \boldsymbol{P}_{n_l-1}) \in \boldsymbol{P}_n(\Sigma)$;

(iv) $\boldsymbol{P}_n(\Sigma)$ is the smallest set satisfying (i)–(iii).

The semantical interpretation of Σ-polynomials as multi-valued functions called *Σ-polynomials* is given in the following definition.

DEFINITION 4. Let $\boldsymbol{P} \in \boldsymbol{P}_n(\Sigma)$, let $\mathfrak{A}$ be a Σ-structure, and let $a_0, \ldots, a_{n-1} \in A$. Then $(\boldsymbol{P})_{\mathfrak{A}}(a_0, \ldots, a_{n-1})$, or simply $P(a_0, \ldots, a_{n-1})$, is a subset of $\mathfrak{A}$ defined as follows:

(i) if $\boldsymbol{P} = \boldsymbol{x}_i$, then $\boldsymbol{P}(a_0, \ldots, a_{n-1}) = \{a_i\}$;

(ii) if $\boldsymbol{P} = \boldsymbol{f}_\gamma(\boldsymbol{P}_0, \ldots, \boldsymbol{P}_{n_\gamma - 1})$, then $P(a_0, \ldots, a_{n-1}) = \{a \mid a = f_\gamma(b_0, \ldots, b_{n_\gamma - 1})$ for some $b_i \in P_i(a_0, \ldots, a_{n-1}),\ i = 0, \ldots, n_\gamma - 1\}$;

(iii) if $\boldsymbol{P} = \Phi^{(l)}(\boldsymbol{P}_0, \ldots, \boldsymbol{P}_{n_l - 1})$ then $P(a_0, \ldots, a_{n-1}) = \{a \mid a$ is a $\Phi - l$ inverse of some $b_0, \ldots, b_{n_l - 1}$ with $b_i \in P_i(a_0, \ldots, a_{n-1}),\ i = 0, \ldots, n_l - 1\}$.

LEMMA 5. *Let $\mathfrak{A}$ be a Σ-structure, $\varnothing \neq H \subseteq A$. Then $a \in [H]_\Sigma$ if and only if for some positive integer n, $\boldsymbol{P} \in \boldsymbol{P}_n(\Sigma)$, and $h_0, \ldots, h_{n-1} \in H$, we have $a \in P(h_0, \ldots, h_{n-1})$.*

Proof. If $a \in [H]_\Sigma$ then, by Lemma 4, $a \in H_i$ for some $i < \omega$ and then the proof of $a \in P(h_0, \ldots, h_{n-1})$ proceeds by an easy induction on i. Conversely, if $a \in P(h_0, \ldots, h_{n-1})$, then we can prove that $a \in H_i$ for some i, by induction on the "rank" of $\boldsymbol{P}$.

COROLLARY. *Let $\mathfrak{A}$ be a Σ-structure and let $\mathfrak{B}$ be a Σ-substructure of $\mathfrak{A}$. Let $\boldsymbol{P} \in \boldsymbol{P}_n(\Sigma)$, $b_0, \ldots, b_{n-1} \in B$. Then $P(b_0, \ldots, b_{n-1}) \subseteq B$.*

The following two lemmas will be used frequently.

LEMMA 6. *Let $\boldsymbol{P} \in \boldsymbol{P}_n(\Sigma)$. Then there exists a formula $r_{\boldsymbol{P}}(\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{n-1}, \boldsymbol{y})$ in $L(\tau)$ such that if $\mathfrak{A}$ is a Σ-structure and $a_0, \ldots, a_{n-1}$, $b \in A$, then $b \in P(a_0, \ldots, a_{n-1})$ if and only if $r_{\boldsymbol{P}}(a_0, \ldots, a_{n-1}, b)$.*

LEMMA 7. *Let $\mathfrak{A}$ be a Σ-structure, let $\mathfrak{B}$ be a Σ-substructure of $\mathfrak{A}$ and let $\boldsymbol{P} \in \boldsymbol{P}_n(\Sigma)$. If $a_0, \ldots, a_{n-1}$, $b \in B$ and $b \in (\boldsymbol{P})_{\mathfrak{A}}(a_0, \ldots, a_{n-1})$ then $b \in (\boldsymbol{P})_{\mathfrak{B}}(a_0, \ldots, a_{n-1})$. In other words, if $r_{\boldsymbol{P}}(a_0, \ldots, a_{n-1}, b)$ in $\mathfrak{A}$, then $r_{\boldsymbol{P}}(a_0, \ldots, a_{n-1}, b)$ in $\mathfrak{B}$.*

Lemmas 6 and 7 follow from Lemma 1 by an easy induction.

Lemma 7 states that $P_{\mathfrak{A}} \subseteq P_{\mathfrak{B}}$. The example given following Theorem 1 shows that $P_{\mathfrak{A}} \neq P_{\mathfrak{B}}$ in general. However, if all sentences in Σ are either universal, or of type $\forall\exists$ (that is, no universal quantifier follows an existential quantifier) then $P_{\mathfrak{A}} = P_{\mathfrak{B}}$ always.

2. **Σ-homomorphisms and slender Σ-subalgebras.** The example of lattices as partially ordered sets (see §1) shows that the usual concept of homomorphism may not preserve algebraic properties, e.g., the homomorphic image of a lattice may not be a lattice or the homomorphic image of a distributive lattice may be nondistributive. Therefore, we need a homomorphism concept which preserves the inverses.

DEFINITION 1. Let $\mathfrak{A}$ and $\mathfrak{B}$ be Σ-structures and let φ be a mapping of A into B. Then φ is called a *Σ-homomorphism* if φ is a homomorphism and if for any positive integer n, $\boldsymbol{P} \in \boldsymbol{P}_n(\Sigma)$, and $a_0, \ldots, a_{n-1} \in A$ we have

$$P(a_0, \ldots, a_{n-1})\varphi = P(a_0\varphi, \ldots, a_{n-1}\varphi).$$

It should be emphasized that Σ-isomorphism *is the same* as isomorphism. Also, if we deal with algebras only, then in Definition 1 the clause "φ is a homomorphism" can be omitted.

Of course, one can give an equivalent definition without the use of Σ-polynomials.

LEMMA 1. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be Σ-structures and let φ be a mapping of A into B. Then φ is a Σ-homomorphism if and only if the following conditions are satisfied*:

(i) *φ is a homomorphism*;

(ii) *if $\Phi \in \Sigma$, $l < e(\Phi)$, $b, a_0, \ldots, a_t \in A$ and b is a $\Phi - l$ inverse of $a_0, \ldots, a_t$ in $\mathfrak{A}$, then $b\varphi$ is a $\Phi - l$ inverse of $a_0\varphi, \ldots, a_t\varphi$ in $\mathfrak{B}$*;

(iii) *if $\Phi \in \Sigma$, $l < e(\Phi)$, $a_0, \ldots, a_t \in A$, $\bar{b} \in B$ and $\bar{b}$ is a $\Phi - l$ inverse of $a_0\varphi, \ldots, a_t\varphi$ in $\mathfrak{B}$, then there exists a $b \in A$ such that b is a $\Phi - l$ inverse of $a_0, \ldots, a_t$ and $b\varphi = \bar{b}$.*

Proof. Let φ be a Σ-homomorphism. Then (i) is satisfied by definition. (ii) and (iii) follow easily by taking $\boldsymbol{P} = \Phi^{(l)}(\boldsymbol{x}_0, \ldots, \boldsymbol{x}_t)$ and applying the definition of Σ-homomorphism. Conversely, if (i)–(iii) are satisfied then we prove $P(a_0, \ldots, a_{n-1})\varphi = P(a_0\varphi, \ldots, a_{n-1}\varphi)$ by induction. If $\boldsymbol{P} = \boldsymbol{x}_i$, the statement is trivial. If $\boldsymbol{P} = \boldsymbol{f}_\gamma(\boldsymbol{P}_0, \ldots, \boldsymbol{P}_{n_\gamma - 1})$, then it follows from (i). If $\boldsymbol{P} = \Phi^{(l)}(\boldsymbol{P}_0, \ldots, \boldsymbol{P}_t)$ it follows from (ii) and (iii).

Some important properties of Σ-homomorphisms are given in the following lemmas.

LEMMA 2. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be Σ-structures and let φ be a Σ-homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$; set $C = A\varphi$. Then $\mathfrak{C}$ is a Σ-substructure of $\mathfrak{B}$.*

LEMMA 3. *Let $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{C}$ be Σ-structures, let φ be a Σ-homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$, and let ψ be a Σ-homomorphism of $\mathfrak{B}$ into $\mathfrak{C}$. Then $\varphi\psi$ is a Σ-homomorphism of $\mathfrak{A}$ into $\mathfrak{C}$.*

A property of homomorphisms (which is very important in proofs concerning free algebras) fails to hold for Σ-homomorphisms. Namely, if φ is a Σ-homomorphism of $\mathfrak{A}$ into $\mathfrak{C}$ and $\mathfrak{B}$ is a Σ-substructure of $\mathfrak{A}$ then φ_B (the restriction of φ to B) is not necessarily a Σ-homomorphism of $\mathfrak{B}$ into $\mathfrak{C}$. Let $b_0, \ldots, b_{n-1} \in B$, $\boldsymbol{P} \in \boldsymbol{P}_n(\Sigma)$; it follows from the corollary to Lemma 5 in §1 that $\boldsymbol{P}_{\mathfrak{A}}(b_0, \ldots, b_{n-1}) \subseteq B$, and from Corollary 1 to Theorem 1 that $\boldsymbol{P}_{\mathfrak{A}}(b_0, \ldots, b_{n-1}) \subseteq \boldsymbol{P}_{\mathfrak{B}}(b_0, \ldots, b_{n-1})$. Whenever $\boldsymbol{P}_{\mathfrak{A}}(b_0, \ldots, b_{n-1}) \neq \boldsymbol{P}_{\mathfrak{B}}(b_0, \ldots, b_{n-1})$, we find that φ_B is not a Σ-homomorphism. This leads us to the definition of slender Σ-substructures.

DEFINITION 2. Let $\mathfrak{B}$ be a Σ-substructure of the Σ-structure $\mathfrak{A}$. Then $\mathfrak{B}$ is called a *slender Σ-substructure* if for any positive integer n, $\boldsymbol{P} \in \boldsymbol{P}_n(\Sigma)$ and $a_0, \ldots, a_{n-1} \in B$ we have that $\boldsymbol{P}_{\mathfrak{A}}(a_0, \ldots, a_{n-1}) = \boldsymbol{P}_{\mathfrak{B}}(a_0, \ldots, a_{n-1})$.

LEMMA 4. *Let $\mathfrak{B}$ be a Σ-substructure of the Σ-structure $\mathfrak{A}$. Then $\mathfrak{B}$ is slender if and only if for $\Phi \in \Sigma$, $l < e(\Phi)$ and $b, a_0, \ldots, a_t \in B$ we have that b is a $\Phi - l$ inverse of $a_0, \ldots, a_t$ in $\mathfrak{B}$ implies that b is a $\Phi - l$ inverse of $a_0, \ldots, a_t$ in $\mathfrak{A}$.*

The proof is again a simple induction based on Definition 3 of §1.

LEMMA 5. *Let $\mathfrak{B}$ be a Σ-substructure of the Σ-structure $\mathfrak{A}$. The following conditions on $\mathfrak{B}$ are equivalent*;

(i) *$\mathfrak{B}$ is slender*;

(ii) *if $\mathfrak{C}$ is any Σ-structure and φ is a Σ-homomorphism of $\mathfrak{A}$ into $\mathfrak{C}$, then φ_B is a Σ-homomorphism of $\mathfrak{B}$ into $\mathfrak{C}$*;

(iii) *if $\mathfrak{C}$ is any Σ-structure and φ is a Σ-homomorphism of $\mathfrak{C}$ into $\mathfrak{B}$, then φ is a Σ-homomorphism of $\mathfrak{C}$ into $\mathfrak{A}$*;

(iv) *if $\mathfrak{C}$ is any Σ-structure and φ is a Σ-homomorphism of $\mathfrak{C}$ onto $\mathfrak{B}$, then φ is a Σ-homomorphism of $\mathfrak{C}$ into $\mathfrak{A}$.*

(v) *$\varphi: x \to x$ is a Σ-homomorphism of $\mathfrak{B}$ into $\mathfrak{A}$.*

Proof. The following implications are obvious: (i) implies (ii), (iii), (iv), and (v); (iii) implies (iv); (iv) implies (v) ($\mathfrak{B}=\mathfrak{C}$); (ii) implies (v) ($\mathfrak{A}=\mathfrak{C}$). Thus it suffices to prove that (v) implies (i); indeed (v) implies that $\boldsymbol{P}_{\mathfrak{B}}(b_0, \ldots, b_{n-1})\varphi=\boldsymbol{P}_{\mathfrak{A}}(b_0, \ldots, b_{n-1})$ $(b_0, \ldots, b_{n-1} \in B)$, that is, $\mathfrak{B}$ is slender.

LEMMA 6. *Let $\mathfrak{B}$ be a slender Σ-substructure of the Σ-structure $\mathfrak{A}$. Then the following conditions hold*:

(i) *let $\mathfrak{C}$ be a Σ-structure and let φ be a Σ-homomorphism of $\mathfrak{C}$ into $\mathfrak{A}$ with $C\varphi \subseteq B$; then φ is a Σ-homomorphism of $\mathfrak{C}$ into $\mathfrak{B}$*;

(ii) *let $\mathfrak{C}$ be a Σ-substructure of $\mathfrak{A}$ with $C \subseteq B$; then $\mathfrak{C}$ is a Σ-substructure of $\mathfrak{B}$*;

(iii) *let $H \subseteq B$; then $[H]_\Sigma$ in $\mathfrak{A}$ equals $[H]_\Sigma$ in $\mathfrak{B}$.*

The proofs are trivial.

3. **Free Σ-structures and the Uniqueness Theorem.** Now we are ready to define free Σ-structures.

DEFINITION 1. Let α be an ordinal. $\mathfrak{F}_\Sigma(\alpha)$ is the *free Σ-structure* with α Σ-generators, if the following conditions are satisfied:

(i) $\mathfrak{F}_\Sigma(\alpha)$ is a Σ-structure;

(ii) $\mathfrak{F}_\Sigma(\alpha)$ is Σ-generated by the elements $x_0, \ldots, x_\gamma, \ldots, \gamma<\alpha$;

(iii) if $\mathfrak{A}$ is a Σ-structure and $a_0, \ldots, a_\gamma, \ldots \in A$ for $\gamma<\alpha$, then the mapping $\varphi: x_\gamma \to a_\gamma$, $\gamma<\alpha$ can be extended to a Σ-homomorphism, $\bar{\varphi}$.

REMARK. The Σ-homomorphism $\bar{\varphi}$ in (iii) need not be unique. Indeed, let $\tau=0$ and let Σ consist of the following two axioms:

$$(x)(y)(z)(u)(x = y \vee x = z \vee x = u \vee y = z \vee y = u \vee z = u)$$
$$(x)(\exists y)(\exists z)(x \neq y \wedge y \neq z \wedge x \neq z).$$

Then a Σ-structure is a 3 element set. Let $A=\{a_0, a_1, a_2\}$, $B=\{b_0, b_1, b_2\}$. Then $\mathfrak{A}=\mathfrak{F}_\Sigma(1)$, e.g., a_0 is a free Σ-generator. The mapping $\varphi: a_0 \to b_0$ has two extensions to Σ-homomorphisms of $\mathfrak{A}$ onto $\mathfrak{B}$, namely, $a_0 \to b_0$, $a_1 \to b_1$, $a_2 \to b_2$ and $a_0 \to b_0$, $a_1 \to b_2$, $a_2 \to b_1$.

Most of the difficulties in the theory of free Σ-structures come from this fact.

The following example is a further illustration of the nonuniqueness of $\bar{\varphi}$.

In this example we deal with algebras of the form $\langle A; \vee, \wedge, 0, 1\rangle$, and Σ consists of the lattice axioms, postulates for 0 and 1 to be the zero and unit, and the following axiom:

$$\begin{aligned}(\mathbf{x})(\exists \mathbf{y})(\exists \mathbf{z})(\mathbf{u})((\mathbf{x} = \mathbf{0} \to \mathbf{y} = \mathbf{z} = \mathbf{1}) \wedge (\mathbf{x} = \mathbf{1} \to \mathbf{y} = \mathbf{z} = \mathbf{0})\\ \wedge ((\mathbf{x} \neq \mathbf{0} \wedge \mathbf{x} \neq \mathbf{1}) \to (\mathbf{y} \neq \mathbf{z} \wedge \mathbf{x} \vee \mathbf{y} = \mathbf{1} \wedge \mathbf{x} \wedge \mathbf{y} = \mathbf{0} \wedge \mathbf{x} \vee \mathbf{z} = \mathbf{1}\\ \wedge \mathbf{x} \wedge \mathbf{z} = \mathbf{0} \wedge ((\mathbf{x} \vee \mathbf{u} = \mathbf{1} \wedge \mathbf{x} \wedge \mathbf{u} = \mathbf{0}) \to \mathbf{u} = \mathbf{y} \vee \mathbf{u} = \mathbf{z})))).\end{aligned}$$

In words: every element $\neq 0$, 1 has exactly two complements. It was shown in a paper of C. C. Chen and the author (J. Algebra (1969)) that $\mathfrak{F}_\Sigma(\alpha)$ exists for all α. If we take $\mathfrak{F}_\Sigma(\omega)$ and map all x_i into any one atom of the five element modular nondistributive lattice, then this map has $2^{\aleph_0}$ extensions to Σ-homomorphisms (this is best possible since $|F_\Sigma(\omega)| = \aleph_0$).

The theory of free Σ-structures is based on the following result which, in a certain sense, is a substitute for the uniqueness of $\bar{\varphi}$.

THEOREM 1. *Let us assume that $\mathfrak{F}_\Sigma(n)$ exists. Then every $\mathbf{P} \in \mathbf{P}_m(\Sigma)$ with $m \leqq n$ is bounded, that is there exists a least positive integer $k_{\mathbf{P}}$ such that if $\mathfrak{A}$ is a Σ-structure, $a_0, \ldots, a_{m-1} \in A$, then*

$$|P(a_0, \ldots, a_{m-1})| \leqq k_{\mathbf{P}}.$$

Proof. Let us assume that Theorem 1 is not true. Then there exist $\mathbf{P} \in \mathbf{P}_m(\Sigma)$ with $m \leqq n$, Σ-structures $\mathfrak{A}_1, \mathfrak{A}_2, \ldots$, and $a_0^t, \ldots, a_{m-1}^t \in A_t$ $(t = 1, 2, \ldots)$ such that

$$|P(a_0^t, \ldots, a_{m-1}^t)| \geqq t \qquad (t = 1, 2, \ldots).$$

STATEMENT. Under these conditions, for every cardinal $\mathfrak{m}$, there exists a Σ-structure $\mathfrak{A}$ and there exist $a_0, \ldots, a_{m-1} \in A$ such that

$$|P(a_0, \ldots, a_{m-1})| \geqq \mathfrak{m}.$$

Proof. Let α be the initial ordinal of cardinality $\mathfrak{m}$ and $\tau' = \tau \oplus (\alpha + m)$; that is we get the type τ' by adjoining the constants $\mathbf{k}_{o(\tau)}, \ldots, \mathbf{k}_{o(\tau)+\gamma}, \ldots, \gamma < \alpha + m$ to τ. Set $\mathbf{l}_0 = \mathbf{k}_{o(\tau)+\alpha}, \ldots, \mathbf{l}_{m-1} = \mathbf{k}_{o(\tau)+\alpha+m-1}$, and let us write $\mathbf{k}_\gamma$ for $\mathbf{k}_{o(\tau)+\gamma}$, $\gamma < \alpha$.

Let H be a finite set of ordinals $< \alpha$; we define a sentence Φ_H of $L(\tau')$ as follows:

$$\Phi_H = \bigwedge (r_{\mathbf{P}}(\mathbf{l}_0, \ldots, \mathbf{l}_{m-1}, \mathbf{k}_\gamma) \mid \gamma \in H) \wedge \bigwedge (\mathbf{k}_\gamma \neq \mathbf{k}_\delta \mid \gamma, \delta \in H, \gamma \neq \delta),$$

where $r_{\mathbf{P}}$ is the formula in $L(\tau)$ which was defined in Lemma 6, §1.

Let Ω be the set of all Φ_H. We claim that there exists a structure $\mathfrak{A}'$ satisfying $\Sigma \cup \Omega$. By the compactness theorem (Theorem A), it suffices to show that $\Sigma \cup \Omega_1$ has a model for all finite $\Omega_1 \subseteq \Omega$. Let $\Omega_1 = \{\Phi_{H_0}, \ldots, \Phi_{H_{t-1}}\}$ and set $H = H_0 \cup \cdots \cup H_{t-1}$. Since Φ_H implies Φ_{H_i}, $i = 0, \ldots, t-1$, it is sufficient to show that $\Sigma \cup \{\Phi_H\}$ has a model. Let $H = \{\gamma_0, \ldots, \gamma_{s-1}\}$. Let $\mathfrak{A}_s'$ be the structure that we get from $\mathfrak{A}_s$ by interpreting $\mathbf{l}_i$ as a_i^s $(i = 0, \ldots, m-1)$ and $\mathbf{k}_{\gamma_0}, \ldots, \mathbf{k}_{\gamma_{s-1}}$ as distinct elements of $P(a_0^s, \ldots, a_{m-1}^s)$; we can do that since $|P(a_0^s, \ldots, a_{m-1}^s)| \geqq s$; let us interpret $\mathbf{k}_\gamma$, $\gamma \neq \gamma_0, \ldots, \gamma_{s-1}$ in an arbitrary manner. It is obvious then that $\mathfrak{A}_s'$ satisfies $\Sigma \cup \{\Phi_H\}$.

Now let $\mathfrak{A}'$ be a model of $\Sigma \cup \Omega$. Let $a_0, \ldots, a_{m-1}$ be the interpretations of $\boldsymbol{l}_0, \ldots, \boldsymbol{l}_{m-1}$ and let $b_0, \ldots, b_\gamma, \ldots,$ be the interpretations of $\boldsymbol{k}_0, \ldots, \boldsymbol{k}_\gamma, \ldots,$ for $\gamma < \alpha$. Then

$$b_0, \ldots, b_\gamma, \ldots \in P(a_0, \ldots, a_{m-1})$$

and $b_\gamma \neq b_\delta$ if $\gamma, \delta < \alpha$ and $\gamma \neq \delta$. Thus $|P(a_0, \ldots, a_{m-1})| \geqq \mathfrak{m}$. Therefore the τ-reduct $\mathfrak{A}$ of $\mathfrak{A}'$ satisfies the requirements, concluding the proof of the statement.

Let $x_0, \ldots, x_{n-1}$ be the Σ-generators of $\mathfrak{F}_\Sigma(n)$. Set $|P(x_0, \ldots, x_{m-1})| = \mathfrak{n}$. If $\mathfrak{A}$ is any Σ-structure and $a_0, \ldots, a_{m-1} \in A$, then there exists a Σ-homomorphism φ of $\mathfrak{F}_\Sigma(n)$ into $\mathfrak{A}$ with $x_0\varphi = a_0, \ldots, x_{m-1}\varphi = a_{m-1}$, thus

$$P(x_0, \ldots, x_{m-1})\varphi = P(a_0, \ldots, a_{m-1})$$

and therefore

$$|P(a_0, \ldots, a_{m-1})| \leqq |P(x_0, \ldots, x_{m-1})| = \mathfrak{n}.$$

Take any cardinal $\mathfrak{m}$ with $\mathfrak{n} < \mathfrak{m}$ and apply the Statement with $\mathfrak{m}$. The arising contradiction, $\mathfrak{m} \leqq \mathfrak{n}$, concludes the proof of Theorem 1.

COROLLARY. *Let us assume that $\mathfrak{F}_\Sigma(n)$ exists. Let $\boldsymbol{P} \in \boldsymbol{P}_k(\Sigma)$, let $\mathfrak{A}$ be a Σ-structure, $a_0, \ldots, a_{k-1} \in A$. If there exist $b_0, \ldots, b_{m-1} \in A$ with $m \leqq n$ such that*

$$a_0, \ldots, a_{k-1} \in [b_0, \ldots, b_{m-1}]_\Sigma,$$

then $P(a_0, \ldots, a_{k-1})$ is finite.

Proof. Since $a_0, \ldots, a_{k-1} \in [b_0, \ldots, b_{m-1}]_\Sigma$, there exist $\boldsymbol{P}_0, \ldots, \boldsymbol{P}_{k-1} \in \boldsymbol{P}_m(\Sigma)$ such that $a_i \in P_i(b_0, \ldots, b_{m-1})$, $i = 0, \ldots, k-1$. Thus

$$P(a_0, \ldots, a_{k-1}) \subseteq P(P_0(b_0, \ldots, b_{m-1}), \ldots, P_{k-1}(b_0, \ldots, b_{m-1}))$$

and the right-hand side is finite by Theorem 1.

THEOREM 2 (THE UNIQUENESS THEOREM). *If the free Σ-structure on α generators, $\mathfrak{F}_\Sigma(\alpha)$ exists, then it is unique up to isomorphism.*

We will prove the following stronger version of Theorem 2.

THEOREM 2′. *Let $\mathfrak{F}_\Sigma(\alpha)$ and $\mathfrak{F}'_\Sigma(\alpha)$ be free Σ-structures, with Σ-generators $x_0, \ldots, x_\gamma, \ldots$ and $x'_0, \ldots, x'_\gamma, \ldots, \gamma < \alpha$, respectively. Let φ be a Σ-homomorphism of $\mathfrak{F}_\Sigma(\alpha)$ into $\mathfrak{F}_\Sigma(\alpha)$ with $x_\gamma\varphi = x'_\gamma$, for $\gamma < \alpha$. Then φ is an isomorphism.*

Since $\mathfrak{F}_\Sigma(\alpha)$ is free and $\mathfrak{F}'_\Sigma(\alpha)$ is a Σ-structure, it follows that such a φ exists; thus Theorem 2′ implies Theorem 2.

Proof. Let $a \in F'_\Sigma(\alpha)$; then there exist $n < \omega$, $\gamma_0, \ldots, \gamma_{n-1} < \alpha$ and $\boldsymbol{P} \in \boldsymbol{P}_n(\Sigma)$ such that $a \in P(x'_{\gamma_0}, \ldots, x'_{\gamma_{n-1}})$. Thus

$$F_\Sigma(\alpha)\varphi \supseteq P(x_{\gamma_0}, \ldots, x_{\gamma_{n-1}})\varphi = P(x'_{\gamma_0}, \ldots, x'_{\gamma_n-1}) \ni a,$$

which means that φ is onto.

Let φ' be a Σ-homomorphism of $\mathfrak{F}'_\Sigma(\alpha)$ into $\mathfrak{F}_\Sigma(\alpha)$ for which $x'_\gamma\varphi' = x_\gamma$ $(\gamma<\alpha)$. Let $\boldsymbol{P}'$ and $\boldsymbol{P}'' \in \boldsymbol{P}_n(\Sigma)$ and $\gamma_0, \ldots, \gamma_{n-1} < \alpha$. Then

$$\begin{aligned}(P'(x_{\gamma_0}, \ldots, x_{\gamma_{n-1}}) &\cup P''(x_{\gamma_0}, \ldots, x_{\gamma_{n-1}}))\varphi' \\ &= (P'(x'_{\gamma_0}, \ldots, x'_{\gamma_{n-1}}) \cup P''(x'_{\gamma_0}, \ldots, x'_{\gamma_{n-1}}))\varphi' \\ &= P(x_{\gamma_0}, \ldots, x_{\gamma_{n-1}}) \cup P''(x'_{\gamma_0}, \ldots, x'_{\gamma_{n-1}}).\end{aligned}$$

Since $P'(x_{\gamma_0}, \ldots, x_{\gamma_{n-1}}) \cup P''(x_{\gamma_0}, \ldots, x_{\gamma_{n-1}})$ is finite by Theorem 1, this implies that φ is 1-1 on this set. Since any two elements of $\mathfrak{F}_\Sigma(n)$ belong to a set of this form, we get that φ (and similarly φ') is a 1-1 and onto homomorphism. To show that φ is an isomorphism we have to prove that

$$r_\gamma(a_0\varphi, \ldots, a_{m_\gamma-1}\varphi) \text{ implies } r_\gamma(a_0, \ldots, a_{m_\gamma-1}).$$

(We can use this condition since φ is 1-1.)

Let $a_i \in P_i(x_{\gamma_0}, \ldots, x_{\gamma_{n-1}})$, $0 \leqq i < m_\gamma$ and form the sets

$$A = \prod (P_i(x_{\gamma_0}, \ldots, x_{\gamma_{n-1}}) \mid 0 \leqq i < m_\gamma)$$

and

$$A' = \prod (P_i(x'_{\gamma_0}, \ldots, x'_{\gamma_{n-1}}) \mid 0 \leqq i < m_\gamma).$$

Let $\varphi^{m_\gamma}\colon A \to A'$ and $(\varphi')^{m_\gamma}\colon A' \to A$ be the maps induced by φ and φ', respectively. Finally, let

$$B = \{\langle b_0, \ldots, b_{m_\gamma-1}\rangle \mid \langle b_0, \ldots, b_{m_\gamma-1}\rangle \in A \text{ and } r_\gamma(b_0, \ldots, b_{m_\gamma-1})\}$$

and

$$B' = \{\langle b_0, \ldots, b_{m_\gamma-1}\rangle \mid \langle b_0, \ldots, b_{m_\gamma-1}\rangle \in A' \text{ and } r_\gamma(b, \ldots, b_{m_\gamma-1})\}.$$

Then A and A' are finite sets, φ^{m_γ}, $(\varphi')^{m_\gamma}$ are 1-1 and onto maps. Furthermore, $B\varphi^{m_\gamma} \subseteq B'$ and $B'(\varphi')^{m_\gamma} \subseteq B$, thus φ^{m_γ} is a 1-1 and onto map between B and B', showing that φ is an isomorphism. This completes the proof of Theorem 2.

COROLLARY. *Let α and β be ordinals with $\bar{\alpha} = \bar{\beta}$. Then if $\mathfrak{F}_\Sigma(\alpha)$ exists, $\mathfrak{F}_\Sigma(\beta)$ also exists and they are isomorphic.*

4. **On the family of free Σ-structures.** Let $E(\Sigma)$ denote the class of all ordinals α for which $\mathfrak{F}_\Sigma(\alpha)$ exists. In this section we will characterize $E(\Sigma)$. The characterization theorem is based on the following result.

THEOREM 1. *Assume that $\mathfrak{F}_\Sigma(\alpha)$ exists; let $x_0, \ldots, x_\gamma, \ldots, \gamma < \alpha$ be a free Σ-generating system of $\mathfrak{F}_\Sigma(\alpha)$. Let β be an ordinal, let $\gamma_\delta < \alpha$ for $\delta < \beta$ such that if $\delta \neq \delta'$ then $\gamma_\delta \neq \gamma_{\delta'}$, and set*

$$B = [\{x_{\gamma_\delta} \mid \delta < \beta\}]_\Sigma.$$

Then $\mathfrak{B}$ is a slender Σ-substructure of $\mathfrak{F}_\Sigma(\alpha)$. Therefore, $\mathfrak{F}_\Sigma(\beta)$ exists and it is isomorphic to $\mathfrak{B}$.

Proof. The second statement follows immediately from the first one and from Lemma 5 (ii) of §2.

In order to simplify our notations, let $\alpha=n<\omega$, $\beta=m(<\omega)$ and $\gamma_i=i$, $i<m$. Thus, we will prove that if $\mathfrak{A}=\mathfrak{F}_\Sigma(n)$ exists and

$$B = [x_0, \ldots, x_{m-1}]_\Sigma,$$

then $\mathfrak{B}$ is slender (the general proof is similar).

First we make a few observations. Let φ be a Σ-homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$ with $x_0\varphi=x_0, \ldots, x_{m-1}\varphi=x_{m-1}$ ($x_m\varphi, \ldots, x_{n-1}\varphi$ can be arbitrary elements of B).

(i) If $\boldsymbol{P} \in \boldsymbol{P}_m(\Sigma)$, then $\boldsymbol{P}_\mathfrak{A}(x_0, \ldots, x_{m-1})=\boldsymbol{P}_\mathfrak{B}(x_0, \ldots, x_{m-1})$.

Indeed, $P_\mathfrak{A}(x_0, \ldots, x_{m-1}) \subseteq P_\mathfrak{B}(x_0, \ldots, x_{m-1})$ by Lemma 7, §1. On the other hand,

$$P_\mathfrak{A}(x_0, \ldots, x_{m-1})\varphi=P_\mathfrak{B}(x_0, \ldots, x_{m-1}),$$

so

$$|P_\mathfrak{A}(x_0, \ldots, x_{m-1})| \geqq |P_\mathfrak{B}(x_0, \ldots, x_{m-1})|.$$

Since by Theorem 1, §3, $P_\mathfrak{A}(x_0, \ldots, x_{m-1})$ is finite, we get the equality.

(ii) φ is onto.

Let $b \in B$; then $b \in P_\mathfrak{B}(x_0, \ldots, x_{m-1})$ for some $\boldsymbol{P} \in \boldsymbol{P}_m(\Sigma)$. Thus

$$b \in \boldsymbol{P}_\mathfrak{B}(x_0, \ldots, x_{m-1}) = \boldsymbol{P}_\mathfrak{A}(x_0, \ldots, x_{m-1})\varphi \subseteq A\varphi.$$

(iii) φ_B is 1-1.

For $\boldsymbol{P}'$ and $\boldsymbol{P}'' \in \boldsymbol{P}_m(\Sigma)$,

$$(\boldsymbol{P}'_\mathfrak{A}(x_0, \ldots, x_{m-1}) \cup \boldsymbol{P}''_\mathfrak{A}(x_0, \ldots, x_{m-1}))\varphi = \boldsymbol{P}'_\mathfrak{B}(x_0, \ldots, x_{m-1}) \cup \boldsymbol{P}''_\mathfrak{B}(x_0, \ldots, x_{m-1}).$$

Combining this with (i), we can argue as in the proof of Theorem 2′, §3.

(iv) φ is an automorphism of $\mathfrak{B}$.

φ_B is a homomorphism; by (ii) it is 1-1 and onto. Thus to prove that it is an automorphism it remains to show that

$$r_\gamma(a_0\varphi, \ldots, a_{m_\gamma-1}\varphi) \text{ implies } r_\gamma(a_0, \ldots, a_{m_\gamma-1}), \quad \text{for } a_0, \ldots, a_{m_\gamma-1} \in B.$$

Let $a_i \in P_i(x_0, \ldots, x_{m-1})$, $0 \leqq i < m$ and set

$$C = \prod (P_i(x_0, \ldots, x_{m-1}) \mid 0 \leqq i < m)$$

and

$$D = \{\langle b_0, \ldots, b_{m_\gamma-1}\rangle \mid \langle b_0, \ldots, b_{m_\gamma-1}\rangle \in C \text{ and } r_\gamma(b_0, \ldots, b_{m_\gamma-1})\}.$$

Then by (i)–(iii) and Theorem 1, §3 the map $\varphi^{m_\gamma}: C \to C$, induced by φ, is 1-1 and onto on C, and C is a finite set. Furthermore, φ is a homomorphism, thus $D\varphi^{m_\gamma} \subseteq D$. Since φ^{m_γ} is 1-1 and D is finite, we get $D\varphi^{m_\gamma}=D$, a statement, equivalent to the one that is to be proved.

Now([3]) let $a_0, \ldots, a_t \in B$, $\Phi \in \Sigma$, $l<e(\Sigma)$ and let $b_0, \ldots, b_{s-1}$ be all the $\Phi-l$ inverses of $a_0, \ldots, a_t$ in $\mathfrak{B}$ (s is finite by the corollary to Theorem 1 of §3). Since φ_B is an automorphism of $\mathfrak{B}$, $b_0\varphi, \ldots, b_{s-1}\varphi$ are the $\Phi-l$ inverses of $a_0\varphi, \ldots, a_t\varphi$

([3]) The original proof was continued using a rather long argument. This simplified version is due to G. H. Wenzel.

in $\mathfrak{B}$. But φ is a Σ-homomorphism, thus by Lemma 1 (iii) of §2, there are s $\Phi-l$ inverses $c_0, \ldots, c_{s-1}$ of $a_0, \ldots, a_t$ in $\mathfrak{A}$ such that $c_0\varphi = b_0\varphi, \ldots, c_{s-1}\varphi = b_{s-1}\varphi$. We get that $\{c_0, \ldots, c_{s-1}\} \subseteq B$, since $\mathfrak{B}$ is a Σ-substructure. Thus (iii) implies $c_0 = b_0, \ldots, c_{s-1} = b_{s-1}$. This means that every $\Phi-l$ inverse in $\mathfrak{B}$ is also a $\Phi-l$ inverse in $\mathfrak{A}$, completing the proof of Theorem 1.

THEOREM 2. *If $\mathfrak{F}_\Sigma(n)$ exists for all $n<\omega$, then $\mathfrak{F}_\Sigma(\omega)$ also exists. In other words, if $n \in E(\Sigma)$, for all $n<\omega$, then $\omega \in E(\Sigma)$.*

Outline of proof. Using the usual construction we form a direct limit $\mathfrak{A}$ of all $\mathfrak{F}_\Sigma(n)$. Theorem 1 is used to prove that $\mathfrak{A}$ is a Σ-structure and Theorem D is used to show that it is Σ-free. Note that the same proof could be used to show the existence of direct limits, provided all φ_{ij} are 1-1 and $\mathfrak{A}_i\varphi_{ij}$ is a slender Σ-substructure of $\mathfrak{A}_j$.

Proof. Let $\mathfrak{F}_\Sigma(n)$ be freely Σ-generated by $x_0^n, \ldots, x_{n-1}^n$ $(n=1, 2, \ldots)$. We can assume that $\mathfrak{F}_\Sigma(n)$ is disjoint to $\mathfrak{F}_\Sigma(m)$ if $n \neq m$.

Let φ_n be a 1-1 Σ-homomorphism of $\mathfrak{F}_\Sigma(n)$ into $\mathfrak{F}_\Sigma(m)$ with $x_i^n\varphi_n = x_i^{n+1}$, $i=0, \ldots, n-1$. For $n \leqq m$, set

$$\varphi_{nm} = \varphi_n \cdots \varphi_{m-1}.$$

Then the Σ-algebras $\mathfrak{F}_\Sigma(n)$ and the Σ-homomorphisms φ_{nm} form a direct limit system. Let $\mathfrak{A}$ denote its direct limit; if $\mathbf{x} \in A$, $\mathbf{x} = \langle x_n, x_{n+1}, \ldots\rangle$, then the mapping $\varphi^n \colon x_n \to \mathbf{x}$ is an embedding of $\mathfrak{F}_\Sigma(n)$ into $\mathfrak{A}$. Set $A_n = F_\Sigma(n)\varphi^n$. Then

$$A = \bigcup (A_n \mid n < \omega), \quad A_1 \subseteq A_2 \subseteq \cdots$$

and $\mathfrak{A}_n \cong \mathfrak{F}_\Sigma(n)$, $n=1, 2, \ldots$.

First we prove that $\mathfrak{A}$ is a Σ-structure. We will verify only that if

$$\Phi = (\mathbf{x})(\exists \mathbf{y})(\mathbf{u})(\exists \mathbf{v})\Psi(\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}) \in \Sigma,$$

then Φ holds in $\mathfrak{A}$.

Let $a \in A$; then $a \in A_n$ for some $n<\omega$. Since $\mathfrak{A}_n$ is a Σ-structure, there exists a $b \in A_n$ such that $(\mathbf{u})(\exists \mathbf{v})\Psi(a, b, \mathbf{u}, \mathbf{v})$ holds in $\mathfrak{A}_n$. To prove that it also holds in $\mathfrak{A}$, take a $c \in A$ and an $m<\omega$, $n \leqq m$, with $a, b, c \in A_m$. Since $\mathfrak{F}_\Sigma(n)\varphi_{nm}$ is a slender Σ-substructure of $\mathfrak{F}_\Sigma(m)$ by Theorem 1, and $A_n = F_\Sigma(n)\varphi_{nm}\varphi^m$, $A_m = F_\Sigma(m)\varphi^m$, we get that $\mathfrak{A}_n$ is a slender Σ-substructure of $\mathfrak{A}_m$. Thus $(\mathbf{u})(\exists \mathbf{v})\Psi(a, b, \mathbf{u}, \mathbf{v})$ in $\mathfrak{A}_m$, hence there exists a $d \in A_m$ with $\Psi(a, b, c, d)$ in $\mathfrak{A}_m$. Therefore, $\Psi(a, b, c, d)$ in $\mathfrak{A}$, so $(\mathbf{u})(\exists \mathbf{v})\Psi(a, b, \mathbf{u}, \mathbf{v})$ in $\mathfrak{A}$, which was to be proved. A similar (but simpler) argument shows that if $a, c \in A$ then also a $\Phi-1$ inverse exists.

Set

$$\mathbf{x}_i = \langle x_i^{i+1}, x_i^{i+2}, \ldots\rangle, \quad \text{for } i = 0, 1, 2, \ldots,$$

then $\mathbf{x}_i \in A$, $A = [\mathbf{x}_0, \mathbf{x}_1, \ldots]_\Sigma$ and $A_n = [\mathbf{x}_0, \ldots, \mathbf{x}_{n-1}]_\Sigma$. Thus $\mathfrak{A}$ is Σ-generated by ω elements.

It remains to show that $\mathfrak{A}$ satisfies (iii) of Definition 1, §3. Let $\mathfrak{B}$ be a Σ-structure, $b_0, b_1, \ldots \in B$. We can assume that A is disjoint to B. We want to construct a Σ-homomorphism φ of $\mathfrak{A}$ into $\mathfrak{B}$ with $\mathbf{x}_i\varphi = b_i$, $i = 0, 1, 2, \ldots$.

Set $C = A \cup B$; we define a relational system with constants on C:

(i) for every $a \in A$, there is a constant $\boldsymbol{k}_a$ and $(\boldsymbol{k}_a)_{\mathfrak{C}} = a$;

(ii) for every $d \in B$, there is a constant $\boldsymbol{l}_d$ and $(\boldsymbol{l}_d)_{\mathfrak{C}} = d$;

(iii) for $\gamma < o_1(\tau)$, r_γ is defined on A and B as it was;

(iv) for $o < n < \omega$, $\boldsymbol{P} \in \boldsymbol{P}_n(\Sigma)$, $\boldsymbol{r_P}$ is defined on A and B as it was; ($\boldsymbol{r_P}$ was defined in Lemma 6, §1);

(v) for $c_0, \ldots, c_{n-1}$ in A or in B and $\boldsymbol{P} \in \boldsymbol{P}_n(\Sigma)$ we define the constants $\boldsymbol{c}(c_0, \ldots, c_{n-1}, \boldsymbol{P}, i)$ for $0 \leqq i < k_{\boldsymbol{P}}$ (of Theorem 1, §3); these are intepreted in $\mathfrak{C}$ such that every element of $P(c_0, \ldots, c_{n-1})$ is the interpretation of one of them.

Let $\mathfrak{C}$ denote the relational system defined by (i)–(v), and let τ^0 be the type of $\mathfrak{C}$. We want to define an additional relation $R(x, y)$ on $\mathfrak{C}$ satisfying the following universal sentences:

(1) $\boldsymbol{R}(\boldsymbol{k}_{\mathbf{x}_i}, \boldsymbol{l}_{b_i}), \qquad i = 0, 1, \ldots;$

(2) $(\boldsymbol{r}(\boldsymbol{k}_{a_0}, \ldots, \boldsymbol{k}_{a_m}) \wedge \boldsymbol{R}(\boldsymbol{k}_{a_0}, \boldsymbol{l}_{d_0}) \wedge \cdots \wedge \boldsymbol{R}(\boldsymbol{k}_{a_m}, \boldsymbol{l}_{d_m})) \rightarrow \boldsymbol{r}(\boldsymbol{l}_{d_0}, \ldots, \boldsymbol{l}_{d_m}),$
where $\boldsymbol{r}$ is some $\boldsymbol{r}_\gamma$ or $\boldsymbol{r_P}$;

(3) $(\boldsymbol{R}(\boldsymbol{k}_{a_0}, \boldsymbol{l}_{d_0}) \wedge \cdots \wedge \boldsymbol{R}(\boldsymbol{k}_{a_{n-1}}, \boldsymbol{l}_{d_{n-1}}) \wedge \boldsymbol{R}(\boldsymbol{k}_a, \boldsymbol{l}_b) \wedge \boldsymbol{r_P}(\boldsymbol{l}_{d_0}, \ldots, \boldsymbol{l}_{d_{n-1}}, \boldsymbol{l}_b)) \rightarrow$
$\boldsymbol{R}(\boldsymbol{c}(a_0, \ldots, a_{n-1}, \boldsymbol{P}, 0), \boldsymbol{l}_b) \vee \cdots \vee \boldsymbol{R}(\boldsymbol{c}(a_0, \ldots, a_{n-1}, \boldsymbol{P}, \boldsymbol{k_P} - 1), \boldsymbol{l}_b);$

(4) $(\boldsymbol{R}(\boldsymbol{k}_a, \boldsymbol{l}_d) \wedge \boldsymbol{R}(\boldsymbol{k}_a, \boldsymbol{l}_{d_1})) \rightarrow \boldsymbol{l}_d = \boldsymbol{l}_{d_1};$

(5) $\boldsymbol{r_P}(\boldsymbol{k}_{\mathbf{x}_0}, \ldots, \boldsymbol{k}_{\mathbf{x}_{n-1}}, \boldsymbol{k}_a) \rightarrow (\boldsymbol{R}(\boldsymbol{k}_a, \boldsymbol{c}(b_0, \ldots, b_{n-1}, \boldsymbol{P}, 0)) \vee \cdots$
$\vee \boldsymbol{R}(\boldsymbol{k}_a, \boldsymbol{c}(b_0, \ldots, b_{n-1}, \boldsymbol{P}, \boldsymbol{k_P} - 1))).$

If R can be defined so as to satisfy (1)–(5) then we can define a mapping φ of A into B by setting $a\varphi = d$ ($a \in A$, $d \in B$) if $R(a, d)$.

By (4), φ is well defined and by (5), φ is defined on the whole of A; (2) and (3) mean that φ is a Σ-homomorphism and by (1), $\mathbf{x}_i\varphi = b_i$.

By Theorem D it is sufficient to prove that R can be defined on every finite subset of C. However, this is trivial, since if H is a finite subset of C, then for some n,

$$H \subseteq [\mathbf{x}_0, \ldots, \mathbf{x}_{n-1}]_\Sigma \cup [b_0, \ldots, b_{n-1}]_\Sigma \cup H',$$

where $H' = H \cap (B - [b_0, b_1, \ldots]_\Sigma)$. It follows from (4) and (5) that no element of H' occurs in (1)–(5); thus it suffices to consider $H'' = H - H'$. Since $\mathfrak{A}_n$ is the free Σ-structure on n Σ-generators, there is a homomorphism ψ of $\mathfrak{A}_n$ into $\mathfrak{B}$ for which $\mathbf{x}_i\psi = b_i$, $i = 0, \ldots, n-1$. Define R on $A_n \cup B$ by $R(a, d)$ if $a\psi = d$. Obviously, R satisfies (1)–(5). This completes the proof of Theorem 2.

The following result is a more complicated version of Theorem 2.

THEOREM 3. *Let α be a limit ordinal. If $\mathfrak{F}_\Sigma(\beta)$ exists for all $\beta < \alpha$, then also $\mathfrak{F}_\Sigma(\alpha)$ exists.*

Sketch of proof. The proof of Theorem 2 started with the construction of a direct limit system. There we had no problem with $\varphi_{ln}\varphi_{nm}=\varphi_{lm}$ (for $l\leqq n\leqq m$) since we defined φ_{ln} as $\varphi_l\cdots\varphi_{n-1}$. However, we cannot do this now. In order to construct the direct limit system, we set

$$C = \bigcup (F_\Sigma(\beta) \mid \beta < \alpha),$$

where the $F_\Sigma(\beta)$ are assumed to be pairwise disjoint. We want to define on C a relation R such that $\varphi_{\beta\gamma}$ for $\beta\leqq\gamma<\alpha$ can be defined by $a\varphi_{\beta\gamma}=b$ for $a\in F_\Sigma(\beta)$ and $b\in F_\Sigma(\gamma)$, if $R(a, b)$. As in the proof of Theorem 2, we can do that applying Theorem D by introducing sufficiently many constants and relations, which satisfy the analogues of (1)–(5), and

$$(6) \qquad (x)(y)(z)((R(x, y) \wedge R(y, z)) \rightarrow R(x, z)).$$

We leave the obvious details to the reader. Then we form the direct limit $\mathfrak{A}$, and we proceed as in the proof of Theorem 2.

Now we are ready to characterize $E(\Sigma)$.

THEOREM 4. *Either there exists a positive integer n such that $\mathfrak{F}_\Sigma(\alpha)$ exists if and only if $\alpha<n$, or $\mathfrak{F}_\Sigma(\alpha)$ exists for every α.*

In other words, either $E(\Sigma)=\{\alpha \mid \alpha<n\}$ or $E(\Sigma)$ is the class of all ordinals.

Proof. Let us assume that there is no n with $E(\Sigma)=\{\alpha \mid \alpha<n\}$. Then for every n there exists an $m\geqq n$ with $m\in E(\Sigma)$. By Theorem 1 this implies $n\in E(\Sigma)$; therefore by Theorem 2, $\omega\in E(\Sigma)$. Let us further assume that for some ordinal δ, $\delta\notin E(\Sigma)$. If δ is the smallest ordinal with $\delta\notin E(\Sigma)$, then by the corollary to Theorem 2, §3, δ is an initial ordinal. Since $\omega<\delta$, δ is a limit ordinal and if $\gamma<\delta$ then $\gamma\in E(\Sigma)$. Thus by Theorem 3, $\delta\in E(\Sigma)$. This contradiction proves Theorem 4.

5. On the existence of free Σ-structures.

Let us recall that $\boldsymbol{P}\in\boldsymbol{P}_n(\Sigma)$ is *bounded* if for some natural number m

$$|P(a_0, \ldots, a_{n-1})| \leqq m$$

for any Σ-structure $\mathfrak{A}$ and $a_0, \ldots, a_{n-1}\in A$. The smallest such integer is denoted by $k_{\boldsymbol{P}}$.

A necessary and sufficient condition for the existence of free Σ-structures is given in the following result:

THEOREM 1. *$\mathfrak{F}_\Sigma(n)$ exists if and only if the following two conditions are satisfied:*
($\mathrm{B_n}$) *every $\boldsymbol{P}\in\boldsymbol{P}_n(\Sigma)$ is bounded;*
($\mathrm{C_n}$) *let $\mathfrak{A}$ and $\mathfrak{B}$ be Σ-structures, let $a_0, \ldots, a_{n-1}\in A$ and $b_0, \ldots, b_{n-1}\in B$. If $A=[a_0, \ldots, a_{n-1}]_\Sigma$ then there exists a Σ-structure $\mathfrak{C}$, Σ-generated by $c_0, \ldots, c_{n-1}$, and there exist Σ-homomorphisms $\varphi\colon C\rightarrow A$ and $\psi\colon C\rightarrow B$ such that $c_i\varphi=a_i$ and $c_i\psi=b_i$ for $0\leqq i<n$.*

Proof. (B_n) is necessary by Theorem 1 of §3. It is obvious that (C_n) is also necessary, since we can always set $\mathfrak{C}=\mathfrak{F}_\Sigma(n)$.

Let us assume that (B_n) and (C_n) are satisfied. Let $\boldsymbol{P}\in \boldsymbol{P}_n(\Sigma)$; ($B_n$) implies that there exists a Σ-structure $\mathfrak{C}_{\boldsymbol{P}}$, Σ-generated by $a_0^{\boldsymbol{P}},\dots,a_{n-1}^{\boldsymbol{P}}$, such that

$$|P(a_0^{\boldsymbol{P}},\dots,a_{n-1}^{\boldsymbol{P}})| = k_{\boldsymbol{P}}.$$

Let $\mathfrak{C}_{\boldsymbol{P}'}$ be a Σ-structure which corresponds to $\boldsymbol{P}'\in \boldsymbol{P}_n(\Sigma)$ and let us apply (C_n) for $\mathfrak{C}_{\boldsymbol{P}}$ and $\mathfrak{C}_{\boldsymbol{P}'}$, obtaining a structure $\mathfrak{C}$ Σ-generated by $c_0,\dots,c_{n-1}$. It is obvious that for $\mathfrak{C}$ both $|P(c_0,\dots,c_{n-1})|$ and $|P'(c_0,\dots,c_{n-1})|$ are maximal.

If $\boldsymbol{P}_0,\dots,\boldsymbol{P}_{k-1}\in \boldsymbol{P}_n(\Sigma)$, then we can always find a minimal upper bound $k_{\boldsymbol{P}_0,\dots,\boldsymbol{P}_{k-1}}$ for $P_0\cup\cdots\cup P_{k-1}$. An obvious induction, combined with the argument given above, yields the following result:

Let H be a nonvoid finite subset of $\boldsymbol{P}_n(\Sigma)$; then there exists a least natural number k_H such that for every Σ-structure $\mathfrak{A}$ and $a_0,\dots,a_{n-1}\in A$ we have

$$\left|\bigcup (P(a_0,\dots,a_{n-1}) \mid \boldsymbol{P}\in H)\right| \leqq k_H.$$

Furthermore, there exists a Σ-structure $\mathfrak{A}_H$ and $a_0^H,\dots,a_{n-1}^H\in A_H$ such that $A_H=[a_0^H,\dots,a_{n-1}^H]_\Sigma$ and if $H'\subseteq H$, $H'\neq\varnothing$, then

$$\left|\bigcup (P(a_0^H,\dots,a_{n-1}^H) \mid \boldsymbol{P}\in H')\right| = k_{H'}.$$

Set $T=\{H \mid H \text{ is finite, } \varnothing\neq H \text{ and } H\subseteq \boldsymbol{P}_n(\Sigma)\}$ and for $H\in T$ let

$$T_H = \{K \mid K\in T \text{ and } H\subseteq K\}.$$

Then $T_{H_1}\cap T_{H_2}=T_{H_1\cup H_2}$ and $T_H\neq\varnothing$, and thus there exists a dual prime ideal $\mathscr{D}$ over T containing all the T_H. Set $\mathfrak{A}=\prod_{\mathscr{D}}(\mathfrak{A}_H \mid H\in T)$. By Theorem C, $\mathfrak{A}$ is a Σ-structure. Let f_i be the function for which $f_i(H)=a_i^H$ for all $H\in T$, $i=0,\dots,n-1$. Then

$$T'_H = \{K \mid \left|\bigcup (P(a_0^K,\dots,a_{n-1}^K) \mid \boldsymbol{P}\in H)\right| = k_H\} \supseteq T_H,$$

so $T'_H\in\mathscr{D}$. Since there is a formula in our language which can express that

$$\left|\bigcup (P(a_0^K,\dots,a_{n-1}^K) \mid \boldsymbol{P}\in H)\right| = k_H,$$

by Theorem C, we conclude that

$$\left|\bigcup (P(f_0^\vee,\dots,f_{n-1}^\vee) \mid \boldsymbol{P}\in H)\right| = k_H$$

for all $H\in T$.

Let $\mathfrak{F}$ be the Σ-substructure of $\mathfrak{A}$, Σ-generated by $f_0^\vee,\dots,f_{n-1}^\vee$. It is obvious that the above equality holds in $\mathfrak{F}$ as well.

Let $\mathfrak{B}$ be any Σ-structure and $b_0,\dots,b_{n-1}\in B$. By (C_n), there exists a Σ-structure $\mathfrak{C}$, Σ-generated by $c_0,\dots,c_{n-1}$, and there exist Σ-homomorphisms $\varphi\colon C\to F$ and $\psi\colon C\to B$ with $c_i\varphi=f_i^\vee$ and $c_i\psi=b_i$, $0\leqq i<n$.

The mapping φ is obviously onto. Let $c, d \in C$ and let us choose $\boldsymbol{P}'$ and $\boldsymbol{P}'' \in \boldsymbol{P}_n(\Sigma)$ with $c \in P'(c_0, \ldots, c_{n-1})$ and $d \in P''(c_0, \ldots, c_{n-1})$ and set $H = \{\boldsymbol{P}', \boldsymbol{P}''\}$. By definition,

$$|P'(c_0, \ldots, c_{n-1}) \cup P''(c_0, \ldots, c_{n-1})| \leqq k_H.$$

On the other hand,

$$(P'(c_0, \ldots, c_{n-1}) \cup P''(c_0, \ldots, c_{n-1}))\varphi = P'(f_0^\vee, \ldots, f_{n-1}^\vee) \cup P''(f_0^\vee, \ldots, f_{n-1}^\vee),$$
$$|P'(f_0^\vee, \ldots, f_{n-1}^\vee) \cup P''(f_0^\vee, \ldots, f_{n-1}^\vee)| = k_H.$$

Thus φ is 1-1. Therefore, if we deal with Σ-algebras then φ is an isomorphism; then this implies that $\varphi^{-1}\psi$ is a Σ-homomorphism of $\mathfrak{F}$ into $\mathfrak{B}$ with $f_i^\vee \varphi^{-1}\psi = b_i$, for $0 \leqq i < n$, establishing that $\mathfrak{F}$ is the free Σ-algebra on n Σ-generators. However, in the general case φ need not be an isomorphism since φ^{-1} need not preserve relations. Let $\mathfrak{F} = \langle A; F, R\rangle$. Using ($\mathrm{B_n}$) and ($\mathrm{C_n}$) and some transfinite method, for instance Theorem B, it can be verified that there exists a "smallest" Σ-structure $\mathfrak{A} = \langle A; F, R\rangle$, such that for all Σ-polynomial symbols $\boldsymbol{P}$ we have $P_{\mathfrak{F}} = P_{\mathfrak{A}}$, for all $f \in F$ we have $(f)_{\mathfrak{F}} = (f)_{\mathfrak{A}}$ and for all $r \in R$, $(r)_{\mathfrak{A}}$ is smallest for all Σ-structures having these properties. For this $\mathfrak{A}$, in place of $\mathfrak{F}$, it is obvious that φ^{-1} is also a homomorphism, completing the proof of Theorem 1.

COROLLARY 1. *All free Σ-algebras exist if and only if the following two conditions are satisfied*:

(B) *all Σ-polynomials are bounded*;

(C) *let $\mathfrak{A}$ and $\mathfrak{B}$ be Σ-structures, let $a_0, a_1, \ldots, a_n, \ldots \in A$, $b_0, b_1, \ldots, b_n, \ldots \in B$ and $A = [a_0, a_1, \ldots, a_n, \ldots]_\Sigma$; then there exists a Σ-structure $\mathfrak{C}$ with $C = [c_0, c_1, \ldots, c_n, \ldots]_\Sigma$ and there exist Σ-homomorphisms $\varphi\colon C \to A$ and $\psi\colon C \to B$ such that $c_i\varphi = a_i$ and $c_i\psi = b_i$, $i = 0, 1, 2, \ldots$.*

Corollary 1 is an obvious combination of Theorem 1 and Theorem 4 of §4.

COROLLARY 2. *Let Σ be universal. Then $\mathfrak{F}_\Sigma(n)$ (that is the free algebra on n generators over Σ) exists if and only if* ($\mathrm{C_n}$) *holds. All free algebras exist if and only if* (C) *holds.*

Indeed, if Σ is universal then all Σ-polynomials are of bound 1, and thus ($\mathrm{B_n}$) is always satisfied.

DEFINITION 1. Σ is said to have property (*P*) if for every Φ in Σ either Φ is universal or $\Phi = (x_0)\cdots(x_{n-1})(\exists y)\Psi(x_0, \ldots, x_{n-1}, y)$, or Φ is positive.

Let $\mathscr{A}$ be a well-ordered inverse limit system of the Σ-structures $\mathfrak{A}_\gamma$, $\gamma < \alpha$; let $\mathfrak{A}_\gamma$ be Σ-generated by $a_0^\gamma, \ldots, a_{n-1}^\gamma$; let the homomorphisms φ_δ^γ ($\delta \leqq \gamma < \alpha$) be Σ-homomorphisms and suppose

$$a_i^\gamma \varphi_\delta^\gamma = a_i^\delta, \quad \text{for } \delta \leqq \gamma < \alpha,\ i = 0, \ldots, n-1.$$

Let $\mathfrak{A}$ be the inverse limit structure of $\mathscr{A}$.

LEMMA 1. *Let $\gamma<\alpha$ and $a\in A_\gamma$. If* $(\mathrm{B_n})$ *is satisfied there exists an* $\boldsymbol{a}\in A$ *with* $\boldsymbol{a}(\gamma)=a$.

Proof. Choose $\boldsymbol{P}\in\boldsymbol{P}_n(\Sigma)$ such that $a\in P(a_0^\gamma,\dots,a_{n-1}^\gamma)$. For $\delta\geqq\gamma$, set

$$U_\delta = \{b \mid b\in P(a_0^\delta,\dots,a_{n-1}^\delta) \text{ and } b\varphi_\gamma^\delta = a\}.$$

Since $a\in P(a_0^\delta,\dots,a_{n-1}^\delta)\varphi_\gamma^\delta$ and φ_γ^δ is a Σ-homomorphism, U_δ is not void. By $(\mathrm{B_n})$ U_δ is finite. Furthermore, $U_\delta\varphi_{\delta'}^\delta\subseteq U_{\delta'}$, if $\gamma\leqq\delta'\leqq\delta<\alpha$. By Theorem B, there exists $a(\delta)\in U_\delta$ for $\delta>\gamma$ such that $a(\delta)\varphi_{\delta'}^\delta=a(\delta')$ if $\gamma\leqq\delta'\leqq\delta<\alpha$. Set $a(\delta)=a\varphi_\delta^\gamma$ if $\delta\leqq\gamma$. Then for $\boldsymbol{a}=\langle a(\gamma)\mid\gamma<\alpha\rangle$ we have that $\boldsymbol{a}\in\mathfrak{A}$ and $a(\gamma)=a$.

THEOREM 2. *If we assume (P) and* $(\mathrm{B_n})$, *then $\mathfrak{A}$ is a Σ-structure.*

Proof. We first verify that if $\Phi=(\boldsymbol{x})(\exists\boldsymbol{y})(\boldsymbol{u})(\exists\boldsymbol{v})\Psi(\boldsymbol{x},\boldsymbol{y},\boldsymbol{u},\boldsymbol{v})\in\Sigma$, and Φ is positive, then Φ holds in $\mathfrak{A}$. Let $\boldsymbol{a}\in A$ and set

$$T_\gamma = \{b \mid b\in A_\gamma \text{ and } b \text{ is a } \Phi-0 \text{ inverse of } a(\gamma)\}.$$

It follows from $(\mathrm{B_n})$ and from the corollary to Theorem 1, §3, that T_γ is finite for all $\gamma<\alpha$ and $T_\gamma\neq\varnothing$. It is obvious that $T_\gamma\varphi_\delta^\gamma\subseteq T_\delta$ if $\delta\leqq\gamma<\alpha$. Thus by Theorem B there exists a $\boldsymbol{b}\in A$ with $b(\gamma)\in T_\gamma$ for all $\gamma<\alpha$, that is,

$$(\boldsymbol{u})(\exists\boldsymbol{v})\Psi(a(\gamma), b(\gamma), \boldsymbol{u}, \boldsymbol{v}) \quad \text{in } \mathfrak{A}_\gamma.$$

We want to prove that $(\boldsymbol{u})(\exists\boldsymbol{v})\Psi(\boldsymbol{a},\boldsymbol{b},\boldsymbol{u},\boldsymbol{v})$ in $\mathfrak{A}$. Let $\boldsymbol{c}\in A$ and set

$$U_\gamma = \{d \mid \Psi(a(\gamma), b(\gamma), c(\gamma), d)\}, \quad \text{for } \gamma<\alpha.$$

Then

$$U_\gamma \subseteq \{d \mid d \text{ is a } \Phi-1 \text{ inverse of } a(\gamma) \text{ and } c(\gamma)\}.$$

Since the right-hand side is finite, U_γ is finite for all $\gamma<\alpha$. Now let $d\in U_\gamma$ and $\delta<\gamma<\alpha$. Then $\Psi(a(\gamma), b(\gamma), c(\gamma), d)$ and since Ψ is positive, $\Psi(a(\delta), b(\delta), c(\delta), d\varphi_\delta^\gamma)$. Thus $U_\gamma\varphi_\delta^\gamma\subseteq U_\delta$. So we can choose $\boldsymbol{d}\in A$ with $d(\gamma)\in U_\gamma$. Therefore, $\Psi(a(\gamma), b(\gamma), c(\gamma), d(\gamma))$ for all $\gamma<\alpha$, which implies $\Psi(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d})$. The existence of $\Phi-1$ inverses is proved by a similar argument.

Now let Φ be universal, $\Phi=(\boldsymbol{x}_0)\cdots(\boldsymbol{x}_{m-1})\Psi(\boldsymbol{x}_0,\dots,\boldsymbol{x}_{m-1})$. Let $\boldsymbol{a}_0,\dots,\boldsymbol{a}_{m-1}\in\mathfrak{A}$; then $\Psi(a_0(\gamma),\dots,a_{m-1}(\gamma))$ for all $\gamma<\alpha$, whence $\Psi(\boldsymbol{a}_0,\dots,\boldsymbol{a}_{m-1})$.

Finally, let $\Phi=(\boldsymbol{x}_0)\cdots(\boldsymbol{x}_{m-1})(\exists\boldsymbol{y})\Psi(\boldsymbol{x}_0,\dots,\boldsymbol{x}_{m-1},\boldsymbol{y})$ and let $\boldsymbol{a}_0,\dots,\boldsymbol{a}_{m-1}\in A$. Set

$$T_\gamma = \{b \mid \Psi(a_0(\gamma),\dots,a_{m-1}(\gamma), b)\}.$$

By $(\mathrm{B_n})$ and from the corollary to Theorem 1, §3, T_γ is finite. Since $T_\gamma\varphi_\delta^\gamma\subseteq T_\delta$ is obvious for $\delta\leqq\gamma<\alpha$, by Theorem B there exists a $\boldsymbol{b}\in A$ with $b(\gamma)\in T_\gamma$ for $\gamma<\alpha$. Thus $\Psi(a_0(\gamma),\dots,a_{m-1}(\gamma), b(\gamma))$ for $\gamma<\alpha$, which implies that $\Psi(\boldsymbol{a}_0,\dots,\boldsymbol{a}_{m-1},\boldsymbol{b})$, completing the proof of Theorem 2.

It is easy to see that the proof of Theorem 2 yields the following result:

COROLLARY. *For* $\mathbf{a}_0, \ldots, \mathbf{a}_{m-1}, \mathbf{b} \in A$ *and* $\mathbf{P} \in \mathbf{P}_m(\Sigma)$, *if* $b(\gamma) \in P(a_0(\gamma), \ldots, a_{m-1}(\gamma))$ *for all* $\gamma < \alpha$, *then* $\mathbf{b} \in P(\mathbf{a}_0, \ldots, \mathbf{a}_{m-1})$.

The converse of this corollary is also true.

LEMMA 2. *Under the conditions of Theorem* 2 *and its corollary, if* $\mathbf{b} \in P(\mathbf{a}_0, \ldots, \mathbf{a}_{m-1})$, *then* $b(\gamma) \in P(a_0(\gamma), \ldots, a_{m-1}(\gamma))$ *for all* $\gamma < \alpha$.

Proof. It is sufficient to prove that if $\mathbf{b}$ is a Φ-inverse of $\mathbf{a}_0, \ldots, \mathbf{a}_{m-1}$, then $b(\gamma)$ is a Φ-inverse of $a_0(\gamma), \ldots, a_{m-1}(\gamma)$ for all $\gamma < \alpha$. If Φ is universal, there is nothing to prove, so let $\Phi = (\boldsymbol{x})(\exists \boldsymbol{y})(\boldsymbol{u})(\exists \boldsymbol{v})\Psi(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{u}, \boldsymbol{v}) \in \Sigma$, let Φ be positive and let $\mathbf{b}$ be a $\Phi - 0$ inverse of $\mathbf{a}$. We have to prove that

$$(\boldsymbol{u})(\exists \boldsymbol{v})\Psi(a(\gamma), b(\gamma), \boldsymbol{u}, \boldsymbol{v}) \quad \text{for all } \gamma < \alpha.$$

Let $c \in A_\gamma$; by Lemma 1 there exists a $\mathbf{c} \in A$ with $c(\gamma) = c$. Since $(\boldsymbol{u})(\exists \boldsymbol{v})\Psi(a, b, \boldsymbol{u}, \boldsymbol{v})$ in $\mathfrak{A}$, there exists a $\mathbf{d} \in A$ with $\Psi(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$. Hence $\Psi(a(\delta), b(\delta), c(\delta), d(\delta))$ for all $\delta \geqq \delta_0$, where $\delta_0 < \alpha$. Since Ψ is positive, we get $\Psi(a(\gamma), b(\gamma), c, d(\gamma))$, completing the proof. The same statement for $\Phi - 1$ inverses is even simpler to prove.

Now let $\Phi = (\boldsymbol{x}_0) \cdots (\boldsymbol{x}_{m-1})(\exists \boldsymbol{y})\Psi(\boldsymbol{x}_0, \ldots, \boldsymbol{x}_{m-1}, \boldsymbol{y})$. Let $\mathbf{b}$ be an inverse of $\mathbf{a}_0, \ldots, \mathbf{a}_{m-1}$. Then $\Psi(\mathbf{a}_0, \ldots, \mathbf{a}_{m-1}, \mathbf{b})$, so $\Psi(a_0(\delta), \ldots, a_{m-1}(\delta), b(\delta))$ holds for all $\delta \geqq \delta_0$, for some $\delta_0 < \alpha$. Choose δ such that $\delta > \max\{\gamma, \delta_0\}$. Since $b(\delta)$ is a $\Phi - 0$ inverse of $a_0(\delta), \ldots, a_{m-1}(\delta)$, it follows that $b(\gamma) = b(\delta)\varphi_\gamma^\delta$ is a $\Phi - 0$ inverse of $a_0(\gamma) = a_0(\delta)\varphi_\gamma^\delta, \ldots, a_{m-1}(\gamma) = a_{m-1}(\delta)\varphi_\gamma^\delta$; that is $\Psi(a_0(\gamma), \ldots, a_{m-1}(\gamma), b(\gamma))$ for all $\gamma < \alpha$, which was to be proved.

Set $\mathbf{a}_0 = \langle a_0^\gamma \mid \gamma < \alpha \rangle, \ldots, \mathbf{a}_{n-1} = \langle a_{n-1}^\gamma \mid \gamma < \alpha \rangle$ and let $\overline{\mathfrak{A}}$ denote the Σ-substructure of $\mathfrak{A}$, Σ-generated by $\mathbf{a}_0, \ldots, \mathbf{a}_{n-1}$.

LEMMA 3. $\overline{\mathfrak{A}}$ *is a slender* Σ-*substructure of* $\mathfrak{A}$.

Proof. We should note that the $\mathbf{a}$ of Lemma 1 is in $\overline{\mathfrak{A}}$. Thus, by repeating the proof of Lemma 2, and restricting $\mathbf{a}, \mathbf{c}$ to $\overline{\mathfrak{A}}$ we get that the conclusion of Lemma 2 holds for $\overline{\mathfrak{A}}$, that is, if $\mathbf{b}$ is a Φ-inverse of $\mathbf{c}_0, \ldots, \mathbf{c}_{m-1}$ in $\overline{\mathfrak{A}}$, then $b(\gamma)$ is a Φ-inverse of $c_0(\gamma), \ldots, c_{m-1}(\gamma)$ in $\mathfrak{A}_\gamma$ for all $\gamma < \alpha$. Thus the corollary to Theorem 2 implies that $\mathbf{b}$ is a Φ-inverse of $\mathbf{c}_0, \ldots, \mathbf{c}_{m-1}$ in $\mathfrak{A}$, which was to be proved.

COROLLARY. *The mapping* $\psi_\gamma \colon \mathbf{c} \to c(\gamma)$ *is a* Σ-*homomorphism of* $\overline{\mathfrak{A}}$ *onto* $\mathfrak{A}_\gamma$.

Now we are ready to prove the main result:

THEOREM 3. *Let us assume* (P) *and* (B_n). *Let* $\mathfrak{A}_\gamma$ *be* Σ-*structures and*

$$A_\gamma = [a_0^\gamma, \ldots, a_{n-1}^\gamma]_\Sigma \quad \text{for } \gamma < \alpha.$$

Let us assume for all $\gamma \leqq \delta < \alpha$ *that there exists a* Σ-*homomorphism* φ_γ^δ *such that* $a_i^\delta \varphi_\gamma^\delta = a_i^\gamma$, $0 \leqq i < n$. *Then there exists a* Σ-*structure* $\mathfrak{A}$ *and there exist* $a_0, \ldots, a_{n-1} \in A$ *such that* $A = [a_0, \ldots, a_{n-1}]_\Sigma$ *and for each* $\gamma < \alpha$ *there exists a* Σ-*homomorphism* ψ_γ *of* $\mathfrak{A}$ *onto* $\mathfrak{A}_\gamma$, *with* $a_i \psi_\gamma = a_i^\gamma$, $0 \leqq i < n$.

Proof. If we have that $\varphi_\gamma^\delta \varphi_\beta^\gamma = \varphi_\beta^\delta$, whenever $\beta \leqq \gamma \leqq \delta < \alpha$, then the $\mathfrak{A}_\gamma$ form an inverse system and we can take the $\mathfrak{A}$ as in Lemma 3 and then by the corollary to Lemma 3, we have the ψ_γ for $\gamma < \alpha$. However, $\varphi_\gamma^\delta \varphi_\beta^\gamma = \varphi_\beta^\delta$ need not hold. We are going to prove that the φ_γ^δ can be replaced by ψ_γ^δ in such a way that we still have $a_i^\delta \psi_\gamma^\delta = a_i^\gamma$ and also $\psi_\gamma^\delta \psi_\beta^\gamma = \psi_\beta^\delta$, for $\beta \leqq \gamma \leqq \delta < \alpha$.

Let us assume that A_γ and A_δ are disjoint if $\gamma \neq \delta$ and let us form

$$C = \bigcup (A_\gamma \mid \gamma < \alpha).$$

We will think of the required family of ψ_γ^δ as a single binary relation $R(x, y)$ on C, where $x\varphi_\gamma^\delta = y$ means $x \in A_\delta$, $y \in A_\gamma$ and $R(x, y)$. Using the same tricks as in the second part of the proof of Theorem 2, §4, we can introduce unary relations $R_\gamma(x)$ for $x \in A_\gamma$ and we can introduce sufficiently many relations and constants such that a system of universal sentences Ω will express that $\boldsymbol{R}(\boldsymbol{x}, \boldsymbol{y}) \wedge \boldsymbol{R}_\delta(\boldsymbol{x}) \wedge \boldsymbol{R}_\gamma(\boldsymbol{y})$ defines a Σ-homomorphism ψ_γ^δ of A_δ onto A_γ with $a_i^\delta \psi_\gamma^\delta = a_i^\gamma$, $0 \leqq i < n$. Let Ω^* be Ω to which we add the sentence

$$(\boldsymbol{x})(\boldsymbol{y})(\boldsymbol{z})((\boldsymbol{R}(\boldsymbol{x}, \boldsymbol{y}) \wedge \boldsymbol{R}(\boldsymbol{y}, \boldsymbol{z})) \rightarrow \boldsymbol{R}(\boldsymbol{x}, \boldsymbol{z})).$$

Let us observe that on every finite subset of C we can define R so as to satisfy Ω^*. Indeed, if H is finite, $H \subseteq C$, then there exist $\gamma_0 < \gamma_1 < \cdots < \gamma_{k-1} < \alpha$ such that $H \subseteq \bigcup (A_{\gamma_i} \mid 0 \leqq i < k)$. Now set $\psi_{\gamma_i}^{\gamma_l} = \varphi_{\gamma_{l-1}}^{\gamma_l} \varphi_{\gamma_{l-2}}^{\gamma_{l-1}} \cdots \varphi_{\gamma_i}^{\gamma_{i+1}}$ for $i \leqq l$ and let $R(x, y)$ mean that $x \in A_{\gamma_l}$, $y \in A_{\gamma_i}$ and $x\psi_{\gamma_i}^{\gamma_l} = y$ for some $0 \leqq i \leqq l < k$. Then R obviously satisfies Ω^*. Thus by Theorem D, R can be defined on C so as to satisfy Ω^*, completing the proof of Theorem 3.

Definition 2. Let $\mathfrak{A}$ be a Σ-structure, Σ-generated by $a_0, \ldots, a_\gamma, \ldots, \gamma < \alpha$. Then $\mathfrak{A}$ is called a *maximally free* Σ*-structure*, in notation, $MF_\Sigma(\alpha)$, with respect to the Σ-generating system $\{a_\gamma \mid \gamma < \alpha\}$ if whenever $\mathfrak{B}$ is a Σ-structure Σ-generated by $b_0, \ldots, b_\gamma, \ldots, \gamma < \alpha$ and φ is a Σ-homomorphism of $\mathfrak{B}$ into $\mathfrak{A}$ with $b_\gamma \varphi = a_\gamma$, for $\gamma < \alpha$, then φ is an isomorphism.

Definition 3. Let K be a set of maximally free Σ-structures on α Σ-generators. K is called a (Σ, α)*-covering system* if for any Σ-structure $\mathfrak{B}$, Σ-generated by $b_0, \ldots, b_\gamma, \ldots, \gamma < \alpha$, there exists an $\mathfrak{A} \in K$ (with the Σ-generating system $a_0, \ldots, a_\gamma, \ldots, \gamma < \alpha$) and a Σ-homomorphism φ of $\mathfrak{A}$ onto $\mathfrak{B}$ with $a_\gamma \varphi = b_\gamma$, for $\gamma < \alpha$.

Corollary 1. *Let us assume* (P) *and* (B_n). *Then there exists a* (Σ, n)*-covering system.*

Proof. Let $\mathfrak{A}$ be a Σ-structure, $A = [h_0, \ldots, h_{n-1}]_\Sigma$. Consider the class of all pairs $\langle \mathfrak{A}_1, H_1 \rangle$, where $\mathfrak{A}_1$ is a Σ-structure, $H_1 = \langle h_0^1, \ldots, h_{n-1}^1 \rangle$, $A = [h_0^1, \ldots, h_{n-1}^1]_\Sigma$ with the property that there exists a Σ-homomorphism φ of $\mathfrak{A}_1$ into $\mathfrak{A}$ with $h_0^1 \varphi = h_0, \ldots, h_{n-1}^1 \varphi = h_{n-1}$. Let us say that $\langle \mathfrak{A}_1, H_1 \rangle$ is isomorphic to $\langle \mathfrak{A}_2, H_2 \rangle$ if there exists an isomorphism φ of $\mathfrak{A}_1$ with $\mathfrak{A}_2$ satisfying $h_i^1 \varphi = h_i^2$, for $i = 0, \ldots, n-1$, where $H_1 = \langle h_0^1, \ldots, h_{n-1}^1 \rangle$ and $H_2 = \langle h_0^2, \ldots, h_{n-1}^2 \rangle$. Let P be a class of such pairs, such that every pair has an isomorphic copy in P and there are no two isomorphic pairs in P. Using (B_n) it is easy to give an upper bound for the cardinality of P, so P is a set. We introduce a binary relation $\leqq$ on P: $\langle \mathfrak{A}_1, H_1\rangle \leqq \langle \mathfrak{A}_2, H_2\rangle$ if there exists a Σ-homomorphism φ of $\mathfrak{A}_2$ into $\mathfrak{A}_1$ such that $h_i^2\varphi = h_i^1$ for $i=0, \ldots, n-1$. Then $\langle P; \leqq\rangle$ is a partially ordered set. The only nontrivial part in checking this is to prove that $\leqq$ is antisymmetric; this is an easy modification of the argument of Theorem 2′ of §3 (the freeness of the algebras involved was used there only to prove (B_n); now we have (B_n) by assumption). Theorem 3 states that Zorn's lemma can be applied to $\langle P; \leqq\rangle$. Any maximal element of $\langle P; \leqq\rangle$ will be maximally free (again use (B_n) and the argument of Theorem 2′ of §3).

It follows from (B_n) that a maximal class of nonisomorphic pairs $\langle \mathfrak{A}, H\rangle$ is a set. Using the above construction, we choose for each $\langle \mathfrak{A}, H\rangle$ an $MF_\Sigma(n)$ containing $\langle \mathfrak{A}, H\rangle$ in $\langle P; \leqq\rangle$ and thus we have a (Σ, n)-covering system.

6. **Strong free Σ-structures and the inverse preserving property.** If K is the class of all groups $\langle G; \cdot, 1\rangle$ defined in the usual way by a Σ, then all free Σ-structures exist and they are the free groups in the usual sense. However, nobody would use the theory of free Σ-structures to prove the existence of free groups. The most convenient way of proving the existence of free groups is the introduction of x^{-1} as an operation because then in Σ the existential quantifiers are eliminated, and in this richer language Σ is equivalent to a universal $\bar{\Sigma}$, to which the simple known methods apply. In this section we will discuss the problem of when it is possible to eliminate the existential quantifiers in Σ such that the resulting $\bar{\Sigma}$ can be used to construct free Σ-structures.

First, we introduce a property of first order axiom systems.

DEFINITION 1. Σ is said to have the *Inverse Preserving Property* (IP) if every Σ-substructure is slender.

THEOREM 1. *The following conditions on Σ are equivalent*:

(i) Σ *has* IP;

(ii) *if* $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ *are Σ-structures*, $\mathfrak{B}$ *is a Σ-substructure of* $\mathfrak{A}$ *and* φ *is a Σ-homomorphism of* $\mathfrak{A}$ *into* $\mathfrak{C}$, *then* φ_B *is a Σ-homomorphism of* $\mathfrak{B}$ *into* $\mathfrak{C}$;

(iii) *if* $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ *are Σ-structures*, $\mathfrak{B}$ *is a Σ-substructure of* $\mathfrak{A}$ *and* φ *is a Σ-homomorphism of* $\mathfrak{C}$ *into* $\mathfrak{B}$, *then* φ *is a Σ-homomorphism of* $\mathfrak{C}$ *into* $\mathfrak{A}$;

(iv) *if* $\mathfrak{B}$ *is a Σ-substructure of* $\mathfrak{A}$, *then* $\varphi: x \rightarrow x$ *is a Σ-homomorphism of* $\mathfrak{B}$ *into* $\mathfrak{A}$.

COROLLARY. *If* Σ *has* IP, *then every Σ-homomorphism* φ *can be written in the form* $\varphi = \psi\chi$ *where* ψ *is an onto Σ-homomorphism and* χ *is a* 1-1 *Σ-homomorphism.*

Thus we see that IP is equivalent to the condition that in the category of Σ-structures with the Σ-homomorphisms, the usual definition of a subobject in terms of the underlying set functor agrees with the definition of a Σ-substructure.

The proofs are trivial consequences of Lemmas 5 and 6 of §2.

We will also need a property of free Σ-structures.

Definition 2. A free Σ-structure is *strong* if the $\bar{\varphi}$ of Definition 1, §3 is always unique.

That is, any mapping of the Σ-generators into a Σ-structure, can be *uniquely* extended to a Σ-homomorphism.

Corollary. *Let $\mathfrak{A}$ be a free Σ-structure on α Σ-generators and let $\mathfrak{B}$ be a free Σ-structure on β generators. If $\bar{\alpha}=\bar{\beta}$, then $\mathfrak{A}$ is strong if and only if $\mathfrak{B}$ is strong.*

This is trivial from the Uniqueness Theorem.

Theorem 2. *If the free Σ-structure $\mathfrak{F}_\Sigma(\omega)$ exists and it is strong, then all free Σ-structures exist and all are strong.*

Proof. The existence of free Σ-structures follows from Theorem 4, §4. It is obvious that if $\alpha=\lim \beta_i$ and each $\mathfrak{F}_\Sigma(\beta_i)$ is strong, then so is $\mathfrak{F}_\Sigma(\alpha)$. It remains to prove that if $\mathfrak{F}_\Sigma(\alpha)$ is strong and $\beta<\alpha$, then $\mathfrak{F}_\Sigma(\beta)$ is strong. Let $x_0,\ldots,x_\gamma,\ldots$, $\gamma<\alpha$ be the Σ-generators of $\mathfrak{F}_\Sigma(\alpha)$. By Theorem 1 of §4 and the corollary to Definition 2, we can assume that $\mathfrak{F}_\Sigma(\beta)=[x_0,\ldots,x_\gamma,\ldots]_\Sigma$, $\gamma<\beta$. Let χ be a Σ-homomorphism of $\mathfrak{F}_\Sigma(\alpha)$ onto $\mathfrak{F}_\Sigma(\beta)$ with $x_\gamma\chi=x_\gamma$ for $\gamma<\beta$ and $x_\gamma\chi=x_0$ for $\beta\leqq\gamma$. If $\mathfrak{F}_\Sigma(\beta)$ is not strong then there exists a Σ-structure $\mathfrak{B}$ and there exist $b_0,\ldots,b_\gamma,\ldots$, $\gamma<\beta$ elements of $\mathfrak{B}$ such that $x_\gamma\to b_\gamma$ $(\gamma<\beta)$ has two extensions to Σ-homomorphisms φ and ψ. Then the mapping $x_\gamma\to b_\gamma$ for $\gamma<\beta$ and $x_\gamma\to b_0$ for $\gamma\geqq\beta$ has two extensions to Σ-homomorphisms, namely $\chi\varphi$ and $\chi\psi$, contradicting that $\mathfrak{F}_\Sigma(\alpha)$ is strong.

Now we are ready to state and to prove the main result.

Theorem 3. *Let us assume that Σ has* IP *and that $\mathfrak{F}_\Sigma(\omega)$ exists and is strong. Then there exists a set of operations $\bar{F}$, containing F, such that on every Σ-structure $\mathfrak{A}=\langle A; F, R\rangle$ we can define the additional operations $f\in\bar{F}-F$, such that the correspondence*

$$\mathfrak{A} = \langle A; F, R\rangle \to \bar{\mathfrak{A}} = \langle A; \bar{F}, R\rangle$$

has the following properties:

(i) *$\mathfrak{A}$ is a Σ-substructure of $\mathfrak{B}$ if and only if $\bar{\mathfrak{A}}$ is a substructure of $\bar{\mathfrak{B}}$*;

(ii) *let ψ map A into B; then ψ is a Σ-homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$ if and only if ψ is a homomorphism of $\bar{\mathfrak{A}}$ into $\bar{\mathfrak{B}}$*;

(iii) *let K denote the class of $\bar{\mathfrak{A}}$; then $\mathfrak{F}_K(\alpha)$ exists for all α.*

Remark. Essentially, what is stated here is a condition under which the category of Σ-algebras and Σ-homomorphisms is isomorphic to a category of structures with homomorphisms, and the underlying set functor of the latter has an adjoint.

Proof. For every $1\leqq n<\omega$ and $\boldsymbol{P}\in\boldsymbol{P}_n(\Sigma)$ we introduce $k_{\boldsymbol{P}}$ n-ary operations, $f_0^{\boldsymbol{P}},\ldots,f_{k_{\boldsymbol{P}}-1}^{\boldsymbol{P}}$ as follows:

Take $\mathfrak{F}_\Sigma(n)$ with the Σ-generators $x_0, \ldots, x_{n-1}$; define $f_i^P(x_0, \ldots, x_{n-1})$, $i<k_P$ such that $P(x_0, \ldots, x_{n-1})=\{f_i^P(x_0, \ldots, x_{n-1}) \mid i<k_P\}$; let $\mathfrak{A}$ be an arbitrary Σ-structure, $a_0, \ldots, a_{n-1} \in A$ and φ a Σ-homomorphism of $\mathfrak{F}_\Sigma(n)$ into $\mathfrak{A}$ with $x_0\varphi=a_0, \ldots, x_{n-1}\varphi=a_{n-1}$. Set

$$f_i^P(a_0, \ldots, a_{n-1}) = f_i^P(x_0, \ldots, x_{n-1})\varphi, \quad \text{for } i < k_P.$$

Set

$$\bar{F} = F \cup \bigcup (\{f_i^P \mid \boldsymbol{P} \in \boldsymbol{P}_n(\Sigma),\ i < k_P\} \mid 1 \leqq n < \omega);$$

and

$$K = \{\langle A; \bar{F}, R\rangle \mid \langle A; F, R\rangle \text{ is a } \Sigma\text{-structure}\}.$$

It is obvious that $\langle A; \bar{F}, R\rangle$ is well defined, since by Theorem 2, φ is unique.

Now we will verify (i)–(iii).

Ad (i). Let $\mathfrak{A}$ be a Σ-substructure of $\mathfrak{B}$, $a_0, \ldots, a_{n-1} \in A$ and let $f \in \bar{F}$ be an n-ary operation. If $f \in F$, then $f(a_0, \ldots, a_{n-1}) \in A$. If $f \notin F$, then $f=f_i^P$ for some $\boldsymbol{P} \in \boldsymbol{P}_n(\Sigma)$ and $i<k_P$. Then

$$f_i^P(x_0, \ldots, x_{n-1}) \in P(x_0, \ldots, x_{n-1})$$

in $\mathfrak{F}_\Sigma(n)$, so

$$f_i^P(a_0, \ldots, a_{n-1}) = f_i^P(x_0, \ldots, x_{n-1})\varphi \in P(x_0, \ldots, x_{n-1})\varphi = P(a_0, \ldots, a_{n-1}) \subseteq A,$$

since φ is an Σ-homomorphism. Thus $\bar{\mathfrak{A}}$ is a substructure of $\bar{\mathfrak{B}}$.

Let $\bar{\mathfrak{A}}$ be a substructure of $\bar{\mathfrak{B}}$; then $\mathfrak{A}$ is a substructure of $\mathfrak{B}$. To prove that it is a Σ-substructure, let $a_0, \ldots, a_{n-1} \in A$ and $\boldsymbol{P} \in \boldsymbol{P}_n(\Sigma)$. If $b \in \boldsymbol{P}_{\mathfrak{B}}(a_0, \ldots, a_{n-1})$ then $b=f(a_0, \ldots, a_{n-1})$ for some $f \in \bar{F}$. Thus $b \in A$.

Ad (ii). Let ψ be a Σ-homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$, $a_0, \ldots, a_{n-1} \in A$, and $f \in \bar{F}$. We want to prove that

$$f(a_0, \ldots, a_{n-1})\psi = f(a_0\psi, \ldots, a_{n-1}\psi).$$

This is obvious if $f \in F$. Let $f \notin F$, that is $f=f_i^P$ for some $\boldsymbol{P} \in \boldsymbol{P}_n(\Sigma)$ and $i<k_P$.

Let φ and χ be the Σ-homomorphisms of $\mathfrak{F}_\Sigma(n)$ into $\mathfrak{A}$ and $\mathfrak{B}$, respectively with $x_i\varphi=a_i$ and $x_i\chi=a_i\psi$, for $i<n$. Since φ and χ are unique (Theorem 2) we get $\chi=\varphi\psi$. Thus

$$f(a_0, \ldots, a_{n-1})\psi = f(x_0, \ldots, x_{n-1})\varphi\psi = f(x_0, \ldots, x_{n-1})\chi = f(a_0\psi, \ldots, a_{n-1}\psi),$$

which was to be proved.

Let ψ be a homomorphism of $\bar{\mathfrak{A}}$ into $\bar{\mathfrak{B}}$; then ψ is a homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$. To prove that ψ is a Σ-homomorphism, take $\boldsymbol{P} \in \boldsymbol{P}_n(\Sigma)$ and $a_0, \ldots, a_{n-1} \in A$. Let us define φ and χ as above. Let $b \in P(a_0, \ldots, a_{n-1})$. Since φ is a Σ-homomorphism, there exists a $u \in P(x_0, \ldots, x_{n-1})$ with $u\varphi=b$. Then $u=f_i^P(x_0, \ldots, x_{n-1})$ for some $i<k_P$. By the definition of f_i^P, we have that $b=f_i^P(a_0, \ldots, a_{n-1})$. Since ψ is a homomorphism, we get that $b\psi=f_i^P(a_0\psi, \ldots, a_{n-1}\psi)$. Again, by the definition of f_i^P, there exists a $v \in F_\Sigma(n)$ with $v=f_i^P(x_0, \ldots, x_{n-1})$ and $v\chi=b\psi$. Since $v \in P(x_0, \ldots, x_{n-1})$ and χ is a Σ-homomorphism, we get that

$$b \in P(a_0, \ldots, a_{n-1}) \text{ implies that } b\psi \in P(a_0\psi, \ldots, a_{n-1}\psi).$$

The converse of this statement can be proved similarly; thus $P(a_0, \ldots, a_{n-1})\psi = P(a_0\psi, \ldots, a_{n-1}\psi)$, which was to be proved.

Ad (iii). It follows from the assumption that $\mathfrak{F}_\Sigma(\alpha)$ exists for all α. (i) and (ii) imply that $\mathfrak{F}_\Sigma(\alpha)$ is $\mathfrak{F}_K(\alpha)$. This completes the proof of Theorem 3.

Theorem 3 is the best possible result, since the following holds:

THEOREM 4. *Let us assume that the conclusions of Theorem* 3 *hold for* Σ. *Then* Σ *has* IP, *and* $\mathfrak{F}_\Sigma(\omega)$ *exists and is strong.*

Proof. $\mathfrak{F}_K(\omega)$ exists by (iii), so by (i) and (ii) $\mathfrak{F}_\Sigma(\omega)$ exists. Since a free algebra over K is always strong, $\mathfrak{F}_\Sigma(\omega)$ is also strong by (ii). Using (i) and (ii), condition (iv) of Theorem 1 can easily be verified; thus by Theorem 1, Σ has IP.

REFERENCES

1. G. Birkhoff, *On the structure of abstract algebras*, Proc. Cambridge Philos. Soc. **31** (1935), 433–454.

2. S. Eilenberg and W. Steenrod, *Foundations of algebraic topology*, Princeton Univ. Press, Princeton, N. J., 1952.

3. T. Frayne, A. C. Morel and D. S. Scott, *Reduced direct products*, Fund. Math. **51** (1962–1963), 195–228.

4. G. Grätzer, *Free algebras over first order axiom systems*, Magyar Tud. Akad. Mat. Kutató Int. Közl. **8** (1963), 193–199.

5. P. J. Higgins, *Algebras with a scheme of operators*, Math. Nachr. **27** (1963), 115–132.

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