DECOMPOSITIONS OF S^3 AND PSEUDO-ISOTOPIES[1]

BY
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In this paper we prove that if G is a cellular upper semicontinuous decomposition of S^3 such that S^3/G is a 3-manifold, then there exists a pseudo-isotopy of S^3 onto itself that shrinks the elements of G to points. The main tool used is the "homeomorphism theorem" of Armentrout [**A**]. A special case of this theorem is stated in Theorem 0 below, but roughly it says that there are homeomorphisms from S^3/G to S^3 that, in some sense, approximate P^{-1} where $P\colon S^3 \to S^3/G$ is the decomposition map.

The terminology used is quite standard (see [**A**] for definitions). We use G to denote the collection of elements of an upper semicontinuous decomposition of S^3 (=Euclidean 3-sphere). We say G is cellular if each element of G is the intersection of a decreasing sequence of 3-cells. We use S^3/G to denote the decomposition space, and $P\colon S^3 \to S^3/G$ to denote the natural map.

If X and Y are topological spaces, then a homotopy from X to Y is a map $H\colon X\times[a,b]\to Y$. If $t\in[a,b]$, we define $H_t\colon X\to Y$ by $H_t(x)=H(x,t)$. If H_t is a homeomorphism for $a\leqq t<b$ then H is called a pseudo-isotopy and if H_t is a homeomorphism for $a\leqq t\leqq b$ then H is called an isotopy. If f and f' are maps from X to Y, then we say that f is homotopic (respectively pseudo-isotopic or isotopic) to f' if there exists a homotopy (respectively pseudo-isotopy or isotopy) $H\colon X\times[a,b]\to Y$ with $H_a=f$ and $H_b=f'$. An isotopy $H\colon X\times[a,b]\to Y$ is called an ε-isotopy if diameter $H(\{x\}\times[a,b])<\varepsilon$ for each $x\in X$. If G is a decomposition of S^3, then to say that there exists pseudo-isotopy of S^3 onto itself that shrinks the elements of G to points means that there exists a pseudo-isotopy $H\colon S^3\times[0,1]\to S^3$ such that

(i) $H_0(x)=x$ for each $x\in S^3$,
(ii) if $g\in G$ then $H_1(g)$ is a point in S^3, and
(iii) if $g, g'\in G$, $g\neq g'$ then $H_1(g)\neq H_1(g')$.

Let M be a 3-manifold (the only one we use in this paper is S^3). A triangulation T of M is a collection of simplexes (homeomorphic images of standard closed simplexes) such that the union of the elements of T is all of M, and if two elements of T intersect, then the intersection is a face of each. A subdivision T' of the triangulation T is a triangulation of M such that each simplex of T' is contained in some simplex of T. If T is a triangulation of M and σ is a simplex of T, then

$N(\sigma, T)=\bigcup \sigma'$ such that $\sigma'\in T$ and $\sigma\cap\sigma'\neq\varnothing$,
$N^2(\sigma, T)=\bigcup \sigma'$ such that $\sigma'\in T$ and $\sigma'\cap N(\sigma, T)\neq\varnothing$,

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$O(\sigma, T) = M - \bigcup \sigma'$ such that $\sigma' \in T$ and $\sigma \cap \sigma' = \varnothing$.

Clearly, $\sigma \subseteq O(\sigma, T) \subseteq N(\sigma, T) \subseteq N^2(\sigma, T)$ and $O(\sigma, T)$ is open. We will also use the following facts. If $\sigma' \nsubseteq N^2(\sigma, T)$ then $O(\sigma, T) \cap O(\sigma', T) = \varnothing$. If T' is a subdivision of T, $\sigma' \in T'$ and $\sigma \in T$ with $\sigma' \subseteq \sigma$ then $O(\sigma', T') \subseteq O(\sigma, T)$.

Finally, if X is a metric space we use ρ_x to denote some fixed metric for X. If $f, f' : Z \to X$ are maps we will also use $\rho_x(f, f')$ to denote the sup $\{\rho_x(f(z), f'(z))\}$. If $X = S^3$ we will use ρ instead of ρ_{S^3}.

THEOREM 0 (ARMENTROUT). *Let G be a cellular upper semicontinuous decomposition of S^3. If S^3/G is a 3-manifold then there exists a homeomorphism $h: S^3/G \to S^3$. Furthermore, if T is a triangulation of S^3/G then h can be chosen so that $h(\sigma) \subseteq P^{-1}(O(\sigma, T))$ for each simplex σ of T.*

LEMMA 1. *Let G be an upper semicontinuous decomposition of S^3. Let $M = S^3/G$ and suppose that $h: M \to S^3$ is a homeomorphism of M onto S^3. Suppose that T is a triangulation of M and that for each simplex $\sigma \in T$, $h(\sigma) \subseteq P^{-1}(O(\sigma, T))$. Then $P^{-1}(O(\sigma, T) \subseteq h(N^2(\sigma, T))$ for each simplex $\sigma \in T$.*

Proof. Let $x \in P^{-1}(O(\sigma, T))$. Then there exists a simplex $\sigma' \in T$ such that $x \in h(\sigma')$. Thus $x \in P^{-1}(O(\sigma', T))$ and hence $O(\sigma, T) \cap O(\sigma', T) \neq \varnothing$. This implies that there exists a simplex $\sigma'' \in T$ such that $\sigma \cap \sigma'' \neq \varnothing$ *and* $\sigma' \cap \sigma'' \neq \varnothing$. Thus $\sigma'' \subseteq N(\sigma, T)$ and $\sigma' \subseteq N^2(\sigma, T)$, so $x \in h(\sigma') \subseteq h(N^2(\sigma, T))$.

THEOREM 1(2). *Let G be a cellular upper semicontinuous decomposition of S^3. If S^3/G is a 3-manifold, then there exists a pseudo-isotopy of S^3 onto itself that shrinks the elements of G to points.*

Proof. The proof will proceed by obtaining a sequence of isotopies, each one of which shrinks the elements of the decomposition a little smaller. The first isotopy is obtained by appealing to degree of a homeomorphism arguments. It will, most likely, move points quite far. The succeeding isotopies will move points less and less and will be obtained by appealing to the ε, δ type theorems of Sanderson **[S]**, Fisher **[F]**, or Kister **[K]**.

For simplicity of notation let S denote S^3 and let M denote S^3/G. Of course, M is homeomorphic to S^3 but we wish to distinguish between them.

Let $\gamma > 0$ be chosen so that if f, f' are maps of M onto M with $\rho_M(f, f') < \gamma$, then f is homotopic to f'. Let T_0 be a triangulation of M such that diameter $O(\sigma, T_0) < \gamma$ for each simplex $\sigma \in T_0$.

Apply Theorem 0 to get a homeomorphism $h_0: M \to S$ such that $h_0(\sigma) \subseteq P^{-1}(O(\sigma, T_0))$ for each simplex $\sigma \in T_0$. (The final stage of the pseudo-isotopy we construct will be $h_0 \circ P$.)

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(2) W. L. Voxman in his Ph.D. thesis at the University of Iowa has extended Theorem 1 to arbitrary 3-manifolds.

By Theorem 4 of **[S]**, Theorem 2 of **[K]**, or Theorem 8 of **[F]** there exists a $\delta_1 > 0$ such that if $h, h' \colon S^3 \to S^3$ are homeomorphisms with $\rho(h, h') < \delta_1$, then h and h' are $\frac{1}{2}$-isotopic.

Let T_1 be a subdivision of T_0 so that for each simplex $\sigma \in T_1$, diameter $h_0(\sigma) < \delta_1/5$. By Theorem 0 there exists a homeomorphism $h_1 \colon M \to S$ such that $h_1(\sigma) \subseteq P^{-1}(O(\sigma, T_1))$ for each $\sigma \in T_1$.

We show now that $h_0 \circ h_1^{-1}$ is isotopic to the identity. By Theorem 3 of **[S]** it suffices to prove that $h_0 \circ h_1^{-1}$ is homotopic to the identity and hence has degree 1. Consider the following diagram.

$$\begin{tikzcd}
M\times\{0\} \cup M\times\{1\} \arrow[r, "h_1 \cup h_0"] \arrow[d, phantom, "\cap"] & S \arrow[d, "P"] \\
M\times[0,1] \arrow[ur, dashed, "J"] \arrow[r, dashed, "H"] & M
\end{tikzcd}$$

The existence of the homotopy H between $P \circ h_1$ and $P \circ h_0$ is proved as follows. Let $y \in M$. Then $y \in \sigma'$ for some simplex $\sigma' \in T_1$ and $\sigma' \subseteq \sigma$ for some simplex $\sigma \in T_0$. Hence $h_0(y) \in P^{-1}(O(\sigma, T_0))$ and $h_1(y) \in P^{-1}(O(\sigma', T_1)) \subseteq P^{-1}(O(\sigma, T_0))$. Thus $P \circ h_1(y)$ and $P \circ h_0(y)$ are both in $O(\sigma, T_0)$ and diameter $O(\sigma, T_0) < \gamma$ and by our choice of γ, the homotopy H exists. The homotopy J is obtained by "lifting" H, using Theorem 1 of **[A-P]**. This lifting does not make the diagram commute, but J does extend $h_1 \cup h_0$. That is

(i) $J(y, 0) = h_1(y)$, and

(ii) $J(y, 1) = h_0(y)$.

Now define a homotopy $K \colon S \times [0, 1] \to S$ as follows:

$$K(x, t) = J(h_1^{-1}(x), t).$$

Then

$$K(x, 0) = J(h_1^{-1}(x), 0) = h_1(h_1^{-1}(x)) = x,$$

and

$$K(x, 1) = J(h_1^{-1}(x), 1) = h_0 \circ h_1^{-1}(x).$$

Thus K is the required homotopy and $h_0 \circ h_1^{-1}$ is isotopic to the identity. Let $H \colon S \times [0, \frac{1}{2}] \to S$ be an isotopy such that $H_0(x) = x$ and $H_{1/2}(x) = h_0 \circ h_1^{-1}(x)$ for all $x \in S$.

By Lemma 1 it follows that for each $\sigma \in T_1$, diameter $H_{1/2}(P^{-1}(O(\sigma, T_1))) < \delta_1$, because $H_{1/2}(P^{-1}(O(\sigma, T_1))) \subseteq H_{1/2}(h_1(N^2(\sigma, T_1))) = h_0(N^2(\sigma, T_1))$ and this latter set has diameter $< \delta_1$. Of course, it follows that for each element g of the decomposition, diameter $H_{1/2}(g) < \delta$, also.

By **[S]** or **[K]** or **[F]**, as above, there exists a $\delta_2 > 0$ such that if $h, h' \colon S^3 \to S^3$ are homeomorphisms with $\rho(h, h') < \delta_2$ then h and h' are $\frac{1}{4}$-isotopic.

Let T_2 be a subdivision of T_1 so that for each simplex $\sigma \in T_2$, diameter $h_0(\sigma) < \delta_2/5$.

By Theorem 0 there exists a homeomorphism $h_2: M \to S$ such that $h_2(\sigma) \subseteq P^{-1}(O(\sigma, T_2))$ for each simplex $\sigma \in T_2$.

We show now that $H_{1/2} \circ h_1 \circ h_2^{-1}$ is $\frac{1}{2}$-isotopic to $H_{1/2} = h_0 \circ h_1^{-1}$. It suffices to show that $\rho(H_{1/2} \circ h_1 \circ h_2^{-1}, H_{1/2}) < \delta_1$. Let $y \in M$. There exists a simplex $\sigma' \in T_2$ such that $y \in \sigma'$ and there exists a simplex $\sigma \in T_1$ such that $\sigma' \subseteq \sigma$. Then $O(\sigma', T_2) \subseteq O(\sigma, T_1)$ and hence each of $h_1(y)$ and $h_2(y)$ are in $P^{-1}(O(\sigma, T_1)) \subseteq h_1(N^2(\sigma, T_1))$. Therefore, if $x = h_2(y)$, then $H_{1/2}(x)$ and $H_{1/2} \circ h_1 \circ h_2^{-1}(x)$ are both in

$$H_{1/2}(h_1(N^2(\sigma, T_1))) = h_0(N^2(\sigma, T_1))$$

which has diameter $< \delta_1$. Thus, $\rho(H_{1/2}, H_{1/2} \circ h_1 \circ h_2^{-1}) < \delta_1$ and they are $\frac{1}{2}$-isotopic.

Let $\tilde{H}: S \times [1/2, 2/3] \to S$ be a $\frac{1}{2}$-isotopy such that $\tilde{H}_{1/2}: (x) = H_{1/2}(x)$ and $\tilde{H}_{2/3}(x) = H_{1/2} \circ h_1 \circ h_2^{-1}(x)$ for all $x \in S$. For convenience we combine H and $\tilde{H}$ into one isotopy,

$$\begin{aligned} H: S \times [0, 2/3] \to S \text{ such that } H_0(x) &= x \text{ and } H_{2/3}(x) = H_{1/2} \circ h_1 \circ h_2^{-1}(x) \\ &= h_0 \circ h_2^{-1}(x) \quad \text{for all } x \in S. \end{aligned}$$

The pattern for all future isotopies was set in the last five paragraphs. By Lemma 1 we have $H_{2/3}(P^{-1}(O(\sigma, T_2))) \subseteq H_{2/3}(h_2(N^2(\sigma, T_2))) = h_0(N^2(\sigma, T_2))$ for each simplex $\sigma \in T_2$. Furthermore, we have by our choice of T_2 that diameter $h_0(N^2(\sigma, T_2)) < \delta_2$. As before, diameter $H_{2/3}(g) < \delta_2$ for each $g \in G$.

By [S], [K], or [F] there exists a $\delta_3 > 0$ such that two homeomorphisms of S^3 onto itself that differ by less than δ_3 are $\frac{1}{8}$-isotopic.

Let T_3 be a subdivision of T_2 so that for each simplex $\sigma \in T_3$, diameter $h_0(\sigma) < \delta_3/5$. Let $h_3: M \to S$ be a homeomorphism such that for each simplex $\sigma \in T_3$, $h_3(\sigma) \subseteq P^{-1}(O(\sigma, T_3))$.

By an argument exactly as above we show that $\rho(H_{2/3} \circ h_2 \circ h_3^{-1}, H_{2/3}) < \delta_2$ and hence there exists a $\frac{1}{4}$-isotopy $\tilde{H}: S \times [\frac{2}{3}, \frac{3}{4}] \to S$ such that $H_{2/3} = \tilde{H}_{2/3}$ and $\tilde{H}_{3/4} = H_{2/3} \circ h_2 \circ h_3^{-1} = h_0 \circ h_3^{-1}$. Again we combine H and $\tilde{H}$ to get an isotopy $H: S \times [0, \frac{3}{4}] \to S$ such that $H_0 = id$ and $H_{3/4} = h_0 \circ h_3^{-1}$. We continue in this same fashion. For each positive integer n we choose a $\delta_n > 0$ such that any two homeomorphisms of S^3 onto S^3 that differ by less than δ_n are $(\frac{1}{2})^n$-isotopic. We pick a subdivision T_n of T_{n-1} such that for each simplex $\sigma \in T_n$ diameter $h_0(\sigma) < \delta_n/5$. We use Theorem 0 to obtain a homeomorphism $h_n: M \to S$ such that for each simplex $\sigma \in T_n$, $h_n(\sigma) \subseteq P^{-1}(O(\sigma, T_n))$. We assume inductively that we have an isotopy $H: S \times [0, (n-1)/n] \to S$ such that $H_0 = id$ and $H_{(n-1)/n} = h_0 \circ h_{n-1}^{-1}$. Then for each $y \in M$, $h_{n-1}(y)$ and $h_n(y)$ are both contained in $h_{n-1}(N^2(\sigma, T_{n-1}))$ for some $\sigma \in T_{n-1}$. $\rho(H_{n-1} \circ h_n^{-1}, H_{(n-1)/n}) < \delta_{n-1}$. Hence there exists a $(\frac{1}{2})^{n-1}$-isotopy:

$$\tilde{H}: S \times \left[\frac{n-1}{n}, \frac{n}{n+1}\right] \to S \text{ such that } \tilde{H}_{(n-1)/n} = H_{(n-1)/n}$$

and

$$\tilde{H}_{n/(n+1)} = H_{(n-1)/n} \circ h_{n-1} \circ h_n^{-1} = h_0 \circ h_n^{-1}.$$

As above, we combine H and $\tilde{H}$ to get $H: S \times [0, n/(n+1)] \to S$ such that $H_0 = id$ and $H_{n/(n+1)} = h_0 \circ h_n^{-1}$.

Thus, for $0 \leqq t < 1$ we get a continuous family of a homeomorphism $H_t: S \to S$ such that

(i) if $(n-1)/n \leqq t \leqq n/(n+1)$ then $\rho(H_t, H_{(n-1)/n}) < (\frac{1}{2})^{n-1}$,

(ii) $\rho(H_{n/(n+1)}, h_0 \circ P) < (\frac{1}{2})^n$.

The first property of the H_t's is clear from the above description of the H's and $\tilde{H}$'s. To prove property (ii), let $x \in S$. Then there exists a simplex $\sigma \in T_n$ such that $P(x) \in \sigma$. Thus both x and $h_n \circ P(x)$ are in $P^{-1}(O(\sigma, T_n)) \subseteq h_n(N^2(\sigma, T_n))$. Thus both $H_{n/(n+1)}(x)$ and $H_{n/(n+1)}(h_n \circ P(x)) = h_0 \circ h_n^{-1} \circ h_n \circ P(x) = h_0 \circ P(x)$ are in $H_{n/(n+1)} \circ h_n(N^2(\sigma, T_n)) = h_0(N^2(\sigma, T_n))$. But this latter set has diameter $< \delta_n \leqq (\frac{1}{2})^n$.

Clearly, if we define $H_1: S \to S$ by $H_1 = h_0 \circ P$, then we get a pseudo-isotopy $H: S \times [0, 1] \to S$ such that $H_0 = id$ and for each element $g \in G$, $H_1(g)$ is a distinct point of S.

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