

A CLASS OF NONLINEAR EVOLUTION EQUATIONS IN A BANACH SPACE

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We treat the nonlinear evolution equation

$$(*) \quad f'(t) = A(t, f(t))f(t)$$

where the unknown function f is from a real number interval into a Banach space X . For suitable real numbers t and vectors x in X , $A(t, x)$ is the infinitesimal generator of a holomorphic semigroup of linear contraction operators in X , and certain regularity requirements are placed on the function $(t, x) \rightarrow A(t, x)$.

After proving a local existence, uniqueness, and stability theorem for (*), we consider the case $A(t, x) = H(x)$ and obtain conditions under which there is a strongly continuous semigroup of nonlinear nonexpansive transformations whose infinitesimal generator is an extension of the transformation $Qx = H(x)x$.

We state our main results in §1 and prove them in §2. In §3, we prove some theorems about linear semigroups in a function space which yield examples of our main results and are of some interest in themselves.

1. The main results. Let X be a complex Banach space. If $0 < \phi \leq \pi/2$, then let $S_\phi = \{z \in C : z = 0 \text{ or } |\arg z| \leq \phi\}$, where C denotes the complex plane. Following [6], we denote by $CH(\phi)$ the collection of all semigroups $\{T(z) : z \in S_\phi\}$ of linear contraction operators in X which are holomorphic on $\text{int}(S_\phi)$ and strongly continuous on S_ϕ . We denote by $GH(\phi)$ the collection of all infinitesimal generators of semigroups in $CH(\phi)$.

Let $[a, b]$ be a closed real number interval, $0 < \phi \leq \pi/2$, and S a closed set in X . Let A be a function from $[a, b] \times S \rightarrow GH(\phi)$ such that the following conditions are satisfied:

(C₁) The operators $A(t, x)$ all have the same domain D_0 .

(C₂) There is a locally bounded nonnegative function K on $[a, b] \times S$ such that

$$\|[I - A(t, y)][I - A(s, x)]^{-1} - I\| \leq K(s, x)(|s - t| + \|x - y\|)$$

for $a \leq s, t \leq b$ and $x, y \in S$, where I denotes the identity transformation on X ,

(C₃) $(\exp [\xi A(t, x)])S \subset S$ for $\xi \geq 0, a \leq t \leq b$, and $x \in S$, where

$$\{\exp [zA(t, x)]; z \in S_\phi\}$$

is the class $CH(\phi)$ semigroup generated by $A(t, x)$.

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In connection with the condition (C_2) , we mention that the invertibility of $I - A(s, x)$ follows from the fact that $A(s, x) \in \text{GH}(\phi)$, see [6], or [2], which serves also as a general reference for semigroups of operators. The fact that $[I - A(t, y)] \cdot [I - A(s, x)]^{-1}$ is bounded follows from (C_1) and [3, Lemma 2, p. 212].

THEOREM 1. *Suppose $x_0 \in D_0 \cap S$ and $a \leq t_0 < b$. Then there is a number c in $(t_0, b]$ such that there is a unique continuously differentiable function f from $[t_0, c]$ into $D_0 \cap S$ satisfying $f(t_0) = x_0$ and*

$$(*) \quad f'(t) = A(t, f(t))f(t) \quad \text{for } t_0 \leq t \leq c.$$

Also, if $\varepsilon > 0$, then there exists $\delta > 0$ such that if $x_1 \in D_0 \cap S$, $t_0 < c_1 \leq c$, $\|x_0 - x_1\| < \delta$, and g is a continuously differentiable function from $[t_0, c_1]$ into $D_0 \cap S$ such that $g(t_0) = x_1$ and $g'(t) = A(t, g(t))g(t)$ for $t_0 \leq t \leq c_1$, then $\|g(t) - f(t)\| < \varepsilon$ for $t_0 \leq t \leq c_1$.

DEFINITION 1.1. A semi-inner product on X means a function $[\cdot, \cdot]$ from $X \times X$ into C such that for each $y \in Y$, $[\cdot, y]$ is a bounded linear functional of norm $\|y\|$, and $[y, y] = \|y\|^2$ (see [5]).

DEFINITION 1.2. If $[\cdot, \cdot]$ is a semi-inner product on X , then a transformation W with domain and range contained in X is said to be *dissipative* (with respect to $[\cdot, \cdot]$) if $\text{Re} [Wx - Wy, x - y] \leq 0$ for $x, y \in D(W)$, the domain of W .

REMARK. Throughout this section, $[\cdot, \cdot]$ will denote a fixed semi-inner product on X , and all results will be independent of the particular semi-inner product used.

THEOREM 2. *Suppose H is a function from S into $\text{GH}(\phi)$ which satisfies conditions (C_1) , (C_2) , and (C_3) ; more precisely, the function $(t, x) \rightarrow H(x)$ satisfies these conditions. Suppose $D_0 \cap S$ is dense in S and define Q on $D_0 \cap S$ by $Qx = H(x)x$. Suppose Q is dissipative. Then there is a unique strongly continuous semigroup $\{T(t); t \geq 0\}$ of nonexpansive nonlinear transformations from S into S such that for each x in $D_0 \cap S$, $T(\cdot)x$ is a continuously differentiable function from $[0, \infty)$ into $D_0 \cap S$, and $(d/dt)T(t)x = QT(t)x$ for $t \geq 0$.*

2. Proof of the main theorems. We will call a function B from a number interval $[0, R]$ into $\text{GH}(\phi)$ *regular* if the following conditions are satisfied:

(R₁) $B(t)$ has domain D_0 for $0 \leq t \leq R$.

(R₂) There is a positive constant L such that

$$\|[I - B(t)][I - B(s)]^{-1} - I\| \leq L|t - s|$$

for $0 \leq s, t \leq R$.

(R₃) $(\exp [\xi B(t)])S \subset S$ for $\xi \geq 0$ and $0 \leq t \leq R$.

We point out that a regular operator function B on $[0, R]$ also satisfies:

(R₄) $\|[I - B(t)][I - B(s)]^{-1}\| \leq 1 + LR$ for $0 \leq s, t \leq R$, where L is as in **(R₂)**.

$$\begin{aligned}
 \text{(R}_5\text{)} \quad & \|[I - B(r)][I - B(s)]^{-1} - [I - B(t)][I - B(s)]^{-1}\| \\
 & \leq \|[I - B(r)][I - B(t)]^{-1} - I\| \cdot \|[I - B(t)][I - B(s)]^{-1}\| \\
 & \leq |r - t|(1 + LR)L.
 \end{aligned}$$

LEMMA 2.1. *Suppose B is a regular operator function on $[0, R]$, and β is a positive nonincreasing function on $[0, R]$ with Lipschitz constant L' . Define the operator function A on $[0, R]$ by $A(t) = \beta(t)[B(t) - I]$.*

Then A satisfies Tanabe's conditions 1° and 2° of [8]. In particular, let $0 < \phi_1 < \phi$ and define

$$\Sigma = \{\lambda \in C : \lambda = 0 \text{ or } |\arg \lambda| \leq \phi_1 + \pi/2\}.$$

Also define

$$M = [\beta(R) \sin(\phi - \phi_1)]^{-1} [(1 - \sin \phi_1)/2]^{-1/2},$$

$$K = (L\beta(0) + L')(1 + LR)/\beta(R),$$

where L is as in (R_2) . Then A satisfies the conditions:

(T_1) $\rho(A(t)) \supset \Sigma$ for $0 \leq t \leq R$, and

$$\|[\lambda I - A(t)]^{-1}\| \leq M/(|\lambda| + 1)$$

for $\lambda \in \Sigma$ and $0 \leq t \leq R$. (If T is an operator in X , then $\rho(T)$ denotes the resolvent set of T .)

(T_2) $\|A(r)A(s)^{-1} - A(t)A(s)^{-1}\| \leq K|r - t|$ for $0 \leq r, s, t \leq R$.

Proof. Let $0 \leq s, t, r \leq R$. Define

$$\Delta_\phi = \{\lambda \in C : \lambda = 0 \text{ or } |\arg \lambda| \geq \phi + \pi/2\}.$$

Then $\beta(t)B(t) \in \text{GH}(\phi)$, and $\Sigma \subset C \setminus \Delta_\phi$, so $\rho(\beta(t)B(t)) \supset \Sigma$ and $\|[\lambda I - \beta(t)B(t)]^{-1}\| \leq 1/d(\lambda, \Delta_\phi)$ for $\lambda \in \Sigma$, see [6]. Also $\lambda I - A(t) = [\lambda + \beta(t)]I - \beta(t)B(t)$, and $\lambda + \beta(t) \in \Sigma$ if $\lambda \in \Sigma$, so $\|[\lambda I - A(t)]^{-1}\| \leq 1/d(\lambda + \beta(t), \Delta_\phi)$ for $\lambda \in \Sigma$. Property (T_1) follows from this and the fact that $d(\lambda + \beta(t), \Delta_\phi) \geq (|\lambda| + 1)/M$ for $\lambda \in \Sigma$.

Let $A_\xi = A(\xi)$, $B_\xi = B(\xi)$, and $\beta_\xi = \beta(\xi)$ for $0 \leq \xi \leq R$. Then

$$A_r A_s^{-1} - A_t A_s^{-1} = \beta_s^{-1} \beta_r [(I - B_r)(I - B_s)^{-1} - (I - B_t)(I - B_s)^{-1}]$$

$$+ \beta_s^{-1} (\beta_r - \beta_t) (I - B_t)(I - B_s)^{-1},$$

so that (T_2) follows from (R_4) and (R_5) .

LEMMA 2.2. *Let F be a function from $[0, R]$ into $[a, b]$ with Lipschitz constant L_1 , and ψ a function from $[0, R]$ into S with Lipschitz constant L_2 . Define the operator function B on $[0, R]$ by*

$$B(t) = A(F(t), \psi(t)).$$

Then B is regular, where we can take the constant L of (R_2) as

$$L = (L_1 + L_2) \sup_{0 \leq t \leq R} K(F(t), \psi(t)),$$

see (C_2) .

Thus if β is positive, nonincreasing, and Lipschitz continuous on $[0, R]$, and we define $A(t) = \beta(t)[B(t) - I]$ for $0 \leq t \leq R$, then A satisfies (T_1) and (T_2) .

If in addition, $\beta(t) - \beta(s) \leq -(t - s)\beta(0)L$ for $0 \leq s \leq t \leq R$, then A also satisfies

(T_3) $\|A(t)A(s)^{-1}\| \leq 1$ for $0 \leq s \leq t \leq R$.

Proof. Only the last statement needs proof, and it follows from the fact that $\beta(t)\beta(s)^{-1} \leq e^{[\beta(t) - \beta(s)]/\beta(s)}$, and $\|(I - B(t))(I - B(s))^{-1}\| \leq e^{L(t - s)}$ for $0 \leq s \leq t \leq R$.

LEMMA 2.3. Let the operator function $A: [0, R] \rightarrow \text{GH}(\phi)$ be as in Lemma 2.1, and let $x_0 \in D_0$. Then there is a unique continuously differentiable function f from $[0, R]$ into D_0 such that $f(0) = x_0$ and $f'(t) = A(t)f(t)$ for $0 \leq t \leq R$.

Proof. Tanabe establishes much more than this in [8].

LEMMA 2.4. Let A , x_0 , and f be as in Lemma 2.3. If $\Delta = \{t_0, \dots, t_n\}$ is a partition of $[0, R]$, then let f_Δ be defined on $[0, R]$ by $f_\Delta(0) = x_0$, and

$$f_\Delta(t) = T_k(t - t_{k-1})f_\Delta(t_{k-1}) \quad \text{for } t_{k-1} \leq t \leq t_k,$$

where $T_k(\xi) = \exp[\xi A(t_k)]$. Then f_Δ converges uniformly to f on $[0, R]$ as the norm of Δ approaches zero.

Proof. Define A_Δ on $[0, R]$ by $A_\Delta(0) = A(t_1)$ and $A_\Delta(t) = A(t_k)$ for $t_{k-1} < t \leq t_k$. Then $f'_\Delta(t) = A_\Delta(t)f_\Delta(t)$ for $t \in [0, R] \setminus \Delta$.

Let $h_\Delta(t) = f(t) - f_\Delta(t)$ for $0 \leq t \leq R$. Then

$$\begin{aligned} h'_\Delta(t) &= [A(t) - A_\Delta(t)]f(t) + A_\Delta(t)h_\Delta(t) \\ &= [I - A_\Delta(t)A(t)^{-1}]f'(t) + A_\Delta(t)h_\Delta(t). \end{aligned}$$

By [4, Lemma 1.3, p. 510], $\|h_\Delta(t)\|(d/dt)\|h_\Delta(t)\| = \text{Re}[h'_\Delta(t), h_\Delta(t)]$ a.e. on $[0, R]$. Thus $(d/dt)\|h_\Delta(t)\| \leq K|\Delta|\Lambda$ a.e. on $[0, R]$, where K is as in (T_2) ,

$$\Lambda = \sup_{0 \leq t \leq R} \|f'(t)\|,$$

and $|\Delta|$ denotes the norm of Δ . We have used the fact that $A_\Delta(t)$ is dissipative, see [5].

LEMMA 2.5. Let A , x_0 , and f be as in Lemma 2.3. Then

$$\|f(t)\| \leq \|f(0)\| \exp \left[- \int_0^t \beta \right]$$

for $0 \leq t \leq R$.

If A satisfies (T_3) , then $\|f'(t)\| \leq \|f'(0)\| \exp[-\int_0^t \beta]$ for $0 \leq t \leq R$.

Proof. Let Δ, f_Δ , and T_k be as in Lemma 2.4. Then $T_k(\xi) = e^{-\xi\beta(t_k)} \exp[\xi\beta(t_k)B(t_k)]$, so that $\|T_k(\xi)\| \leq e^{-\xi\beta(t_k)}$. Therefore,

$$\|f_\Delta(t)\| \leq \|f(0)\| \exp \left[-\beta(t_k)(t - t_{k-1}) - \sum_{j=1}^{k-1} \beta(t_j)(t_j - t_{j-1}) \right]$$

for $t_{k-1} \leq t \leq t_k$, and the first conclusion follows.

Define $X_k = T_k(t_k - t_{k-1})$, $A_k = A(t_k)$, and $\beta_k = \beta(t_k)$ for $k = 0, 1, \dots, n$.

If $t_{k-1} < t < t_k$, then

$$\begin{aligned} f'_\Delta(t) &= A_k f_\Delta(t) \\ &= A_k T_k(t - t_{k-1}) X_{k-1} \cdots X_1 \\ &= T_k(t - t_{k-1}) A_k A_{k-1}^{-1} A_{k-1} X_{k-1} \cdots X_1 \\ &= T_k(t - t_{k-1}) A_k A_{k-1}^{-1} X_{k-1} \cdots X_1 A_1 A_0^{-1} A_0 x_0, \end{aligned}$$

and the second conclusion follows.

LEMMA 2.6. Let A , x_0 , and f be as in Lemma 2.3, but add the condition that $x_0 \in S$. Then $(\exp [\int_0^t \beta])f(t) \in S$ for $0 \leq t \leq R$.

Proof. Let Δ and f_Δ be as in Lemma 2.4. From (R_3) and the construction of f_Δ , we get

$$\left(\exp \left[(t-t_{k-1})\beta(t_{k-1}) + \sum_{j=1}^{k-1} \beta(t_j)(t_j-t_{j-1}) \right] \right) f_\Delta(t) \in S$$

for $t_{k-1} \leq t \leq t_k$.

2.7. **Proof of Theorem 1.** Choose $\delta > 0$ so that $K(t, x)$ (see condition (C_2)) is bounded for $|t-t_0| \leq \delta$, $\|x-x_0\| \leq \delta$. Let K_0 be an upper bound for $K(t, x)$ on this set, with $K_0 > 1$, $(1/\delta)$. Let

$$\begin{aligned} y_0 &= A(t_0, x_0)x_0, & \gamma &= 2K_0(1+2\|x_0\| + \|y_0\|), \\ c &= \min [b, t_0 + (1/2\gamma)], & R &= -\gamma^{-1} \ln (1-\gamma(c-t_0)). \end{aligned}$$

We will need the following two inequalities, which follow immediately from the above definitions:

$$(2.7.1) \quad R(2\|x_0\| + \|y_0\|) \leq \delta,$$

$$(2.7.2) \quad c-t_0 \leq \delta.$$

Define F from $[0, R]$ onto $[t_0, c]$ by $F(\tau) = t_0 + \gamma^{-1}[1 - e^{-\gamma\tau}]$. Define β on $[0, R]$ by $\beta(\tau) = e^{-\gamma\tau}$, and define G from $[t_0, c]$ onto $[0, R]$ by $G(t) = -\gamma^{-1} \ln [1 - \gamma(t-t_0)]$. Then

$$(2.7.3) \quad F(G(t)) = t, \quad G(F(\tau)) = \tau,$$

$$(2.7.4) \quad G'(t)\beta(G(t)) = 1,$$

$$(2.7.5) \quad \int_0^\tau \beta = F(\tau) - t_0.$$

Define α on $[0, R]$ by $\alpha(\tau) = \exp (\int_0^\tau \beta)$.

We intend to solve (*) by first solving

$$(**) \quad g'(\tau) = \beta(\tau)[A(F(\tau), \alpha(\tau)g(\tau)) - I]g(\tau),$$

and then making the substitution $f(t) = e^{t-t_0}g(G(t))$. (2.7.3), (2.7.4), and (2.7.5) are the pertinent identities for showing that this yields a solution of (*).

We define inductively the sequence $\{g_n\}$ of functions on $[0, R]$ as follows:

$$g_0(\tau) = x_0, \quad g_{n+1}(0) = x_0, \quad g'_{n+1}(\tau) = A_n(\tau)g_{n+1}(\tau),$$

where

$$A_n(\tau) = \beta(\tau)[A(F(\tau), \psi_n(\tau)) - I], \quad \psi_n(\tau) = \alpha(\tau)g_n(\tau).$$

We see that this inductive definition is possible by Lemmas 2.2, 2.3, and 2.6.

We will need the fact that each of the operator functions A_n has property (T_3) ; in fact this is the reason for our change of variable. Define $B_n(\tau) = A(F(\tau), \psi_n(\tau))$ for $0 \leq \tau \leq R$, and $n=0, 1, 2, 3, \dots$. Then each B_n is regular by Lemma 2.2. For each n , let $L^{(n)}$ denote the least constant L that will work in condition (R_2) for B_n . Notice that

$$[\beta(\tau) - \beta(\sigma)]/\beta(0) \leq -\gamma B(R)(\tau - \sigma) \leq -(\gamma/2)(\tau - \sigma)$$

for $0 \leq \sigma \leq \tau \leq R$. Thus by Lemma 2.2, A_n satisfies (T_3) if $L^{(n)} \leq (\gamma/2)$. In order to show this we will need

$$(2.7.6) \quad |F'(\tau)| = |e^{-\gamma\tau}| \leq 1,$$

$$(2.7.7) \quad |\alpha'(\tau)| = \left| \exp \left[-\gamma\tau + \int_0^\tau \beta \right] \right| \leq 1,$$

$$(2.7.8) \quad |F(\tau) - t_0| \leq \delta.$$

Thus, we have $L_0 \leq K_0(1 + \|x_0\|) < \gamma/2$ since $\|\psi_0(\tau) - x_0\| < R\|x_0\| \leq \delta$, $\|\psi'_0(\tau)\| \leq \|x_0\|$, so that A_0 has property (T_3) .

Suppose A_n has property (T_3) . Then

$$\begin{aligned} \psi'_{n+1}(\tau) &= \alpha(\tau)g'_n(\tau) + \alpha'(\tau)g_n(\tau), \\ \|\psi'_{n+1}(\tau)\| &\leq \|g'_n(0)\| + \|x_0\| \leq 2\|x_0\| + \|y_0\| \end{aligned}$$

by Lemma 2.5, and (2.7.7). Therefore,

$$(2.7.9) \quad \|\psi_{n+1}(\tau) - x_0\| \leq \delta$$

by (2.7.1). Thus $L^{(n+1)} \leq K_0(1 + 2\|x_0\| + \|y_0\|) = \gamma/2$ by (2.7.6), (2.7.8), and Lemma 2.2. Thus A_{n+1} also has property (T_3) .

Thus, we have

$$(2.7.10) \quad \|g'_n(\tau)\| \leq (\|y_0\| + \|x_0\|)/\alpha(\tau)$$

for $0 < \tau \leq R$, and $n=0, 1, 2, \dots$ by Lemma 2.5.

For each $n=1, 2, 3, \dots$, define h_n on $[0, R]$ by $h_n(\tau) = g_{n+1}(\tau) - g_n(\tau)$. Then

$$\begin{aligned} h'_n(\tau) &= A_n(\tau)g_{n+1}(\tau) - A_{n-1}(\tau)g_n(\tau) \\ &= [A_n(\tau) - A_{n-1}(\tau)]g_{n+1}(\tau) + A_{n-1}(\tau)h_n(\tau) \\ &= [I - A_{n-1}(\tau)A_n(\tau)^{-1}]g'_{n+1}(\tau) + A_{n-1}(\tau)h_n(\tau). \end{aligned}$$

By [4, Lemma 1.3, p. 510], we have

$$\|h_n(\tau)\|(d/d\tau)\|h_n(\tau)\| = \operatorname{Re} [h'_n(\tau), h_n(\tau)]$$

a.e. on $[0, R]$, so that $(d/d\tau)\|h_n(\tau)\| \leq K_0(\|x_0\| + \|y_0\|)\|h_{n-1}(\tau)\|$ a.e. on $[0, R]$. We have used the fact that $A_{n-1}(\tau)$ is dissipative (see [5], property (C_2) , (2.7.8), (2.7.9), and (2.7.10).

Therefore, $\{g_n\}$ converges uniformly to a function g on $[0, R]$. Also

$$\|g(\tau) - g(\sigma)\| \leq (\|x_0\| + \|y_0\|)|\tau - \sigma|$$

for $0 \leq \sigma, \tau \leq R$. Let $\psi(\tau) = \alpha(\tau)g(\tau)$, $0 \leq \tau \leq R$. Then

$$\|\psi(\tau) - \psi(\sigma)\| \leq (2\|x_0\| + \|y_0\|)|\tau - \sigma|,$$

and $\|\psi(\tau) - x_0\| \leq \delta$ for $0 \leq \tau \leq R$. Define the operator function A on $[0, R]$ by $A(\tau) = \beta(\tau)[A(F(\tau), \psi(\tau)) - I]$. Then A has properties (T₁), (T₂), and (T₃) (we will not need (T₃)). Define u on $[0, R]$ by $u(0) = x_0$, $u'(\tau) = A(\tau)u(\tau)$.

We wish to show that $u = g$. Let $u_n = u - g_n$. An argument similar to the one used to show that $\{h_n\}$ converges to 0 will show that $\{u_n\}$ converges to 0. Therefore g satisfies (**), and the function f defined on $[t_0, c]$ by $f(t) = e^{t-t_0}g(G(t))$ satisfies (*). Note also that $f(t) = \psi(G(t))$, $\|f(t) - x_0\| \leq \delta$ for $y_0 \leq t \leq c$.

Suppose $x_1 \in D_0 \cap S$, $t_0 < c_1 \leq c$, and v is a continuously differentiable function from $[t_0, c_1]$ into $D_0 \cap S$ such that $v(t_0) = x_1$ and $v'(t) = A(t, v(t))v(t)$ for $t_0 \leq t \leq c_1$. Define w on $[t_0, c_1]$ by $w(t) = f(t) - v(t)$. Then

$$\begin{aligned} w'(t) &= [A(t, f(t)) - A(t, v(t))]f(t) + A(t, v(t))w(t) \\ &= ([I - A(t, v(t))][I - A(t, f(t))]^{-1} - I)(f(t) - f'(t)) + A(t, v(t))w(t), \\ (d/dt)\|w(t)\| &\leq K_0\|w(t)\|(\|f(t)\| + \|f'(t)\|) \end{aligned}$$

a.e. on $[t_0, c_1]$. The stability claim, and hence the uniqueness claim, follow from this differential inequality.

2.8. Proof of Theorem 2. First we mention that if we prove that for each $x \in D_0 \cap S$, there exists a continuously differentiable function f from $[0, \infty)$ into $D_0 \cap S$ such that $f(0) = x$ and $f'(t) = Qf(t)$ for $t \geq 0$, then the rest of the theorem follows in routine manner. We define $T_0(t)x = f(t)$ for $x \in D_0 \cap S$ and $t \geq 0$. The fact that T_0 is nonexpansive on $D_0 \cap S$ follows from the fact that Q is dissipative. Thus each $T_0(t)$ has a unique extension to a nonexpansive transformation $T(t)$ from S into S . $\{T(t); t \geq 0\}$ is the desired semigroup.

Now we return to the first question. Let $x_0 \in D_0 \cap S$. Then by Theorem 2, there is a number $c > 0$ such that there is a unique continuously differentiable function f from $[0, c]$ into $D_0 \cap S$ such that $f(0) = x_0$ and $f'(t) = Qf(t)$ for $0 \leq t \leq c$. Let ζ denote the supremum of the set of all such numbers c , and suppose that $\zeta < \infty$. Let f denote the unique continuously differentiable function from $[0, \zeta)$ into $D_0 \cap S$ such that $f(0) = x_0$ and $f'(t) = Qf(t)$ for $0 \leq t < \zeta$.

If $0 < h < \zeta$, then define f_h on $[0, \zeta - h]$ by $f_h(t) = f(t+h) - f(t)$. Then $f_h'(t) = Qf(t+h) - Qf(t)$, and $(d/dt)\|f_h(t)\| \leq 0$ a.e. on $[0, \zeta - h]$ since Q is dissipative, so that $\|(1/h)[f(t+h) - f(t)]\| \leq \|(1/h)(f(h) - f(0))\|$ for $0 \leq t < \zeta - h$. Therefore $\|f'(t)\| \leq \|f'(0)\|$ for $0 \leq t \leq \zeta$, and thus $x_1 = \lim_{t \rightarrow \zeta} f(t)$ exists. Therefore $f([0, \zeta))$ is relatively compact and $K(t, f(t))$ (see (C₂)) is bounded on $[0, \zeta)$. Using this and the fact that $f(t)$ and $f'(t)$ are also bounded on $[0, \zeta)$, we see by examining the argument for Theorem 1 that there is a positive constant η such that for each t in $[0, \zeta)$, there is a unique continuously differentiable function g from $[t, t + \eta]$ into $D_0 \cap S$ such that $g(t) = f(t)$ and $g'(s) = Qg(s)$ for $t \leq s \leq t + \eta$. Simply take $\zeta - \eta < t < \zeta$, and use the corresponding function g to extend f beyond ζ .

3. Semigroups in a function space. Let E be a set, $B(E)$ the Banach space of bounded complex valued functions on E with supremum norm, and Y a closed real or complex subspace of $B(E)$. We denote by Ω the collection of all positive bounded functions p on E which are bounded away from zero and have the property that $pY \subset Y$.

If Y is complex, we take $\text{CH}(\phi)$ and $\text{GH}(\phi)$ as defined in §1, with $X=Y$. If Y is a real Banach lattice, then CP denotes the collection of all strongly continuous semigroups of linear positive contraction operators in Y , and GP denotes the collection of infinitesimal generators of such semigroups. In either case, G denotes the collection of all infinitesimal generators of strongly continuous semigroups of linear contraction operators in Y .

If $y \in Y$, then $\gamma(y)$ denotes a multiplicative linear functional on $B(E)$ such that $|\langle y, \gamma(y) \rangle| = \|y\|$. We define the semi-inner product $[\cdot, \cdot]$ on $Y \times Y$ by $[x, y] = \langle x, \gamma(y) \rangle \langle y, \gamma(y) \rangle^*$, where $*$ denotes complex conjugation. All reference to a semi-inner product in this section will be to this one just defined. One special property of $[\cdot, \cdot]$ which is useful to us is that $[px, y] = \langle p, \gamma(y) \rangle [x, y]$ for $x, y \in Y$, $p \in B(E)$, $px \in Y$. Also, if Y is a real Banach lattice, then $[\cdot, \cdot]$ has the special properties required in [7]. That is, $[\cdot, y]$ is a positive linear functional if $y \geq 0$, and $[x, x^+] = \|x^+\|^2$ for each x in Y , where x^+ denotes the positive part of x .

By Definition 1.2, a linear operator A in Y is dissipative if $\text{Re}[Ay, y] \leq 0$ for $y \in D(A)$. Following [6], in case Y is complex, we say that a linear operator A in Y is ϕ -sectorial if $e^{i\theta}A$ is dissipative for $|\theta| \leq \phi$. Following [7], in case Y is a real lattice, we say that a linear operator A in Y is *dispersive* if $[Ax, x^+] \leq 0$ for all $x \in D(A)$.

LEMMA 3.1. *A linear operator A in Y is in $(\text{G}, \text{GP}, \text{GH}(\phi))$ if and only if $D(A)$ is dense in Y , the range of $I-A$ is all of Y , and A is (dissipative, dispersive, ϕ -sectorial).*

Proof. The proof of this lemma is contained in [5], [7], and [6], respectively. We merely state the lemma here for reference in proving the next theorem, which is a generalization of the author's earlier theorem in [1].

THEOREM 3.1. *Suppose $A \in \text{G}$, and $A = A_1 + \cdots + A_n$, where each A_j has domain $D(A)$, and each A_j has a closed extension. If each A_j is (dissipative, dispersive, ϕ -sectorial), and $p_1, \dots, p_n \in \Omega$, then $p_1A_1 + \cdots + p_nA_n \in (\text{G}, \text{GP}, \text{GH}(\phi))$.*

Proof. $p_1A_1 + \cdots + p_nA_n$ is easily seen to be (dissipative, dispersive, ϕ -sectorial). Thus by Lemma 3.1, we need only show that the range of $I - (p_1A_1 + \cdots + p_nA_n)$ is all of Y .

We will first prove that the range of $I - (p_1A_1 + A_2 + \cdots + A_n)$ is all of Y . By [3, Lemma 2, p. 212], the operator $U_1 = A_1(I - A)^{-1}$ is bounded. Since $F(p_1)Y \subset Y$ for every polynomial F , then $p_1^{(1/m)} \in \Omega$ for every positive integer m by the classical Weierstrass theorem. Choose m so that $\|1 - p_1^{(1/m)}\| < \|U_1\|^{-1}$, and let $r = p_1^{1/m}$. Then

$$I - (rA_1 + A_2 + \cdots + A_n) = I - A + (1-r)A_1 = (I + (1-r)U_1)(I - A).$$

Thus the range of $I - (rA_1 + A_2 + \cdots + A_n)$ is all of Y . Replacing A_1 by $rA_1, r^2A_1,$ etc., we see that the range of $I - (p_1A_1 + A_2 + \cdots + A_n)$ is all of Y .

Now we consider the operator $A' = A_2 + p_1A_1 + A_3 + \cdots + A_n$ and repeat the previous argument to prove that the range of $I - (p_1A_1 + p_2A_2 + A_3 + \cdots + A_n)$ is all of Y . Repeating this process proves the theorem.

EXAMPLE. Let E denote real Euclidean n -space, and let Y denote any of the subspaces of $B(E)$ in which the Laplacian operator generates a strongly continuous semigroup. The semigroup will then consist of contraction operators and will be in $\text{CH}(\phi)$ if Y is complex, in CP if Y is a real lattice. Let A denote the Laplacian operator in Y , and for each $j = 1, \dots, n$, let A_j denote the restriction of $(\partial^2/\partial s_j^2)$ to the domain of A .

LEMMA 3.2. *Let A be in G with $A = A_1 + \cdots + A_n$, where each A_j has domain $D(A)$, each A_j has a closed extension, and each A_j is dissipative. Define the function P from $\Omega^{(n)}$ into G by $P(p) = p_1A_1 + \cdots + p_nA_n$.*

Then there is a locally bounded nonnegative function K on $\Omega^{(n)}$ such that

$$\| [I - P(q)][I - P(p)]^{-1} - I \| \leq \left(\sum \|q_i - p_i\| \right) K(p)$$

for $p, q \in \Omega^{(n)}$.

Proof. If $p, q \in \Omega^{(n)}$, then

$$[I - P(q)][I - P(p)]^{-1} - I = [P(p) - P(q)][I - P(p)]^{-1} = \sum (p_i - q_i)A_i[I - P(p)]^{-1}.$$

There we can take $K(p) = \max_i \|A_i[I - P(p)]\|^{-1}$.

To see that $K(p)$ is locally bounded, notice that

$$A_i(I - P(r))^{-1} = A_i(I - P(p))^{-1} \left(I + \sum (p_i - r_i)A_i(I - P(p))^{-1} \right)^{-1},$$

so that $K(r) \leq K(p)/(1 - K(p) \sum \|p_i - r_i\|)$ for

$$K(p) \sum \|p_i - r_i\| < 1.$$

THEOREM 3.3. *Let B be in $\text{GH}(\phi)$ with $B = B_1 + \cdots + B_n$, where each B_j has a closed extension, each B_j has domain $D(A) = D_0$, and each B_j is ϕ -sectorial. Let S be a closed set in Y , $[a, b]$ a closed interval, and p a Lipschitz continuous function from $[a, b] \times S$ into $\Omega^{(n)}$. Define the operator function A from $[a, b] \times S$ into $\text{GH}(\phi)$ by $A(t, x) = \sum p_i(t, x)A_i$. Then A satisfies conditions (C_1) and (C_2) .*

Proof. This follows from Lemma 3.2.

There are a variety of ways in which the set S in Theorem 3.2 could be chosen in order that the operator function A will satisfy (C_3) , and it seems inappropriate to state any theorems about this. It is not quite so easy to choose H and S so that Q will be dissipative as in Theorem 2, but we will indicate one way in which it can be done.

Let Y be complex, let Y_0 denote the space of real functions in Y , and suppose Y_0 is a lattice. Let $A \in \text{GH}(\phi)$, $A = A_1 + \cdots + A_n$ as in Theorem 3.2. Let $D_{00} = D_0 \cap Y_0$,

$A^0 = A|_{D_{00}}$, $A_j^0 = A_j|_{D_{00}}$, and suppose that A^0 , A_j^0 satisfy the portion of Theorem 3.1 dealing with positive semigroups. Let Y_{00} denote the nonpositive functions in Y_0 , let $S_0 = \bigcap (A_j^0)^{-1} Y_{00}$, and let S denote the closure of S_0 . Let p_1, \dots, p_n be Lipschitz continuous accretive ($-p_j$ dissipative) functions from S into Ω . Define H from S onto $\text{GH}(\phi)$ by $H(x) = \sum p_i(x)A_i$. Then the hypothesis of Theorem 2 is satisfied. This can all be done taking Y, A, A_j as in the example after Theorem 3.1 which dealt with the Laplacian operator.

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