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FIBERING 3-MANIFOLDS THAT ADMIT FREE Z_k ACTIONS

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1. **Introduction.** In a recent paper [3], Kwun showed that closed orientable 3-manifolds which double-cover themselves fiber over the circle (some technical restrictions are placed on the manifolds). In doing so he applied a criterion for fibering due to Stallings [6]. In this paper we extend Kwun's approach to show that certain 3-manifolds admitting a free Z_k action fiber over the circle.

Recall that a free Z_k action on a space M is an effective action on M by a cyclic group of order k with only the identity having fixed points. A *proper* Z_k action is one with the property that a generator of the action is homotopic to the identity homeomorphism. M^* will be used to denote the orbit space M/Z_k. We let $p\colon M \to M^*$ be the projection map throughout this paper. Singular homology and cohomology with integer coefficients will be used exclusively. All manifolds are connected.

Our goal is to establish the following

THEOREM. *Let M be a compact, connected, orientable, irreducible 3-manifold with* Bd M *either empty or connected. If M admits a proper free Z_k action, for some prime $k \geqq 2$, such that $H_1(M^*; Z)$ has no element of order k then M can be fibered over the circle.*

Examples of 3-manifolds admitting proper free Z_k actions are plentiful. One ready source of nontrivial examples is the class of closed 3-manifolds admitting effective $SO(2)$ actions. In this class we find that the lens space $L(\beta, -\alpha (\mathrm{mod}\ \beta))$ admits proper free Z_k actions for $(k, \alpha)=1$. Moreover this space does not fiber over the circle and the first homology group of the orbit space always has elements of order k. Other examples from this class indicate that we cannot drop from our theorem the requirement that the Z_k action be proper. For a classification of $SO(2)$ actions on 3-manifolds the reader is referred to [4].

2. **Partitioning** M. In this section we present some lemmas leading to the main result which we prove in §§3 and 4. The first lemma is a collection of well-known facts.

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LEMMA 1. *Let M be a compact orientable irreducible* 3-*manifold such that* Bd M *is either empty or connected. If either $\pi_1(M)$ is infinite or the genus of* Bd M *is positive then M is a $K(\pi, 1)$ space and $\pi=\pi_1(M)$ has no elements of finite order.*

The triple (M, p, M^*) will be considered as a principal Z_k-bundle in the sequel. For properties of bundles the reader is referred to [2] and [7].

LEMMA 2. *Let M be a compact* 3-*manifold admitting a free Z_k action, where $k \geqq 2$ is prime. If $H_1(M^*; Z)$ has no k-torsion then there is a bundle map*

$$\bar{g}\colon (M, p, M^*) \to (S^1, p', S^1),$$

where p' is the standard k to 1 *covering projection of S^1.*

Proof. There is a map $f\colon M^* \to L_\infty$ such that the principal Z_k-bundle $\xi=(M, p, M^*)$ is induced by the universal bundle $\eta=(S^\infty, p_\infty, L_\infty)$. We take as η the bundle obtained by the following construction. Let (S^i, p_i, L_i) be the standard k to 1 covering of the generalized lens space $L_i=L(k; 1, \ldots, 1)$ by S^i [5, p. 88]. Consider the diagram

$$\begin{array}{ccccccc} S^1 & \subset & S^3 & \subset & S^5 & \subset & \cdots \\ {\scriptstyle p_1}\downarrow & & {\scriptstyle p_3}\downarrow & & {\scriptstyle p_5}\downarrow & & \\ L_1 & \subset & L_3 & \subset & L_5 & \subset & \cdots \end{array}$$

where S^{i+2} is regarded as the double suspension of S^i. Define S^∞ and L_∞ to be the unions $\bigcup S^i$ and $\bigcup L_i$ respectively, with the weak topology. S^∞ is contractible, hence η is a universal Z_k-bundle [7, p. 101].

The set of homotopy classes $[M^*, L_\infty]$ is in a one-to-one correspondence with the homomorphisms from $\pi_1(M^*)$ into Z_k since L_∞ is a $K(Z_k, 1)$ space [1, p. 198]. Thus there is a homomorphism $\varphi\colon \pi_1(M^*) \to Z_k$ that completely determines the homotopy class of f. ξ is not the trivial bundle so φ is onto. Moreover Z_k being abelian implies that φ must factor through $H_1(M^*)$, the abelianization of $\pi_1(M^*)$. Since $H_1(M^*)$ is finitely generated and has no k torsion it follows that $H_1(M^*)$ has a free part mapping onto Z_k through which φ can be factored. Let

$$\pi_1(M^*) \xrightarrow{\beta} Z \xrightarrow{\gamma} Z_k$$

be such that $\varphi=\gamma\beta$.

Because S^1 and L_∞ are $K(\pi, 1)$ spaces there are maps $c\colon S^1 \to L_\infty$ and $g\colon M^* \to S^1$ corresponding to γ and β respectively. We may just as well suppose $f=cg$. The induced Z_k-bundle $c^*(\eta)$ is equivalent to the standard k-sheeted covering of S^1 by itself. Since $\xi \simeq f^*(\eta) \simeq g^*(c^*(\eta))$ we have shown the existence of a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\bar{g}} & S^1 \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle p'} \\ M^* & \xrightarrow{g} & S^1 \end{array}$$

where p' is the standard k to 1 covering of S^1 and $\bar{g}$ is the required bundle map.

Suppose M admits a free Z_k action generated by the homeomorphism h. We call (U, T) an *h-partition* of M if T is the disjoint union of connected two-sided 2-manifolds regularly embedded in M (i.e. Bd $T = T \cap$ Bd M) such that the following conditions are satisfied:

(1) $M = \bigcup_{i=1}^{k} (h^i(U) \cup h^i(T))$,
(2) U is open in M,
(3) Fr $U = T \cup h(T)$,
(4) $h^i(U) \cap h^j(U) = \varnothing$ if $i \not\equiv j \pmod{k}$,
(5) $h^i(T) \cap h^j(T) = \varnothing$ if $i \not\equiv j \pmod{k}$.

LEMMA 3. *Let M be a compact orientable* 3*-manifold admitting a free Z_k action and such that if* Bd $M \neq \varnothing$ *then* Bd M *is connected. If there is a bundle map from* (M, p, M^*) *to* (S^1, p', S^1), *where p' is the standard k to* 1 *covering of S^1, then there is a compact, connected, orientable, polyhedral* 2*-manifold T such that T determines an h-partition (U, T) of M with the properties that U is connected, and if* Bd $M \neq \varnothing$ *then* Bd T *is connected and does not separate* Bd M.

Proof. Consider the diagram below, the existence of which is given (for clarity we denote the base space of the standard S^1 covering by S_0^1):

$$\begin{array}{ccc} M & \xrightarrow{\bar{g}} & S^1 \\ \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle p'} \\ M^* & \xrightarrow{g} & S_0^1 \end{array}$$

We may suppose there is a point $a \in S_0^1$ such that $g^{-1}(a)$ is the disjoint union of polyhedral compact orientable 2-manifolds regularly embedded in M^*. Let $p'^{-1}(a) = \{a_i\}_{i=1}^{k} \subset S^1$. By a suitable choice of labeling we may let A be the arc in S^1 with endpoints a_1 and a_2 such that no other a_i lies on A. Let $U = \bar{g}^{-1}(A - \{a_1, a_2\})$ and $T = \bar{g}^{-1}(a_1)$.

There is a homeomorphism h generating the Z_k action such that (U, T) forms an h-partition. The difficulty arises in that this h-partition does not necessarily satisfy the connectedness conditions of the lemma. So where necessary we will alter the partition (U, T). For convenience of notation we still let (U, T) denote the altered or new partition.

First we adjust the frontiers of the components of U. Suppose a component U_1

of U has at least two components T_1 and T_2 of T on its frontier. Let C be a polyhedral arc in Cl U_1 with one end on T_1 and the other end on T_2 but otherwise lying in Int U_1. Let N be a closed tubular neighborhood of C in Cl U_1 such that $N \cap T_1$ and $N \cap T_2$ are closed disks. We obtain a new h-partition by replacing U with the set

$$(U-N) \cup h[\text{Int}_M N \cup \text{Int}_T ((T_1 \cup T_2) \cap N)]$$

and change T by replacing $T_1 \cup T_2$ with

$$\text{Cl}\,[T_1 \cup T_2 \cup \text{Bd}\, N - \text{Int}_T (N \cap (T_1 \cup T_2))].$$

The number of components of T decreases and the number of components of U does not increase. Repetition of this process and a similar one for $h(T)$ yields an h-partition (U, T) such that each component of U has at most one component of T and at most one component of $h(T)$ on its frontier.

Suppose now a component U_1 of U is such that $F_rU=T_1\subset T$, where T_1 is a component of T. Crossing T_1 from U_1, one enters $h^{k-1}(U)$. Whenever this occurs take as a new U the set $(U-U_1) \cup h(\text{Cl}\, U)$. Note that $F_r(h(U_1))\subset \text{Cl}\, U$. Obtain a new T by dropping T_1 from the old one. Repeating this process and a similar one for $h(T)$, i.e. replace U by $(U-U_i) \cup h^{-1}(\text{Cl}\, U_i)$ where necessary, we obtain a new (U, T) partition where each component of U has precisely one component of T and one component of $h(T)$ on its frontier.

Let T_1 denote a component of T. T_1 lies on the frontier of a unique component U_1 of U. Let $T_1^{(1)}\subset h(T)$ be the component of $h(T)$ on $F_r(U_1)$. Then $T_1^{(1)}$ lies on the frontier of a unique component $U_1^{(1)}$ of $h(U)$. We continue the construction of this chain of sets, letting $T_1^{(2)}$ be the component of $h^2(T)$ on $F_rU_1^{(1)}$. As before, let $U_1^{(2)}$ denote the unique component of $h^2(U)$ with $T_1^{(2)}$ on its frontier. Repetition yields a sequence of components $T_1, U_1, T_1^{(1)}, U_1^{(1)}, T_1^{(2)}, \ldots, T_1^{(k-1)}, U_1^{(k-1)}, T_2, U_2, T_2^{(1)}, U_2^{(1)}, T_2^{(2)}, \ldots, T_2^{(k-1)}, U_2^{(k-1)}, \ldots, T_i^{(j)}, U_i^{(j)}, \ldots, T_n^{(k-1)}, U_n^{(k-1)}$, where $T_i\subset T$, $U_i\subset U$, $T_i^{(j)}\subset h^j(T)$, $U_i^{(j)}\subset h^j(U)$, $F_rU_i^{(j)}=T_i^{(j)} \cup T_i^{(j+1)}$, and $T_i^{(k)}$ is identified with T_{i+1}. This sequence must return to T_1, say at the nth stage, and we identify $T_n^{(k)}$ with T_1 completing the cycle.

All of the components of $h^i(U)$, $i=1, \ldots, k$, must have appeared as M is connected. From the construction it is clear that $h_1(T_1)$ appears $1/k$th of the way through the cycle and from there on the sequence is nothing more than the first $1/k$th part under repeated application of h. Let K be the union of everything appearing in the first $1/k$th part of the sequence. As a new U we take the set $K-(T_1 \cup h(T_1))$ and as a new T just the set T_1.

This version of (U, T) is an h-partition with U and T connected. If Bd $M=\varnothing$ we are done.

On the other hand, if Bd $M\neq\varnothing$ we must adjust (U, T) so that Bd T is connected. Bd T consists of disjoint simple closed curves $\{C_i\}_{i=1}^n$ lying on Bd M. If $n\geqq 2$ a polygonal arc A in Bd M can be found with endpoints a and b lying on different curves C_i and C_j such that $A-\{a, b\}$ lies entirely in either U or $h^{-1}(U)$. Suppose $A-\{a, b\}\subset U$ (if it lies in $h^{-1}(U)$ a similar argument is applied). Let N be a closed tubular neighborhood lying in Cl U, containing A in its boundary, and meeting T in two closed disks D_i, D_j such that $D_i \cap \text{Bd } M$, $D_j \cap \text{Bd } M$ are arcs in C_i, C_j respectively. Furthermore N is chosen so there is a homeomorphism f mapping $A\times[0, 1]$ onto $N \cap \text{Bd } M$ with $f(A\times 1/2)=A$. For a new U take the set

$$(U-N) \cup h[\text{Int } N \cup f(A\times(0, 1)) \cup \text{Int } (D_i \cup D_j)].$$

Replace T by the set

$$T \cup \text{Bd } N-[f(A\times(0, 1)) \cup \text{Int } (D_i \cup D_j)].$$

This operation decreases the number of components of Bd T by one but does not disturb the connectedness of U and T. Eventually this gives rise to an h-partition (U, T) with Bd T, U, and T all connected. Note that Bd $M-$Bd T is connected in this final form of our h-partition.

We are now ready to proceed to the proof of the main result. We will consider only manifolds without boundary in the next section, deferring the consideration of manifolds with boundary until §4.

3. M closed.

THEOREM 4. *If M is an irreducible closed orientable 3-manifold such that for some prime $k\geqq 2M$ admits a proper free Z_k action and $H_1(M^*; Z)$ has no element of order k, then M can be fibered over the circle.*

Proof. Lemmas 2 and 3 imply the existence of an h-partition (U, T) of M with U connected and T a connected, orientable, closed, polyhedral 2-manifold. Let A be an arc with one endpoint $x_0 \in T$ and the other endpoint $h(x_0) \in h(T)$ but otherwise lying in U.

We can find a retraction r_0 of Cl U onto A with $r_0^{-1}(x_0)=T$ and $r_0^{-1}(h(x_0))=h(T)$. Define the retraction r_i of $h^i(\text{Cl } U)$ onto $h^i(A)$ by $r_i=h^i r_0 h^{-i}$ for $i=1,\ldots,k-1$. The r_i's define a retraction $r\colon M\to F$, where $F=\bigcup_{i=1}^{k} h^i(A)$, such that $rh=hr$. Hence r induces a retraction $r'\colon M^*\to p(F)$. Let f be r' followed by a homeomorphism of $p(F)$ onto S^1. There is a bundle map $\bar{f}$ such that the following diagram commutes, where p' is the standard k to 1 covering of S^1:

$$\begin{array}{ccc} M & \xrightarrow{\bar{f}} & S^1 \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ M^* & \xrightarrow{f} & S^1_0 \end{array}$$

Now using the hypothesis that the Z_k action is proper we have that $h\simeq 1_M$. Restricting the homotopy yields a map

$$G_0\colon T\times[0, 1]\to M$$

such that $G_0(x, 0)=x$ and $G_0(x, 1)=h(x)$. Define maps $G_i\colon T\times[i, i+1]\to M$ by $G_i(x, t)=h^iG_0(x, t-i)$ for $i=1,\ldots,k-1$. Then $G_i(x, i+1)=G_{i+1}(x, i+1)=h^{i+1}(x)$. From the G_i's we obtain a map

$$G\colon T\times S^1\to M, \quad \text{where } S^1 \text{ is considered as } [0, k]/\{0, k\}.$$

Let $\deg G=d\geqq 0$. We show that $d\neq 0$. Let $i\colon T\to M$ be the inclusion map and consider the homology sequence for the pair (M, T). It follows from the freeness of $H_2(M, T)$ that the sequence $0\to H_2(T)\xrightarrow{i_*} H_2(M)$ is split exact. Using the universal coefficient theorem for cohomology we see that $H^2(M)\xrightarrow{i^*} H^2(T)\to 0$ is exact.

Let $\alpha\in H_3(T\times S^1)$ and $\beta\in H_3(M)$ be generators such that $G_*\alpha=d\beta$. Using Poincaré duality we get the commutative diagram

$$\begin{array}{ccccc}
H^2(T\times y_0) & \xrightarrow{\cong} & H^2(T) & & \\
\uparrow{\scriptstyle i'^*} & & \uparrow{\scriptstyle i^*} & & \\
H^2(T\times S^1) & \xleftarrow{G^*} & H^2(M) & & \\
\downarrow{\scriptstyle \alpha\cap} & & \downarrow{\scriptstyle d\beta\cap} & & \\
H_1(T\times S^1) & \xrightarrow{G_*} & H_1(M) & \xrightarrow{\bar f_*} & H_1(S^1)
\end{array}$$

where $i'\colon T\times y_0\to T\times S^1$ is the inclusion.

Using the Künneth formulas we write

$$H^2(T\times S^1) = P\oplus Q, \qquad H_1(T\times S^1) = R\oplus S,$$

where $P\cong H^2(T)\otimes H^0(S^1)$, $Q\cong H^1(T)\otimes H^1(S^1)$, $R\cong H_0(T)\otimes H_1(S^1)$ and $S\cong H_1(T)\otimes H_0(S^1)$. Since i^* is an epimorphism, for each $u\in P$ there is a $u'\in Q$ such that $u+u'\in \operatorname{Im} G^*$. Under the Poincaré duality isomorphism P maps onto R. Since $\bar f_*G_*(S)=0$ we have $\bar f_*G_*(R)=\bar f_*(dH_1(M))$.

We may identify R with $\pi_1(x_0\times S^1)$ by taking $x_0\in T$ (for convenience we still let $\{x_0\}=F\cap T$). R is generated by the element represented by a loop ω going around $x_0\times S^1$ exactly once. $G\omega$ is a loop starting from $x_0\in T$, passing through $h(x_0)$ $1/k$th of the way around, and thereafter repeating itself $k-1$ times under the action of h^i, $i=1,\ldots,k-1$. Thus $\bar fG\omega$ will start from some point $s_0\in S^1$, wrap around some number of times, then reach $s_0+e^{i(2\pi j/k)}$ when $G\omega$ is j/kth of the way around, for $j=1, 2,\ldots, k$. So under $\bar fG$, the first $1/k$th part of ω is wrapped around $S^1 m+1/k$ times. Therefore ω is wrapped around $km+1$ times. Thus

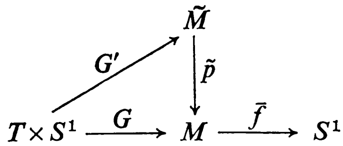
$$\bar f_\# G_\#([\omega])\in\pi_1(S^1)$$

is an integer congruent to 1 modulo k (letting $\pi_1(S^1)=Z$). Since

$$\bar f_*G_*(R) = \bar f_*(dH_1(M)),$$

after identifying $H_1(S^1)$ with $\pi_1(S^1)$, it follows that $km+1$ is a multiple of $d=\deg G$. Therefore $d\neq 0$.

Let $\tilde{M}\xrightarrow{\tilde{p}} M$ be a covering of M corresponding to the subgroup π' of $\pi_1(M)$, where $\pi'=\operatorname{Im}(G_\#\colon \pi_1(T\times S^1)\to\pi_1(M))$. If G' is a lifting of G we get the commutative diagram



$\tilde{p}$ is a finite-to-one covering projection since $\deg G\neq 0$. $G'_\#\colon \pi_1(T\times S^1)\to\pi_1(M)$ is an epimorphism, so every loop in $\tilde{M}$ will circle around S^1 under the map $\bar{f}\tilde{p}$ some n number of times, where n is a multiple of $km+1$.

Let the circle C be the component of $p^{-1}(F)$ that contains the basepoint (we may assume basepoints have been chosen nicely). Let μ be a nonsingular loop on C. $\deg\tilde{p}$ divides d and $\tilde{p}\mu$ circles around F at most $\deg\tilde{p}$ times (in absolute value). Thus $\bar{f}\tilde{p}\mu$ circles around S^1 at most $\deg\tilde{p}$ times. It follows that $\deg\tilde{p}\leqq d\leqq n\leqq\deg\tilde{p}$, hence $\deg\tilde{p}=d$. So $\tilde{p}$ is a d to 1 covering projection and $[\pi_1(M),\pi']=d$.

Let γ be an element of $\pi_1(M)$ represented by a loop circling around F exactly once. Consider the cosets $\pi',\gamma\pi',\ldots,\gamma^{d-1}\pi'$. These cosets are distinct since $\bar{f}_\#(\gamma^i\pi')\equiv i \pmod d$ and their disjoint union is all of $\pi_1(M)$. Moreover π' is the kernel of a homomorphism from $\pi_1(M)$ onto Z_d and so is a normal subgroup.

We have $\pi_1(T\times S^1)\cong\pi_1(T)\times\pi_1(S^1)$ naturally split, thus $\pi_1(\tilde{M})\cong G'_\#(\pi_1(T))\times G'_\#(\pi_1(S^1))$. If we let $K=\ker(\bar{f}_\#\colon\pi_1(M)\to\pi_1(S^1))$, it is clear that $K\subset\pi'$. Since $\tilde{p}_\#$ is a monomorphism, we have $K=\tilde{p}_\#(\ker((\bar{f}\tilde{p})_\#\colon\pi_1(\tilde{M})\to\pi_1(S^1)))$. Considering the effect of $(\bar{f}\tilde{p})_\#$ and $G_\#$ we see that $K\cong\tilde{p}_\# G'_\#(\pi_1(T))$. Hence K is finitely generated. Moreover, $\pi_1(M)/K\cong Z$ and by Lemma 1 $\pi_1(M)$ has no elements of finite order. Application of Stalling's theorem [6] completes the proof that M can be fibered over the circle.

4. *M* with boundary.

THEOREM 5. *If M is an irreducible compact orientable* 3*-manifold with connected boundary such that for some prime $k\geqq 2$ M admits a proper free Z_k action and $H_1(M^*;Z)$ has no elements of order k, then M can be fibered over the circle and* Bd $M\approx S'\times S'$.

Proof. The proof of this theorem follows that of Theorem 4 very closely so the obvious overlap will be omitted. There exists an h-partition (U,T) of M with U connected, T a connected, orientable, compact, polyhedral 2-manifold such that Bd $T=T\cap$ Bd M is connected and does not separate Bd M.

We proceed as before to define a map $G: T\times S^1 \to M$. However we require a different diagram for the argument that $\deg G=d$ is not zero. First note that $H^2(\text{Bd } M, \text{Bd } T)\cong Z$ and consider the following commutative diagram:

$$\begin{array}{ccccccccc}
H^1(\text{Bd } M) & \xrightarrow{j^*} & H^1(\text{Bd } T) & \longrightarrow & H^2(\text{Bd } M, \text{Bd } T) & \xrightarrow{\approx} & H^2(\text{Bd } M) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & & & & & \\
H^2(M, \text{Bd } M) & \xrightarrow{i^*} & H^2(T, \text{Bd } T) & & & & & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

where i and j are inclusion maps. It follows from the fact that j^* is an epimorphism that i^* is an epimorphism.

Let $\alpha\in H_3(T\times S^1, \text{Bd } T\times S^1)$ and $\beta\in H_3(M, \text{Bd } M)$ be generators such that $G_*\alpha=d\beta$. Consider the following diagram:

$$\begin{array}{ccccc}
H^2(T\times y_0, \text{Bd } T\times y_0) & \xrightarrow{\approx} & H^2(T, \text{Bd } T) & & \\
\uparrow{\scriptstyle i'^*} & & \uparrow{\scriptstyle i^*} & & \\
H^2(T\times S^1, \text{Bd } T\times S^1) & \xleftarrow{G^*} & H^2(M, \text{Bd } M) & & \\
\downarrow{\scriptstyle \alpha\cap} & & \downarrow{\scriptstyle d\beta\cap} & & \\
H_1(T\times S^1) & \xrightarrow{G_*} & H_1(M) & \xrightarrow{\bar{f}_*} & H_1(S^1)
\end{array}$$

where $i': (T\times y_0, \text{Bd } T\times y_0)\subset(T\times S^1, \text{Bd } T\times S^1)$. Write $H^2(T\times S^1, \text{Bd } T\times S^1)=P\oplus Q$, where $P\cong H^2(T, \text{Bd } T)\otimes H^0(S^1)$ and $Q\cong H^1(T, \text{Bd } T)\otimes H^1(S^1)$. As in the previous proof write $H_1(T\times S^1)=R\oplus S$. It follows that $\bar{f}_*G_*(R)=\bar{f}_*(dH_1(M))$. The remainder of the argument to show that M fibers over the circle is the same as that for Theorem 4.

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