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INFINITE DEFICIENCY IN FRÉCHET MANIFOLDS

BY
T. A. CHAPMAN

Abstract. Denote the countable infinite product of lines by s, let X be a separable metric manifold modeled on s, and let K be a closed subset of X having Property Z in X, i.e. for each nonnull, homotopically trivial, open subset U of X, it is true that $U\backslash K$ is nonnull and homotopically trivial. We prove that there is a homeomorphism h of X onto $X\times s$ such that $h(K)$ projects onto a single point in each of infinitely many different coordinate directions in s. Using this we prove that there is an embedding of X as an open subset of s such that K is carried onto a closed subset of s having Property Z in s. We also establish stronger versions of these results.

1. **Introduction.** Let s denote the countable infinite product of open intervals and let I^∞ denote the Hilbert cube, i.e. the countable infinite product of closed intervals. A *Fréchet manifold* (*or F-manifold*) is defined to be a separable metric space having an open cover by sets each homeomorphic to s.

Let α be a set of positive integers and for the remainder of the paper let T denote s or I^∞. A subset K of T is said to be *deficient with respect to* α *in* T provided that for each of the coordinate intervals which is indexed by an element of α, K projects onto a single interior point of the interval. If α is infinite, then we say that K has *infinite deficiency in* T. More generally let X be a space and let α be as given above. A subset K of $X\times T$ is said to be *deficient with respect to* α *in* $X\times T$ provided that $\pi_T(K)$ is deficient with respect to α in T, where π_T is the projection of $X\times T$ onto T. We say that K has *infinite deficiency in* $X\times T$ provided that $\pi_T(K)$ has infinite deficiency in T. We remark that if X is any F-manifold, then in [4] it is proved that X, $X\times s$, and $X\times I^\infty$ are all homeomorphic.

A closed set K in a space X is said to have *Property* Z *in* X if, for each nonempty, homotopically trivial, open set U in X, it is true that $U\backslash K$ is nonempty and homotopically trivial. In [2] it is proved that a closed subset K of T has Property Z in T if and only if there exists a homeomorphism h of T onto itself such that $h(K)$ has infinite deficiency in T.

If X is any F-manifold and K is a closed subset of $X\times T$ having infinite deficiency in $X\times T$, then it is easy to show that K has Property Z in $X\times T$. This follows from the apparatus used in the proof of Theorem 9.1 of [2], which says that infinite deficiency in T implies Property Z in T. The question of Property Z implying

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infinite deficiency in $X \times s$ was raised at the conference on infinite-dimensional topology held at Cornell University, January 5–7, 1969, and it appears as Problem 11 in the report of that meeting, *Problems in the topology of infinite-dimensional manifolds*. In this paper we solve Problem 11 by proving the following theorem.

THEOREM 1. *Let X be any F-manifold and let $\{K_i\}_{i=1}^{\infty}$ be a collection of closed subsets of $X \times T$, with each K_i having Property Z in $X \times T$. Then there is a homeomorphism h of $X \times T$ onto itself and a collection $\{\alpha_i\}_{i=1}^{\infty}$, where the α_i's are pairwise-disjoint, infinite sets of positive integers, such that for all $i>0$, $h(K_i)$ is deficient with respect to α_i in $X \times T$.*

As an easy consequence of Theorem 1 and the preceding comments we obtain the following characterization of Property Z in F-manifolds.

COROLLARY. *Let X be any F-manifold and let K be a closed subset of X. A necessary and sufficient condition for K to have Property Z in X is that there exists a homeomorphism of X onto $X \times T$ carrying K onto a set having infinite deficiency in $X \times T$.*

In [8] David W. Henderson proved that any F-manifold can be embedded as an open subset of s. In Problem 10 of *Problems in the topology of infinite-dimensional manifolds* the following question was raised: If, for each $i>0$, K_i is a closed subset of an F-manifold X and each K_i has Property Z in X, is there an open embedding $h: X \to s$ such that, for each $i>0$, $h(K_i)$ is closed in s? In our next theorem we use Henderson's open embedding theorem, modify it with our Theorem 1, and obtain a solution to Problem 10.

THEOREM 2. *Let X be any F-manifold and let $\{K_i\}_{i=1}^{\infty}$ be a collection of closed subsets of X, with each K_i having Property Z in X. Regard $s=s_1 \times s_2$, where s_1 and s_2 are copies of s, and let $f: X \to s_1$ be any open embedding. Then there exists an embedding $h: X \to s$ such that $h(X)=f(X) \times s_2$ and $h(K_i)$ is a closed subset of s having infinite deficiency in s, for all $i>0$.*

We note that if X is any F-manifold not homeomorphic to s, then there is no open embedding $h: X \to s$ such that $h(X)$ is closed in s. Thus the assumption of Property Z in Theorem 2 cannot be omitted.

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2. **Preliminaries.** We will regard the Hilbert cube I^∞ as a canonical compactification of s in which $I^\infty = \prod_{i=1}^{\infty} I_i$ and $s = \prod_{i=1}^{\infty} I_i^0$, where for each $i>0$ we have $I_i=[-1, 1]$ and $I_i^0=(-1, 1)$. The metric we use for I^∞ and s is given by

$$d((x_i), (y_i)) = \left(\sum_{i=1}^{\infty} 2^{-i} \cdot (x_i - y_i)^2\right)^{1/2},$$

where (x_i), $(y_i) \in I^\infty$. Whenever no confusion arises we will use d to denote the metric of any space under consideration.

Let N denote the set of positive integers and for each $i \in N$ let τ_i be the projection of I^∞ onto I_i. If $\alpha \subset N$ define $I^\alpha = \prod_{i\in\alpha} I_i$, $s^\alpha = \prod_{i\in\alpha} I_i^0$, and let τ_α be the projection of I^∞ onto I^α. For each $i>0$ let $W_i^+ = \tau_i^{-1}(1)$, $W_i^- = \tau_i^{-1}(-1)$, and $W_i = W_i^+ \cup W_i^-$. We call W_i^+ and W_i^- the *endslices* of I^∞ in the i-direction.

A subset of I^∞ of the form $\prod_{i=1}^\infty J_i$ is called a *closed basic set in* I^∞ provided that J_i is a closed subinterval of I_i, for each i, and $J_i = I_i$ for all but finitely many i. *An open basic set in* s is the intersection of a closed basic set in I^∞ with s.

A homeomorphism h of I^∞ onto itself is said to be a *β^*-homeomorphism* provided that $h(s)=s$. The *pseudo-boundary* of I^∞ is $B(I^\infty)=I^\infty \backslash s$ and the *pseudo-interior* of I^∞ is s.

If $\{f_i\}_{i=1}^\infty$ is a sequence of homeomorphisms of a space X onto itself for which the sequence $\{f_i \circ f_{i-1} \circ \cdots \circ f_1\}_{i=1}^\infty$ converges pointwise to a homeomorphism f of X onto itself, then we call f the *infinite left product* of $\{f_i\}_{i=1}^\infty$ and write $f = L\prod_{i=1}^\infty f_i$.

We list below three convergence procedures that we will need to insure the existence of an infinite left product of homeomorphisms. The first of these is Lemma 2.1 of [2] and the second is Theorem 2 of [5]. The third is an easy consequence of the apparatus used in [5] to establish the second.

Convergence Procedure A. *For each homeomorphism g of a compact metric space X onto itself and each $\varepsilon > 0$ let*

$$\eta(g, \varepsilon) = \text{g.l.b.}\,\{d(g(x), g(y)) \mid d(x, y) \geqq \varepsilon\}.$$

If $\{f_i\}_{i=1}^\infty$ is a sequence of homeomorphisms of X onto itself such that

$$d(f_i, \text{id}) < \min\left((3^{-i}), (3^{-i})\cdot\eta(f_{i-1} \circ \cdots \circ f_1, 2^{-i})\right),$$

for all $i > 1$, then $f = L\prod_{i=1}^\infty f_i$ exists.

Convergence Procedure B. *Let $\mathscr{U}$ be a countable star-finite open cover of any space X. (By star-finite cover we mean a cover such that the closure of each member of the cover intersects the closures of only finitely many other members of the cover.) There exists an ordering $\{U_i\}_{i=1}^\infty$ of the elements of $\mathscr{U}$ such that for any sequence $\{f_i\}_{i=1}^\infty$ of homeomorphisms of X onto itself, where f_i is the identity on $X \backslash U_i$ for all $i>0$, $f = L\prod_{i=1}^\infty f_i$ exists. Moreover, we can assign a positive integer n_i to each U_i, independent of the choice of $\{f_i\}_{i=1}^\infty$, such that $n_i \leqq n_{i+1}$ and*

$$f(U_i) = (f_{n_i} \circ f_{n_i - 1} \circ \cdots \circ f_1)(U_i),$$

for all $i > 0$.

Convergence Procedure B′. *Let $\mathscr{U}$ be a countable star-finite open cover of any space X and let $\{U_i\}_{i=1}^\infty$, $\{n_i\}_{i=1}^\infty$ be as in Convergence Procedure* B. *If Y is any space and $\{f_i\}_{i=1}^\infty$ is any sequence of homeomorphisms of $X \times Y$ onto itself such that f_i is* *the identity on* $(X\backslash U_i)\times Y$, *for all* $i>0$, *then* $f=L\prod_{i=1}^{\infty} f_i$ *exists. Moreover we have*

$$f(U_i\times Y) = (f_{n_i}\circ\cdots\circ f_1)(U_i\times Y),$$

for all $i>0$.

3. **Two technical lemmas.** We will need a result on extensions of homeomorphisms in I^∞, where the given homeomorphism and its extension are required to lie in the same neighborhood of the identity.

LEMMA 1. *Let* K_1, K_2 *be compact subsets of* I^∞ *having Property Z in* I^∞ *and let* $\varepsilon>0$ *be given. If* $h\colon K_1\to K_2$ *is an onto homeomorphism such that* $h(K_1\cap s)=K_2\cap s$ *and* $d(h,\mathrm{id})<\varepsilon$, *then h can be extended to a* β^**-homeomorphism H such that* $d(H,\mathrm{id})<\varepsilon$.

A homeomorphism extension theorem for I^∞ was first established by Anderson in [2], and modified with an "ε-condition" in [6] by Barit and in [7] by Bessaga and Pełczyński. If the "ε-condition" is omitted from the statement of our Lemma 1, then we obtain a statement which is equivalent to a homeomorphism extension theorem of Torunczyk, which is given as Theorem 6 of [9]. If the extension theorem of Barit or that of Bessaga and Pełczyński (loc. cit.) is used, then our Lemma 1 follows routinely from the proof of Torunczyk's extension theorem.

LEMMA 2. *Let* $\{F_i\}_{i=1}^\infty$ *be a collection of relatively closed subsets of s, with each* F_i *having Property Z in s, let W be a finite union of endslices of* I^∞, *and let* $\{\alpha_i\}_{i=1}^\infty$ *be a collection of pairwise-disjoint, infinite sets of positive integers. Then there is a* β^**-homeomorphism h and an infinite, increasing sequence* $\{n_i\}_{i=1}^\infty$ *of positive integers such that h is the identity on W and* $h(\bigcup_{i=1}^\infty F_i)=\bigcup_{i=1}^\infty K_i$, *where each* K_i *is a relatively closed subset of s having deficiency with respect to* α_{n_i}.

Proof. We will construct h as an infinite left product of homeomorphisms $\{h_n\}_{n=1}^\infty$. Convergence will be assured by Convergence Procedure A by noting that whenever $h_1,\ldots,h_n$ have been constructed, h_{n+1} can be constructed arbitrarily close to the identity. We will only explicitly construct h_1 and h_2, as the construction of h_2 will essentially constitute the inductive step.

Let $\beta_1=\{i_1,i_2,\ldots\}$ be an infinite, increasing sequence of positive integers such that $N\backslash\beta_1$ is infinite. We will construct a β^*-homeomorphism h_1 such that $h_1(F_1)=\bigcup_{n=1}^\infty A_n$, where each A_n is a relatively closed subset of s having deficiency with respect to α_{i_n}, and $h_1|W=\mathrm{id}$.

From Theorem 8.2 of [2] we know that $\bar{F}_1$, the closure of F_1 in I^∞, has Property Z in I^∞. Using Theorem II A of [3] and Theorem 8.5 of [2] there is a homeomorphism f_1 of I^∞ onto itself such that $f_1(s\cup\bar{F}_1)=s$. Since $f_1(W\cap\bar{F}_1)$ is a compact subset of s we can use Theorem II A of [3] to obtain a homeomorphism f_2 of I^∞ onto itself such that

$$f_2[f_1(W\cap\bar{F}_1)\cup B(I^\infty)] = B(I^\infty).$$

Using Lemma 1 we can clearly get a β^*-homeomorphism f_3 such that $f_3\circ f_2\circ f_1|W=\mathrm{id}$.

Now we note that $f_3 \circ f_2 \circ f_1(\bar{F}_1 \backslash W) = \bigcup_{n=1}^{\infty} C_n$, a countable union of compact subsets of s. Let $\{\gamma_n\}_{n=1}^{\infty}$ be a collection of pairwise-disjoint, infinite sets of positive integers such that for each n, $\gamma_n \subset N \backslash \bigcup_{n=1}^{\infty} \alpha_{i_n}$. For each n write $\gamma_n = \gamma_n' \cup \gamma_n''$ such that $\gamma_n' \cap \gamma_n'' = \varnothing$ and both are infinite, and where from Theorem 3.5 of [1] we can get a β^*-homeomorphism f_4 such that $f_4 | W = \text{id}$ and $f_4(C_n)$ has deficiency with respect to $\alpha_{i_n} \cup \gamma_n'$.

Using Theorem 5.3 of [1] there is a homeomorphism f_5 of I^∞ onto itself such that

$$f_5[f_4 \circ f_3 \circ f_2 \circ f_1((\bar{F}_1 \cap B(I^\infty)) \backslash W) \cup B(I^\infty)] = B(I^\infty),$$

f_5 is the identity on W, and $\tau_i \circ f_5 = \tau_i$, for all $i \notin \bigcup_{n=1}^{\infty} \gamma_n'$. Then we put $h_1 = f_5 \circ f_4 \circ f_3 \circ f_2 \circ f_1$, which is obviously a β^*-homeomorphism satisfying $h_1 | W = \text{id}$. It is also clear that $h_1(F_1) = \bigcup_{n=1}^{\infty} A_n$, where $A_n = f_5 \circ f_4(C_n) \cap s$ is a relatively closed subset of s which is deficient with respect to α_{i_n}, for all n.

We must be able to construct h_2 arbitrarily close to the identity. Thus let $\varepsilon > 0$ be given and let l be a positive integer such that $(\sum_{i=l}^{\infty} 2^{2-i})^{1/2} < \varepsilon/6$. Let $\beta_2 = \{j_1, j_2, \ldots\}$ be an infinite, increasing sequence of positive integers such that $\beta_2 \subset N \backslash \beta_1$, $N \backslash (\beta_1 \cup \beta_2)$ is infinite, and $i \in \bigcup_{n=1}^{\infty} \alpha_{j_n}$ implies that $i \geqq l$. We will construct a β^*-homeomorphism h_2 such that $d(h_2, \text{id}) < \varepsilon$, $h_2 \circ h_1(F_2 \backslash F_1) = \bigcup_{n=1}^{\infty} B_n$, where each B_n is a relatively closed subset of s having deficiency with respect to α_{j_n}, and h_2 is the identity on $W_1 \cup h_1(W_1) \cup W \cup h_1(\bar{F}_1)$.

From Theorem IIA of [3] there is a homeomorphism g_1 of I^∞ onto itself such that $g_1(h_1(\bar{F}_1 \cup \bar{F}_2) \cup B(I^\infty)) = B(I^\infty)$ and $d(g_1, \text{id}) < \varepsilon/12$. Using Theorem 8.5 of [2] there is a homeomorphism g_2 of I^∞ onto itself such that $g_2(g_1 \circ h_1(\bar{F}_1 \cup \bar{F}_2) \cup s) = s$. Examining the apparatus used in [2] to obtain g_2 we find that we may additionally require that $d(g_2, \text{id}) < \varepsilon/12$. Put $A = W_1 \cup h_1(W_1) \cup W$ and note that

$$g_2 \circ g_1[h_1(\bar{F}_1 \cap B(I^\infty)) \cup (A \cap h_1(\bar{F}_2))]$$

is a σ-compact subset of s. Thus using Lemma IIA of [3] there is a homeomorphism g_3 of I^∞ onto itself such that

$$g_3\{g_2 \circ g_1[h_1(\bar{F}_1 \cap B(I^\infty)) \cup (A \cap h_1(\bar{F}_2))] \cup B(I^\infty)\} = B(I^\infty)$$

and $d(g_3, \text{id}) < \varepsilon/6$. Using Lemma 1 there is a β^*-homeomorphism g_4 such that $g_4 \circ g_3 \circ g_2 \circ g_1$ is the identity on $h_1(\bar{F}_1) \cup A$ and $d(g_4, \text{id}) < \varepsilon/3$.

Let $g_4 \circ g_3 \circ g_2 \circ g_1(h_1(\bar{F}_2) \backslash (h_1(\bar{F}_1) \cup A)) = \bigcup_{n=1}^{\infty} D_n$, a countable union of compact subsets of s. Let $\{\delta_n\}_{n=1}^{\infty}$ be a collection of pairwise-disjoint, infinite sets of positive integers such that $\delta_n \subset N \backslash \bigcup_{n=1}^{\infty} (\alpha_{i_n} \cup \alpha_{j_n})$ and $m \in \delta_n$ implies that $m \geqq l$, for all $n > 0$. Then write each $\delta_n = \delta_n' \cup \delta_n''$ such that $\delta_n' \cap \delta_n'' = \varnothing$ and both are infinite, and where from Theorem 3.5 of [1] there is a β^*-homeomorphism g_5 such that $g_5(D_n)$ has deficiency with respect to $\alpha_{j_n} \cup \delta_n'$ and $d(g_5, \text{id}) < \varepsilon/6$. If we examine the apparatus used in [1] to obtain g_5 we find that we may additionally require that $g_5 | (h_1(\bar{F}_1) \cup A) = \text{id}$.

Now let g_6 be a homeomorphism of I^∞ onto itself such that

$$g_6\{[g_5 \circ g_4 \circ g_3 \circ g_2 \circ g_1(h_1(\bar{F}_2) \cap B(I^\infty))\backslash(h_1(\bar{F}_1) \cup A)] \cup B(I^\infty)\} = B(I^\infty),$$
$$g_6|(h_1(\bar{F}_1) \cup A) = \mathrm{id}, \qquad d(g_6, \mathrm{id}) < \varepsilon/6,$$

and $\tau_i \circ g_6 = \tau_i$, for all $i \notin \bigcup_{n=1}^{\infty} \delta_n'$. Put $h_2 = g_6 \circ g_5 \circ g_4 \circ g_3 \circ g_2 \circ g_1$, which is obviously a β^*-homeomorphism satisfying $h_2|(h_1(\bar{F}_1) \cup A) = \mathrm{id}$. It is clear that $h_2 \circ h_1(F_2\backslash F_1) = \bigcup_{n=1}^{\infty} B_n$, where $B_n = g_6 \circ g_5(D_n) \cap s$ is a relatively closed subset of s which is deficient with respect to α_{j_n}, for all n. This completes the construction of h_2 and the proof of the lemma.

4. Separating closed sets with Property Z in F-manifolds.

LEMMA 3. *Let $\{F_i\}_{i=1}^{\infty}$, $\{K_j\}_{j=1}^{\infty}$ be collections of relatively closed subsets of s, each having Property Z in s, and let W be a finite union of endslices of I^∞. Then there exists a B^*-homeomorphism h such that $h|W = \mathrm{id}$ and $h(\bigcup_{j=1}^{\infty} K_j) \cap (\bigcup_{i=1}^{\infty} F_i) = \varnothing$.*

Proof. Using Lemma 2 there exists a β^*-homeomorphism f_1, a collection $\{\alpha_i\}_{i=1}^{\infty}$ of pairwise-disjoint, infinite subsets of N, and relatively closed subsets $\{C_i\}_{i=1}^{\infty}$ of s, each C_i having deficiency with respect to α_i, such that $f_1|W = \mathrm{id}$ and $f_1(\bigcup_{i=1}^{\infty} F_i) = \bigcup_{i=1}^{\infty} C_i$.

Applying Lemma 2 to $\{K_j\}_{j=1}^{\infty}$ we obtain a β^*-homeomorphism f_2, a collection $\{\beta_j\}_{j=1}^{\infty}$ of pairwise-disjoint, infinite sets of positive integers such that each β_j is constructed by choosing exactly one element from each α_i, and a collection $\{D_j\}_{j=1}^{\infty}$ of relatively closed subsets of s such that each D_j is deficient with respect to β_j, $f_2|W = \mathrm{id}$, and $f_2(\bigcup_{j=1}^{\infty} K_j) = \bigcup_{j=1}^{\infty} D_j$.

We will now construct a β^*-homeomorphism g such that $g|W = \mathrm{id}$ and $g(D_j) \cap C_i = \varnothing$, for all i and j. For each i and j let $g_{i,j}$ be a β^*-homeomorphism which satisfies the following conditions

(1) $\tau_k \circ g_{i,j} = \tau_k$, for $k \notin \alpha_i \cap \beta_j$,

(2) $\tau_{\alpha_i \cap \beta_j} \circ g_{i,j}(D_j) \neq \tau_{\alpha_i \cap \beta_j}(C_i)$,

(3) $g_{i,j}|W = \mathrm{id}$, and

(4) $g_{i,j}(W_n^+) = W_n^+$ and $g_{i,j}(W_n^-) = W_n^-$, for all $n > 0$.

It is clear that (3) can be achieved by phasing out the motion isotopically as we approach W. Then we define

$$g = L \prod_{n=1}^{\infty} (g_{1,n} \circ g_{2,n-1} \circ \cdots \circ g_{n,1}),$$

which satisfies our requirements. It is clear that $h = f_1^{-1} \circ g \circ f_2$ satisfies the conditions of the lemma.

LEMMA 4. *Let $\{F_i\}_{i=1}^{\infty}$ and $\{K_j\}_{j=1}^{\infty}$ be collections of closed subsets of any F-manifold X, with each F_i and K_j having Property Z in X. Then there is a homeomorphism h of X onto itself such that $h(\bigcup_{j=1}^{\infty} K_j) \cap (\bigcup_{i=1}^{\infty} F_i) = \varnothing$.*

Proof. Using Henderson's open embedding theorem let $f\colon X \to s$ be an embedding of X as an open subset of s. From Theorem 1 of [5] there is a star-finite open cover $\{U_k\}_{k=1}^{\infty}$ of $f(X)$, where each U_k is a basic open subset of s contained in $f(X)$. Moreover let us assume that the U_k's are indexed as in Convergence Procedure B.

Note that for each $k>0$, $\overline{U}_k$ is a Hilbert cube, where $\overline{U}_k$ is the closure of U_k in I^∞, and U_k may be regarded as the relative pseudo-interior of $\overline{U}_k$. It is obvious that the topological boundary of U_k in s is contained in a finite union of endslices of the relative pseudo-boundary of $\overline{U}_k$. We can then apply Lemma 3 to obtain a homeomorphism g_1 of $f(X)$ onto itself such that $g_1|f(X)\backslash U_1 = \mathrm{id}$ and

$$g_1\Big(\bigcup_{j=1}^{\infty} f(K_j)\Big) \cap \Big(\bigcup_{i=1}^{\infty} f(F_i)\Big) \cap U_1 = \varnothing.$$

Suppose now that homeomorphisms $\{g_k\}_{k=1}^{n}$ of $f(X)$ onto itself have been defined such that $g_k|f(X)\backslash U_k = \mathrm{id}$ and

$$(g_k \circ \cdots \circ g_1)\Big(\bigcup_{j=1}^{\infty} f(K_j)\Big) \cap \Big(\bigcup_{i=1}^{\infty} f(F_i)\Big) \cap U_k = \varnothing,$$

for $1 \leqq k \leqq n$. Once more apply Lemma 3 to obtain a homeomorphism g_{n+1} of $f(X)$ onto itself such that $g_{n+1}|f(X)\backslash U_{n+1} = \mathrm{id}$ and

$$(g_{n+1} \circ \cdots \circ g_1)\Big(\bigcup_{j=1}^{\infty} f(K_j)\Big) \cap \Big(\bigcup_{i=1}^{\infty} f(F_i)\Big) \cap U_{n+1} = \varnothing.$$

It is clear that $g = L\prod_{k=1}^{\infty} g_k$ defines a homeomorphism of $f(X)$ onto itself such that $g(\bigcup_{j=1}^{\infty} f(K_j)) \cap (\bigcup_{i=1}^{\infty} f(F_i)) = \varnothing$. Then $h = f^{-1} \circ g \circ f$ satisfies the requirements of our lemma.

5. **Proof of Theorem 1.** We break Theorem 1 up into two lemmas, the first treating the case $T = I^\infty$ and the second treating the case $T = s$.

LEMMA 5. *Let X be any F-manifold and let $\{K_i\}_{i=1}^{\infty}$ be a collection of closed subsets of $X \times I^\infty$, with each K_i having Property Z in $X \times I^\infty$. Then there is a homeomorphism h of $X \times I^\infty$ onto itself and a collection $\{\alpha_i\}_{i=1}^{\infty}$, where the α_i's are pairwise-disjoint, infinite sets of positive integers, such that for all $i>0$, $h(K_i)$ is deficient with respect to α_i in $X \times I^\infty$.*

Proof. For each positive integer j let

$$F_j^+ = \{(x,y) \in X \times I^\infty \mid \tau_j(y) = 1\}, \qquad F_j^- = \{(x,y) \in X \times I^\infty \mid \tau_j(y) = -1\},$$

and $F_j = F_j^+ \cup F_j^-$. A straightforward and standard argument shows that each F_j has Property Z in $X \times I^\infty$. Thus applying Lemma 4 to $X \times I^\infty$ there is a homeomorphism f_1 of $X \times I^\infty$ onto itself such that $f_1(\bigcup_{i=1}^{\infty} K_i) \cap (\bigcup_{j=1}^{\infty} F_j) = \varnothing$.

Let j and m be any positive integers and for each $x \in X \times \prod_{i \neq j} I_i$ let $U_x \times (\delta_x, 1]$

be an open subset of $(X\times\prod_{i\neq j} I_i)\times I_j$ containing $(x, 1)$ such that $(U_x\times(\delta_x, 1])\cap f_1(K_m)=\varnothing$. Note that $\{U_x \mid x\in X\times\prod_{i\neq j} I_i\}$ is an open cover of $X\times\prod_{i\neq j} I_i$. Since we have already noted that $X\times\prod_{i\neq j} I_i$ is homeomorphic to X, and is hence an F-manifold, we can apply Lemma 5 of [5] to get a star-finite open refinement $\{U_i\}_{i=1}^{\infty}$ of $\{U_x \mid x\in X\times\prod_{i\neq j} I_i\}$ which covers $X\times\prod_{i\neq j} I_i$. Moreover we may assume that the U_i's are indexed as in Convergence Procedure B. It is clear that for each $i>0$ there is a number δ_i such that $1/2\leqq\delta_i\leqq 1$ and $(U_i\times(\delta_i, 1])\cap f_1(K_m)=\varnothing$.

Now construct a closed cover $\{C_i\}_{i=1}^{\infty}$ of $X\times\prod_{i\neq j} I_i$ such that $C_i\subset U_i$, for all $i>0$. For each $i>0$ let $\varphi_i\colon X\times\prod_{i\neq j} I_i\to[0, 1]$ be a continuous function such that $\varphi_i(x)=1$ for $x\in C_i$ and $\varphi_i(x)=0$ for $x\in(X\times\prod_{i\neq j} I_i)\backslash U_i$. For each $i>0$ we now construct a homeomorphism g_i of $(X\times\prod_{i\neq j} I_i)\times I_j$ onto itself which slides points linearly in the I_j-direction as follows: if $x\in X\times\prod_{i\neq j} I_i$, then $\{x\}\times I_j$ is taken linearly onto itself such that $g_i(x, \delta_i)=(x, \delta_i(1-\varphi_i(x))+(1/2)\varphi_i(x))$. Applying Convergence Procedure B$'$ to $\{g_i\}_{i=1}^{\infty}$ let $g=L\prod_{i=1}^{\infty} g_i$, which is a homeomorphism of $X\times I^{\infty}$ onto itself satisfying $g\circ f_1(K_m)\subset X\times\prod_{i\neq j} I_i\times[-1, 1/2]$. Using similar techniques we can construct a homeomorphism g' of $X\times I^{\infty}$ onto itself which slides points only in the I_j-direction such that $g'\circ g\circ f_1(K_m)\subset X\times\prod_{i\neq j} I_i\times[-1/2, 1/2]$.

Choose a collection $\{\beta_i\}_{i=1}^{\infty}$ of pairwise-disjoint, infinite sets of integers such that $N=\bigcup_{i=1}^{\infty}\beta_i$. Then we can successively use the above construction to obtain a homeomorphism f_2 of $X\times I^{\infty}$ onto itself such that $\tau_j\circ\pi_{I^{\infty}}\circ f_2\circ f_1(K_i)\subset[-1/2, 1/2]$, for all $j\in\beta_i$ and all $i>0$.

Using Theorem 3.5 of [1] there is a homeomorphism d_i of I^{β_i} onto itself such that $d_i(\prod_{j\in\beta_i}[-1/2, 1/2]_j)$ has deficiency with respect to an infinite subset α_i of β_i, for all $i>0$. Grouping these together we obtain a homeomorphism d of $X\times I^{\infty}=X\times I^{\beta_1}\times I^{\beta_2}\times\cdots$ onto itself defined by $d=(\mathrm{id}, d_1, d_2, \ldots)$. Then $h=d\circ f_2\circ f_1$ is a homeomorphism of $X\times I^{\infty}$ onto itself such that $h(K_i)$ is deficient with respect to α_i, for all $i>0$.

LEMMA 6. *Let X be any F-manifold and let $\{K_i\}_{i=1}^{\infty}$ be a collection of closed subsets of $X\times s$, with each K_i having Property Z in $X\times s$. Then there is a homeomorphism h of $X\times s$ onto itself and a collection $\{\alpha_i\}_{i=1}^{\infty}$, where the α_i's are pairwise-disjoint, infinite sets of positive integers, such that for all $i>0$, $h(K_i)$ is deficient with respect to α_i in $X\times s$.*

Proof. Using Theorem 9.3 of [1] there is a homeomorphism f of $s=\prod_{i=1}^{\infty} I_i^0$ onto $s^{\alpha}\times I^{\beta}$, where α is the set of even positive integers and β is the set of odd positive integers. Let $\bar{f}$ be the homeomorphism of $X\times s$ onto $X\times s^{\alpha}\times I^{\beta}$ defined by $\bar{f}(x, y)=(x, f(y))$, where $(x, y)\in X\times s$.

From Lemma 5 there is a homeomorphism g of $(X\times s^{\alpha})\times I^{\beta}$ onto itself such that $\pi_{I^{\beta}}\circ g\circ\bar{f}(K_i)$ is deficient with respect to β_i, where the β_i's are pairwise-disjoint, infinite, and $\beta=\bigcup_{i=1}^{\infty}\beta_i$. Now write $\alpha=\bigcup_{i=1}^{\infty}\alpha_i'$, where the α_i''s are pairwise-disjoint and infinite. Furthermore write $\alpha_i'=\bigcup_{j=1}^{\infty}\alpha_{i,j}$, where for each $i>0$ the $\alpha_{i,j}$'s are pairwise-disjoint and infinite, and write $\beta_i=\{m_{i,1}, m_{i,2}, \ldots\}$, an increasing sequence of integers.

We note that for each i and j we have

$$\tau_k \circ \pi_{s^\alpha \times I^\beta} \circ g \circ \bar{f}(K_i) \subset s^{\alpha_{i,j}} \times \{a_{i,j}\},$$

where $a_{i,j} \in I^0_{m_{i,j}}$ and $k=\alpha_{i,j} \cup \{m_{i,j}\}$. Theorem 9.3 of [1] gives a homeomorphism of $\prod_{i=1}^{\infty} I_i^0 \times [-1, 1]$ onto $\prod_{i=1}^{\infty} I_i^0 \times (-1, 1)$. If we examine the apparatus used to prove this we find that for each i and j we can get a homeomorphism $d_{i,j}$ of $s^{\alpha_{i,j}} \times I_{m_{i,j}}$ onto $s^{\alpha_{i,j}} \times I^0_{m_{i,j}}$ such that $d_{i,j}(s^{\alpha_{i,j}} \times \{a_{i,j}\}) = s^{\alpha_{i,j}} \times \{a_{i,j}\}$. This then gives a homeomorphism d of $X \times s^\alpha \times I^\beta$ onto $X \times s^\alpha \times s^\beta$ such that $d \circ g \circ \bar{f}(K_i)$ is deficient with respect to β_i, for all $i>0$. Thus with $h=d \circ g \circ \bar{f}$ and $\{\alpha_i\}_{i=1}^{\infty}=\{\beta_i\}_{i=1}^{\infty}$ the proof of the lemma is complete.

6. **Proof of Theorem 2.** Using Theorem 1 there is a homeomorphism f_1 of X onto $X \times s$ and a collection $\{\alpha_i\}_{i=1}^{\infty}$ of pairwise-disjoint, infinite sets of positive integers such that for each $i>0$, $f_1(K_i)$ is deficient with respect to α_i in $X \times s$.

Applying Henderson's open embedding theorem let $f_2: X \to s_1$ be an open embedding, where $s_1=\prod \{I_i^0 \mid i \text{ odd}\}$. If $s_2=\prod \{I_i^0 \mid i \text{ even}\}$ then there is a homeomorphism f_3 of s onto s_2 which is defined coordinatewise. Thus $f_4=f_2 \times f_3$ is an open embedding of $X \times s$ into $s=s_1 \times s_2$ and there is a collection $\{\beta_i\}_{i=1}^{\infty}$ of pairwise-disjoint, infinite sets of positive even integers such that $f_4 \circ f_1(K_i)$ is deficient with respect to β_i, for all $i>0$.

If $\mathrm{Cl}\,(f_4 \circ f_1(K_i))$ is the closure of $f_4 \circ f_1(K_i)$ in I^∞, then $\mathrm{Cl}\,(f_4 \circ f_1(K_i)) \backslash f_4 \circ f_1(K_i) = \bigcup_{j=1}^{\infty} C_{i,j}$, a countable union of compact subsets of I^∞, with each $C_{i,j}$ having deficiency with respect to β_i, for all $i>0$. Using Theorem 5.3 of [1] there is a homeomorphism g of I^∞ onto itself such that $g[(\bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} C_{i,j}) \cup B(I^\infty)] = B(I^\infty)$ $\tau_i \circ g=\tau_i$, for i odd, and $g \circ f_4 \circ f_1(K_i)$ has infinite deficiency, for all $i>0$. Clearly $h=g \circ f_4 \circ f_1$ is the required embedding.

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LOUISIANA STATE UNIVERSITY,
BATON ROUGE, LOUISIANA