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# COUNTABLE PARACOMPACTNESS AND WEAK NORMALITY PROPERTIES

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In [4], Dowker proved that a normal space $X$ is countably paracompact if and only if its product with the closed unit interval is normal. In this paper, we prove an analogue of Dowker's theorem. Specifically, we define the term *δ-normal* and then prove the following:

THEOREM 1. *A topological space is countably paracompact if and only if its product with the closed unit interval is δ-normal.*

After proving this theorem, we obtain similar results for the topological spaces studied in [7] and [11]. Also, cogent examples are given and the relation this note bears to the work of others is discussed.

We shall follow the terminology of [5] except we shall assume separation properties for a space only when these assumptions are explicitly stated.

For an infinite cardinal $\mathfrak{m}$, a set $A$ in a topological space will be called a *$G_{\mathfrak{m}}$-set* (respectively, a *regular $G_{\mathfrak{m}}$-set*) provided it is the intersection of at most $\mathfrak{m}$ open sets (respectively, at most $\mathfrak{m}$ closed sets whose interiors contain $A$). If $\mathfrak{m}=\aleph_0$, we shall use the familiar terms $G_\delta$-set and regular $G_\delta$-set.

It is clear that the zero-set of any continuous real valued function is a regular $G_\delta$-set and that the intersection of no more than $\mathfrak{m}$ such zero-sets is a regular $G_{\mathfrak{m}}$-set. In the remaining part of this paper, we shall use these facts without explicitly mentioning them.

DEFINITION. For an infinite cardinal $\mathfrak{m}$, a topological space is *$\mathfrak{m}$-normal* if each pair of disjoint closed sets, one of which is a regular $G_{\mathfrak{m}}$-set, have disjoint neighborhoods. For $\mathfrak{m}=\aleph_0$, we shall use the more suggestive term *δ-normal.*

Note that a normal space is $\mathfrak{m}$-normal and that a regular space is normal if and only if it is $\mathfrak{m}$-normal for every infinite cardinal $\mathfrak{m}$. On the other hand, a compact $T_1$-space that is not Hausdorff is $\mathfrak{m}$-normal for every infinite cardinal but yet it fails to be normal.

Recall that a space is *$\mathfrak{m}$-paracompact* if each open cover having cardinal less than or equal to $\mathfrak{m}$ has a locally finite open refinement. Characterizations of $\mathfrak{m}$-paracompact spaces may be found in [14] and [8].

THEOREM 2. *Each $\mathfrak{m}$-paracompact space is $\mathfrak{m}$-normal.*

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**Proof.** Suppose $X$ is $\mathfrak{m}$-paracompact and let $A$ and $B$ be disjoint closed sets such that $B$ is a regular $G_{\mathfrak{m}}$-set. Then there is a family $\mathscr{G}$, having cardinal less than or equal $\mathfrak{m}$, consisting of open neighborhoods of $B$ such that $B$ is the intersection of $\{\bar{G} : G \in \mathscr{G}\}$. Let $\Gamma$ be the set of all finite nonempty subfamilies of $\mathscr{G}$. For $\alpha \in \Gamma$, define $G_\alpha$ to be the intersection of $\{G : G \in \alpha\}$. Then $B \subset G_\alpha$ and $B = \bigcap \bar{G}_\alpha$. The family of sets $X \backslash A \cap \bar{G}_\alpha$ is a directed open cover of $X$. By Theorem 5 in [8], there exists a locally finite open cover $\{V_\alpha\}$ such that $\bar{V}_\alpha \cap A \cap \bar{G}_\alpha = \varnothing$ for all $\alpha$ in $\Gamma$. Now let $U$ be the union of all sets $V_\alpha \backslash \bar{G}_\alpha$ and $V$ be the union of the sets $G_\alpha \backslash \bigcup \{\bar{V}_\beta : \beta \text{ is not a subset of } \alpha\}$. Clearly, $U$ is open, contains $A$ and is disjoint from $V$. Since $\{V_\alpha\}$ is locally finite it follows that $V$ is open. For $x \in B$, let $\gamma = \bigcup \{\beta : x \in \bar{V}_\beta\}$. Since the cover $\{V_\alpha\}$ is locally finite, $\gamma \in \Gamma$ and $x \notin \bigcup \{\bar{V}_\beta : \beta \not\subset \gamma\}$; whence $x \in V$. Therefore $B \subset V$ and the proof is complete.

For the important special case where $\mathfrak{m} = \aleph_0$, we have:

THEOREM 3. *Each countably paracompact space is* $\delta$*-normal.*

The *local weight* of a topological space is the least cardinal $\mathfrak{m}$ such that each point has a neighborhood base consisting of at most $\mathfrak{m}$ elements.

THEOREM 4. (i) *A Hausdorff* $\mathfrak{m}$*-normal space having local weight* $\leqq \mathfrak{m}$, *is regular.*

(ii) *A Hausdorff* $\mathfrak{m}$*-normal space having cardinal* $\leqq \mathfrak{m}$, *is regular.*

**Proof.** Under the hypotheses in each case, a singleton is a regular $G_{\mathfrak{m}}$-set.

For emphasis, we state the following special case:

THEOREM 5. *If a* $\delta$*-normal Hausdorff space is either countable or satisfies the first axiom of countability, then it is regular.*

COROLLARY 6 (AULL [1]). *Each countably paracompact, first countable, Hausdorff space is regular.*

EXAMPLE. For each infinite cardinal $\mathfrak{m}$, there is an $\mathfrak{m}$-normal, Hausdorff space which is not regular. Given $\mathfrak{m}$, let $w_\alpha$ be the least ordinal having cardinal greater than $\mathfrak{m}$. Denote by $W^*$ the set of ordinals less than or equal to $w_\alpha$ and by $W$, the set $W^* \backslash \{w_\alpha\}$. In $W^* \times W^* \backslash \{(w_\alpha, w_\alpha)\}$, identify all points of $W \times \{w_\alpha\}$. The quotient space $X$ is Hausdorff but not regular (the images in $X$ of the upper edge and the diagonal are not separated by disjoint open sets). Nonetheless, $X$ is $\mathfrak{m}$-compact; hence it is $\mathfrak{m}$-paracompact and $\mathfrak{m}$-normal.

Throughout this paper $I$ will denote the closed unit interval. In the proof of Theorem 1, we shall use the following lemma.

LEMMA 7. *For any topological space* $X$, *the following are equivalent*:

(a) $X$ *is countably paracompact.*

(b) *If* $g$ *is a strictly positive lower semicontinuous function on* $X$, *then there exist real valued functions* $l$ *and* $u$ *with* $l$ *lower semicontinuous and* $u$ *upper semicontinuous such that* $0 < l(x) \leqq u(x) \leqq g(x)$ *for all* $x \in X$.

(c) *If $A$ is a closed subset of $X \times I$ and $K$ is closed in $I$ such that $A$ and $X \times K$ are disjoint, then $A$ and $X \times K$ have disjoint neighborhoods.*

**Proof.** The equivalence of (a) and (c) is due to Tamano (Theorem 3.9 in [16]) while the equivalence of (a) and (b) is an easy consequence of Theorem 10 in [7].

**Proof of Theorem 1.** If $X$ is countably paracompact, then $X \times I$ is countably paracompact (Theorem 1 in [4]). By Theorem 3 above $X \times I$ is $\delta$-normal. Conversely, suppose $K$ is closed in $I$. Since $I$ is metrizable, $K$ is a regular $G_\delta$-set. Therefore $X \times K$ is a regular $G_\delta$-set in $X \times I$. In view of Lemma 7, the $\delta$-normality of $X \times I$ will imply that $X$ is countably paracompact.

THEOREM 8. *A closed continuous image of an $\mathfrak{m}$-normal space is $\mathfrak{m}$-normal.*

**Proof.** Observe that a continuous inverse image of a regular $G_{\mathfrak{m}}$-set is regular $G_{\mathfrak{m}}$. Using this fact, the standard proof that a closed continuous image of a normal space is normal, becomes applicable here.

REMARK. In general, it is not true that preimages of $\mathfrak{m}$-normal spaces are $\mathfrak{m}$-normal even for perfect maps (i.e., continuous closed maps for which the preimages of compact sets are compact). Note that in view of Theorem 1 and the fact that a space $X$ is always the perfect image of $X \times I$, it follows that if every perfect preimage of $X$ is $\delta$-normal, then $X$ is countably paracompact.

Many of the standard examples of nonnormal spaces, also, fail to be $\delta$-normal. Here we give a partial list of such examples. (i) The space $S \times S$ where $S$ is the reals with the half-open interval topology [15]. (ii) The space $X \times Y$ constructed by Michael [12]. (iii) The space $R^R$ where $R$ is the space of reals. (iv) The spaces constructed in problems 3K, 5I, 6P, 6Q of [5]. We shall use the space $S \times S$ to illustrate a technique that can be used to verify that these spaces are not $\delta$-normal. In $S \times S$ let $A$ be the set of points $(x, -x)$ where $x$ is rational and $B$ be the set of such points for irrational $x$. Then $A$ is closed and $B$ is a regular $G_\delta$-set while these sets do not have disjoint neighborhoods. To show this, one can exploit the fact that the irrationals are not an $F_\sigma$-set in the reals (cf. [12]).

DEFINITION. A space will be called *$\delta$-normally separated* if each closed set and each zero set disjoint from it are completely separated. A space will be termed *weakly $\delta$-normally separated* if each regular closed set (i.e., the closure of an open set) and zero-set disjoint from it are completely separated.

REMARK. The properties of being $\delta$-normal and $\delta$-normally separated are, unfortunately, not comparable for arbitrary topological spaces. In a space where every regular $G_\delta$-set is a zero-set, $\delta$-normal separation implies $\delta$-normality, but not conversely (see the example at the end of this paper). On the other hand, Hewitt's example [6] of an infinite regular Hausdorff space on which each continuous real-valued function is constant, is a $\delta$-normally separated space which is not $\delta$-normal (cf. Remark following Theorem 13). The author does not know whether among completely regular spaces, $\delta$-normal separation implies $\delta$-normality.

Clearly each normal space is δ-normally separated. Likewise, δ-normal separation implies weak δ-normal separation and the converse is true for δ-normal spaces.

The concept of δ-normal separation is not a new one. P. Zenor introduced this idea in [17] and used the term *Property Z.*

The δ-normal separation of a space $X$ can be characterized in terms of properties of the ring $C(X)$ of a real-valued continuous function on $X$.

THEOREM 9. *A topological space $X$ is δ-normally separated if and only if for each $f \in C(X)$ and each closed set $A$ on which $f$ is strictly positive, there exists a unit $u$ of the ring $C(X)$ such that $fu$ is identically one on $A$.*

**Proof.** Assume $X$ is δ-normally separated and that $f$ and $A$ have the given properties. Then there exists a nonnegative element $h$ of $C(X)$ which vanishes on $A$ and assumes the value 1 everywhere on the zero-set of $f$. Then the ring inverse of $|f|+h$ is the desired unit. The converse is obvious.

We shall now proceed to state and prove the analogue of Theorem 1 for the δ-normal separation and weak δ-normal separation properties. To achieve this, we need to recall the definitions of *cb*-spaces and weak *cb*-spaces. A space $X$ is a *cb-space* (respectively, *weak cb-space*) provided every locally bounded real valued function on $X$ (respectively, every locally bounded lower semicontinuous function on $X$) is bounded above by a continuous function. Information concerning *cb*-spaces and weak *cb*-spaces may be found in [7] and [11], respectively. In comparing Theorem 1 with Theorem 11 below, it is useful to remember that a space is *cb* if and only if it is weak *cb* and countably paracompact.

LEMMA 10. (a) *Each cb-space is δ-normally separated.*

(b) *Each weak cb-space is weakly δ-normally separated.*

**Proof.** We shall prove (a) and make parenthetical comments to indicate the proof of (b). Let $A$ be closed and $Z$ be a zero-set disjoint from $A$. Given a nonnegative function $h$ in $C(X)$ such that $Z$ is the zero-set of $h$, define $g(x)=1+h(x)$ for $x$ not in $A$ and $g(x)=h(x)$ for $x$ belonging to $A$. Then $g$ is lower semicontinuous (normal lower semicontinuous if $A$ is regular closed). Clearly $g$ is strictly positive. By Theorem 1 in [7] (Theorem 3.1 in [11] for (b)) there is a strictly positive real valued continuous function $f$ such that $f \leqq g$. Then the function $h/f$ completely separates $A$ and $Z$.

THEOREM 11. *Let $X$ be a topological space. Then*

(a) *$X$ is a cb-space if and only if $X \times I$ is δ-normally separated.*

(b) *$X$ is a weak cb-space if and only if $X \times I$ is weakly δ-normally separated.*

**Proof.** The necessity follows from Lemma 10 above; the sufficiency from Corollary 12 and Theorem 13 in [9].

REMARK, In both Theorems 1 and 11, $I$ may be replaced by any infinite compact metric space. Also, note that Theorem 10 in [9] implies that a variation of (a) in the above theorem is valid when $I$ is replaced by an infinite product of intervals.

COROLLARY 12. (a) *Each countably compact space is both $\delta$-normal and $\delta$-normally separated.*

(b) *A completely regular, pseudocompact space is weakly $\delta$-normally separated.*

**Proof.** Since countably compact spaces are *cb* (Corollary 3 in [7]) and completely regular pseudocompact spaces are weak *cb* (Corollary 3.8 in [11]), this theorem follows immediately from Lemma 10.

It is well known that a normal pseudocompact Hausdorff space is countably compact. In [17], Zenor shows that normality may be replaced by $\delta$-normal separation. Here we show the condition can be further weakened to $\delta$-normality.

THEOREM 13. *A completely regular space is countably compact if and only if it is $\delta$-normal and pseudocompact.*

**Proof.** By Corollary 12 above, a pseudocompact $\delta$-normal space is also $\delta$-normally separated. This theorem now follows from Zenor's result (Theorem 3 in [17]).

REMARK. In Theorem 13, it is essential that the space be completely regular; for there exist regular, countably paracompact, Hausdorff spaces, that are not countably compact, on which every real valued function is constant [10]. Such a space can be obtained by altering slightly the construction used by Hewitt in [6].

For a completely regular space $X$, let $\upsilon X$ denote the Hewitt realcompactification. In [5, p. 120], it is noted that the normality of $X$ and of $\upsilon X$ are independent of each other. The same is true for $\delta$-normality and $\delta$-normal separation. To see this, first, let $X$ be a completely regular pseudocompact space that is not countably compact (the Tychonoff plank will do nicely). Then $\upsilon X$ is compact and hence is both $\delta$-normal and $\delta$-normally separated, but $X$ has neither of these properties. On the other hand, let $P$ be the product $R^R$ of $\mathfrak{c}$ ($\mathfrak{c}=\text{card } R$) copies of the reals $R$ and let $X$ be an associated $\Sigma$-product. Then $X$ is normal, and $\upsilon X=P$ (see [3]) but $P$ is not countably paracompact. Whence it follows from Theorem 1 that $P$ is not $\delta$-normal and from Theorem 11 that $P$ is not $\delta$-normally separated.

The situation for weak $\delta$-normal separation is entirely different as Theorems 14 and 17 below will show.

THEOREM 14. *If a completely regular Hausdorff space $X$ is weakly $\delta$-normally separated then $\upsilon X$ is as well.*

**Proof.** If $A$ is regular closed in $\upsilon X$ and $Z$ is a zero-set in $\upsilon X$, then $A \cap X$ is regular closed in $X$ and $Z \cap X$ is a zero-set in $X$. Moreover, $A$ and $Z$ are the closures in $\upsilon X$ of $A \cap X$ and $Z \cap X$ respectively (for the latter see 8.8(b) in [5]). If $f$ is a continuous real valued function on $X$ which completely separates $A \cap X$ and $Z \cap X$, then its extension to $\upsilon X$ clearly separates $A$ from $Z$.

COROLLARY 15. *Any product of complete separable metric spaces is weakly $\delta$-normally separated.*

**Proof.** In [3], it is proved that any such product is $\upsilon X$ for some normal space $X$.

In order to obtain a partial converse of Theorem 14, we prove the following lemma which seems to be of independent interest. A point $x$ of a space $X$ is a *q-point* [13] if it has a sequence $\{U_n\}$ of neighborhoods such that if $\{x_n\}$ is a sequence of distinct points with $x_n \in U_n$, then this sequence has an accumulation point.

LEMMA 16. *If every point of $\upsilon X \setminus X$ is a q-point of $\upsilon X$, then every pair of disjoint sets $A$, $Z$ where $A$ is regular closed in $X$ and $Z$ is a zero-set in $X$, have disjoint closures in $\upsilon X$.*

**Proof.** Suppose on the contrary that $p$ belongs to the closure of both $A$ and $Z$ and let $\{U_n\}$ be a sequence of open neighborhoods of $p$ given by the definition of $q$-points. Let $G$ denote the interior of $A$ in $X$ and $f \in C(X)$ be a function whose zero-set is $Z$. By our assumption $p$ belongs to the closure of $G \cap \{x : |f(x)| < 1/n\}$ (call this set $H_n$) for each positive integer $n$. Whence $U_n \cap H_n$ is nonempty for each $n$. Pick $x_n$ from this set. Clearly, we may assume that the $x_n$ are distinct. Since $A$ and $Z$ are disjoint, it follows that $\{H_n\}$ is locally finite. Thus the sequence $\{x_n\}$ has no accumulation point. But this is impossible since $p$ is a $q$-point.

THEOREM 17. *If $\upsilon X$ is locally compact (more generally, if each point in $\upsilon X \setminus X$ has a compact neighborhood in $\upsilon X$), then $X$ is weakly $\delta$-normally separated if and only if $\upsilon X$ has the same property.*

A converse for Theorem 14 is not possible, without some sort of restriction on $\upsilon X$. This is shown in the example below.

It is natural to ask what relation the above results bear to the well known unanswered question [4, p. 221]: *Must the product of a normal Hausdorff space with the closed unit interval be normal*? In this regard, first, observe that if $X$ is normal, then $X \times I$ is normal provided it is $\delta$-normal. This fact suggests the following question: *If $X$ is a regular, $\delta$-normal space, must $X \times I$ be $\delta$-normal*? Except for noting that without the assumption that the space is regular, the answer to this question is negative (see p. 221 in [4]), the author has not obtained any significant clues concerning the answer to this question. On the other hand, the answer to the corresponding question for $\delta$-normal separation is negative. This is the substance of the following example.

EXAMPLE. Let $X$ and $X^*$ be the spaces constructed on pp. 240, 241 of [11]. There it is pointed out that $X$ is locally compact, countably paracompact but not a *cb*-space while $X^*$ is $\sigma$-compact but not locally compact and that $X^* = \upsilon X$. It is a simple matter (using Theorem 9 and the special relation that $X$ bears to $X^*$) to show that $X$ is also $\delta$-normally separated. Since $X$ is not a *cb*-space, it follows from Theorem 11 that $X \times I$ is not $\delta$-normally separated (or even weakly $\delta$-normally separated). It is, however, $\delta$-normal.

Also, in view of Theorem 2.8 in [2], note that $\upsilon(X\times I)=\upsilon X\times I$. Now since $\upsilon X\times I=X^*\times I$ is Lindelöf and regular, it is normal. Nonetheless $X\times I$ fails to be weakly $\delta$-normally separated. This shows that the restriction on $\upsilon X$ in Theorem 17 cannot be entirely suppressed.

In [14], Morita obtained the following generalization of Dowker's theorem [4]: A space $X$ is $\mathfrak{m}$-paracompact and normal if and only if $X\times I^{\mathfrak{m}}$ is normal. In view of Morita's result, it is natural to ask: What condition on $X$ is necessary and sufficient for $X\times I^{\mathfrak{m}}$ to be $\mathfrak{m}$-normal? Theorem 2 implies that $\mathfrak{m}$-paracompactness of $X$ is a sufficient condition; however the author has been unable to determine whether $\mathfrak{m}$-paracompactness is also necessary. The chief stumbling block is the lack of a characterization of $\mathfrak{m}$-paracompactness similar to that for countable paracompactness given by Tamano (Lemma 7 above).

In contrast to the obstacles encountered in attempting to obtain an analogue of Theorem 1 for uncountable cardinals, Theorem 11 (as pointed out in the Remark following that theorem) can be extended by merely giving an appropriate meaning to the term $\mathfrak{m}$-normal separation. Specifically, define a space to be $\mathfrak{m}$-*normally separated* provided the intersection of any family consisting of at most $\mathfrak{m}$ zero-sets is completely separated from any closed set disjoint from it. Then for any space $X$, $X\times I^{\mathfrak{m}}$ is $\mathfrak{m}$-normally separated if and only if $X$ is an $H(\mathfrak{m})$-space in the sense of [9].

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