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A COLLECTION OF SEQUENCE SPACES

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Abstract. This paper concerns a collection of sequence spaces we shall refer to as d_α spaces. Suppose $\alpha=(\alpha_1, \alpha_2, \ldots)$ is a bounded number sequence and $\alpha_i \neq 0$ for some i. Suppose $\mathcal{P}$ is the collection of permutations on the positive integers. Then d_α denotes the set to which the number sequence $x=(x_1, x_2, \ldots)$ belongs if and only if there exists a number $k>0$ such that

$$h_\alpha(x) = \operatorname{lub}_{p\in\mathcal{P}} \sum_{i=1}^{\infty} |x_{p(i)}\alpha_i| < k.$$

h_α is a norm on d_α and (d_α, h_α) is complete.

We classify the d_α spaces and compare them with l_1 and m. Some of the d_α spaces are shown to have a semishrinking basis that is not shrinking. Further investigation of the bases in these spaces yields theorems concerning the conjugate space properties of d_α. We characterize the sequences β such that, given α, $d_\beta = d_\alpha$. A class of manifolds in the first conjugate space of d_α is examined. We establish some properties of the collection of points in the first conjugate space of a normed linear space S that attain their maximum on the unit ball in S. The effect of renorming c_0 and l_1 with h_α and related norms is studied in terms of the change induced on this collection of functionals.

Introduction. The d_α spaces were studied by W. L. C. Sargent [6] in 1960 and more recently by W. Ruckle [5] and D. J. H. Garling [2]. Some of the results in §§I and III appear in one or more of the above papers, as will be indicated.

Throughout this paper if S is a linear space and g is a norm on S then (S, g) will denote S with the norm g. The symbol $(S, g)^*$ denotes the first conjugate space of (S, g) and g^* denotes the conjugate norm on $(S, g)^*$ induced by g. If H is a subset of S then $L(H)$ denotes the linear span of H. The symbol $N(S)$ denotes the origin in S and $U(S, g)$ denotes the unit ball in (S, g). The term basis will refer to a Schauder basis.

I. d_α spaces.

Definition 1.1. Suppose n is a positive integer. Then x_0^n denotes the number sequence $(x_1, x_2, \ldots)$ such that $x_i=1$ if $i\leqq n$ and $x_i=0$ otherwise.

Definition 1.2. Suppose $\alpha\in m$. Then $B(\alpha)$ denotes the number sequence $(B_1(\alpha), B_2(\alpha), \ldots)$ defined as follows: for each i,

$$\begin{aligned} B_i(\alpha) &= h_\alpha(x_0^1) && \text{if } i = 1, \\ &= h_\alpha(x_0^i)-h_\alpha(x_0^{i-1}) && \text{if } i > 1. \end{aligned}$$

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DEFINITION 1.3. Z denotes the number sequence collection to which the sequence $a=(a_1, a_2, \ldots)$ belongs if and only if a is nonincreasing, $a_1>0$ and for each i, $a_i \geqq 0$. $Z_0=c_0 \cap Z$ and $Z_1=l_1 \cap Z$.

The following lemma was found to be very useful in the investigation of the d_α spaces.

LEMMA 1.1. *Suppose* $\alpha \in m$. *Then* $B(\alpha) \in Z$ *and* $d_\alpha=d_{B(\alpha)}$. *Moreover if* $x \in d_\alpha$ *then* $h_\alpha(x)=h_{B(\alpha)}(x)$.

Thus in the investigation of these spaces we need only consider the sequences in Z.

THEOREM 1.1 [2]. $d_\alpha=m$ *if and only if* $\alpha \in l_1$.

THEOREM 1.2 [2]. $d_\alpha=l_1$ *if and only if* $\alpha \in m-c_0$.

Thus the spaces fall naturally into three categories, (1) those that are l_1, (2) those that are m and (3) those that are "between" l_1 and m.

OBSERVATION. If $d_\alpha=m$ then h_α is equivalent to the ordinary norm $|\cdot|_m$ on m, and if $d_\alpha=l_1$ then h_α is equivalent to the ordinary norm $|\cdot|_1$ on l_1.

DEFINITION 1.4. $e=e_1, e_2, \ldots$ denotes the point sequence in m such that for each i, $e_i=(e_1^i, e_2^i, \ldots)$, $e_i^i=1$ and $e_j^i=0$ if $i\neq j$. If e is a basis for a normed linear space (S, g) then $b=b_1, b_2, \ldots$ denotes the point sequence in $(S, g)^*$ that is bi-orthogonal to e. G_e denotes the closure of the linear span of b. If e is a basis for (S, g) and $f\in(S, g)^*$ then the number sequence $(f_1, f_2, \ldots)$ is defined by $f_i=f(e_i)$ for each i.

DEFINITION 1.5. T_1 denotes the linear transformation from $(l_1, |\cdot|_1)^*$ to m defined by $T_1(f)=(f_1, f_2, \ldots)$ for each $f\in(l_1, |\cdot|_1)^*$.

It is well known that T_1 is a congruence (isometry) from $[(l_1, |\cdot|_1)^*, |\cdot|_1^*]$ to $(m, |\cdot|_m)$. The following theorem shows that this relationship between l_1 and m does not necessarily exist between the d_α spaces that are l_1 and those that are m.

THEOREM 1.3. *Suppose* $\alpha \in Z_1$. *Then each two of the following statements are equivalent.*

(1) *There exists a point* $\beta \in Z-Z_0$ *such that* T_1 *is a congruence from* $[(d_\beta, h)^*, h_\beta^*]$ *to* (d_α, h_α).

(2) $\alpha_2=0$.

(3) *There exists a number c such that if* $x\in d_\alpha$ *then* $h_\alpha(x)=c\cdot|x|_m$.

Proof. Suppose $\beta\in Z-Z_0$. $d_\alpha=m$ and $d_\beta=l_1$ and h_α is equivalent to $|\cdot|_m$ and h_β is equivalent to $|\cdot|_1$. Hence $(d_\beta, h_\beta)^*=(l_1, |\cdot|_1)^*$ and T_1 is a reversible linear transformation from $[(d_\beta, h_\beta)^*, h^*]$ onto (d_α, h_α). Now suppose T_1 is a congruence. Suppose further that for each positive integer n, $f^n=T_1^{-1}(\beta_1, \beta_2, \ldots, \beta_n, 0, 0, \ldots)$. Since T_1 is a congruence $h_\alpha(T_1(f^n))=h_\beta^*(f^n)$. But $h_\beta^*(f^n)=1$ so $h_\alpha(T_1(f^1))=\beta_1\alpha_1=1$. Thus $\beta_1=1/\alpha_1$ and $h_\alpha(T_1(f^2))=\beta_1\alpha_1+\beta_2\alpha_2=1$. Since $\beta_2\neq 0$ then $\alpha_2=0$ and (1) implies (2).

Suppose now that $\alpha_2=0$ and $f=(f_1, f_2, \ldots) \in d_\alpha$. Then $h_\alpha(f)=\alpha_1 \cdot |f|_m$. So (2) implies (3).

Now suppose (3). $e_1 \in d_\alpha$ and $h_\alpha(e_1)=\alpha_1=c \cdot |e|_m=c$. So $\alpha_1=c$. Let $\beta=(\beta_1, \beta_2, \ldots)$ such that for each i, $\beta_i=1/\alpha_1$. Then $\beta \in Z-Z_0$, $d_\beta=l_1$ and if $x \in d_\beta$ then $h_\beta(x) =(1/\alpha_1)|x|_1$. So T_1 is a congruence.

II. **Bases in the d_α spaces.** The following are some of the properties that a point sequence $p_1, p_2, \ldots$ may have and are listed here for easy reference.

DEFINITION 2.1. Suppose (S, g) is a normed linear space, W is the set of positive integers and Q is the collection of all finite subsets of W. Suppose further that $p=p_1, p_2, \ldots$ is a sequence each term of which is a point of S.

(i) p is orthogonal means that if each of H and K is in Q, $H \subseteq K$ and $a_1, a_2, \ldots$ is a number sequence, then $g(\sum_{i \in H} a_i p_i) \leqq g(\sum_{i \in K} a_i p_i)$.

(ii) p is strictly orthogonal means that p is orthogonal and if each of H and K is in Q, and $H \subseteq K$ and $a_1, a_2, \ldots$ is a number sequence, then the following two statements are equivalent.

(1) $g(\sum_{i \in H} a_i p_i)=g(\sum_{i \in K} a_i p_i)$.

(2) $H=K$ or $H \neq K$ and if $i \in K-H$ then $a_i=0$.

(iii) p is strictly coorthogonal means that if each of H and K is in Q, and $H \subseteq K$, and $a_1, a_2, \ldots$ is a number sequence then

$$g\Big(\sum_{i \in W-K} a_i p_i\Big) \leqq g\Big(\sum_{i \in W-H} a_i p_i\Big)$$

and the following two statements are equivalent.

(1) $g(\sum_{i \in W-K} a_i p_i)=g(\sum_{i \in W-H} a_i p_i)$.

(2) $H=K$, or $H \neq K$ and if $i \in K-H$ then $a_i=0$.

(iv) If p is a basis, p is unconditional means that if $x \in S$ and $x=\sum_{i=1}^{\infty} x_i p_i$ and if $r \in \mathscr{P}$, then $x=\sum_{i=1}^{\infty} x_{r(i)} p_{r(i)}$.

(v) If p is a basis, p is semishrinking means that there exists a number $c>0$ such that

(1) $0<\text{glb}_i\,(g(p_i)) \leqq \text{lub}_i\,(g(p_i))<c$, and

(2) if $f \in (S, g)^*$ then $\lim_{n \to \infty} f(p_n)=0$.

(vi) If p is a basis, p is shrinking means that if $q=q_1, q_2, \ldots$ is the point sequence in $(S, g)^*$ that is biorthogonal to p and $y_1, y_2, \ldots$ is a bounded point sequence in S such that for each j, $\lim_{n \to \infty} q_j(y_n)=0$, then if $f \in (S, g)^*$, $\lim_{n \to \infty} f(y_n)=0$.

THEOREM 2.1. *Suppose $\alpha \in Z-Z_1$. Then the point sequence $e=e_1, e_2, \ldots$ in (d_α, h_α) has the following properties*:

(1) *e is orthogonal*;

(2) *e is strictly orthogonal*;

(3) *e is strictly coorthogonal*;

(4) *e is a basis*;

(5) *e is unconditional*;

(6) *e is boundedly complete.*

That e is orthogonal, strictly orthogonal, and strictly coorthogonal is easily verified. Garling [2] has shown that the linear span of e, $L(e)$, is dense in d_α and so it follows that since for each i, $e_i \neq N(d_\alpha)$ and since e is orthogonal, that e is an unconditional basis for d_α. That e is boundedly complete is obvious.

It may be noted that the collection of d_α spaces can be enlarged as follows: if $\alpha=(\alpha_1, \alpha_2, \ldots)$ is a bounded number sequence and $\alpha_i \neq 0$ for some i and if $k \geqq 1$, then $d_{\alpha,k}$ denotes the set to which the number sequence $x=(x_1, x_2, \ldots)$ belongs only in the case that there exists a number c such that

$$h_{\alpha,k}(x) = \operatorname*{lub}_{p \in \mathscr{P}} \left[\sum_{i=1}^{\infty} |x_{p(i)}\alpha_i|^k \right]^{1/k} < c.$$

In this case, results similar to Lemma 1.1, Theorem 1.1, Theorem 1.2 and Theorem 2.1 may still be obtained. Theorem 1.1 becomes $d_{\alpha,k}=m$ if and only if $\alpha \in l_k$. Theorem 1.2 becomes $d_{\alpha,k}=l_k$ if and only if $\alpha \in m-c_0$. Again, if $d_{\alpha,k}=m$ then $h_{\alpha,k}$ is equivalent to $|\cdot|_m$ and if $d_{\alpha,k}=l_k$ then $h_{\alpha,k}$ is equivalent to the ordinary norm on l_k, $|\cdot|_k$.

Here then, we have spaces some of which are l_k, some m and some "between" l_k and m. The remainder of this paper deals with the d_α (i.e. $d_{\alpha,1}$) spaces.

A. Pełczyński and W. Szlenk [3], answering a question of I. Singer, constructed an example of a normed linear space with a basis that was semishrinking but not shrinking. J. R. Retherford [4] has shown that the space (d), which is d_α with $\alpha_i=1/i$, also has a basis that is semishrinking but not shrinking.

THEOREM 2.2. *Suppose that $\alpha \in Z_0-Z_1$. Then the basis e for (d_α, h_α) is semishrinking but not shrinking.*

Proof. If $\alpha \in Z_0-Z_1$ then there exists a point $x=(x_1, x_2, \ldots)$ in d_α such that for each i, $x_i \geqq x_{i+1} \geqq 0$ and $x \notin l_1$. Suppose $f \in (d_\alpha, h_\alpha)^*$ and that $\lim_{i\to\infty} f_i \neq 0$. Then there exists a number $c>0$ and a subsequence $f_{n_1}, f_{n_2}, \ldots$ of $f_1, f_2, \ldots$ such that for each i, $|f_{n_i}| \geqq c$. Let $y=(y_1, y_2, \ldots)$ be the point of d_α such that $y_i=0$ if $i \neq n_j$ for every j and $y_i=x_j \cdot |f_{n_j}|/f_{n_j}$ if $i=n_j$ for some j. So if N is a number there exists an integer s such that

$$N < c \cdot \sum_{i=1}^{s} x_i \leqq \sum_{i=1}^{s} |f_{n_i}| \cdot x_i = \sum_{i=1}^{n_s} f_i y_i.$$

So $f \notin (d_\alpha, h_\alpha)^*$ and we have a contradiction. Hence $\lim_{i\to\infty} f_i=0$. For each i, $h_\alpha(e_i)=\alpha_1$ so e is semishrinking.

For each positive integer n, let $S_n=\sum_{i=1}^{n} \alpha_i$ and $y_n=(1/S_n)\cdot\sum_{i=1}^{n} e_i$. Then $h_\alpha(y_n)=1$. Let F denote the point of $(d_\alpha, h_\alpha)^*$ defined as follows: if $x \in d_\alpha$ and $x=(x_1, x_2, \ldots)$, then $F(x)=\sum_{i=1}^{\infty} x_i\alpha_i$. For each n, $F(y_n)=1$. But if $b=b_1, b_2, \ldots$ is the point sequence in $(d_\alpha, h_\alpha)^*$ biorthogonal to e and if j is a positive integer then $\lim_{n\to\infty} b_j(y_n)=0$. So e is not shrinking.

COROLLARY 2.1. *Suppose $\alpha \in Z$. Then $[(d_\alpha, h_\alpha)^*, h_\alpha^*]$ is not separable.*

Proof. It is well known, for instance [1, p. 77], that if $p=p_1, p_2, \ldots$ is an unconditional basis for a normed linear complete space (S, g) then p is shrinking if and only if $[(S, g)^*, g^*]$ is separable. Thus if $\alpha \in Z-Z_1$, since e is an unconditional basis that is not shrinking we have that $[(d_\alpha, h_\alpha)^*, h_\alpha^*]$ is not separable. If $\alpha \in Z_1$ then (d_α, h_α) is isomorphic to $(m, |\cdot|_m)$ and thus $[(d_\alpha, h_\alpha)^*, h_\alpha^*]$ is not separable.

DEFINITION 2.2. Suppose (S, g) is a normed linear space and that H is a linear manifold in $(S, g)^*$. J^H denotes the transformation from S into $(H, g^*)^*$ defined as follows: if $x \in S$ and $f \in H$ then $[J^H(x)](f)=f(x)$.

THEOREM 2.3. *Suppose $\alpha \in Z-Z_1$. Then J^{G_e} is a congruence.*

Proof. (d_α, h_α) is complete and e is an unconditional basis for (d_α, h_α) that is boundedly complete. Thus it follows from a result of Singer [7] that J^{G_e} is a congruence.

THEOREM 2.4. *Suppose $\alpha \in Z-Z_1$ and b is the point sequence in $(d_\alpha, h_\alpha)^*$ biorthogonal to e. Then b is*

(1) *orthogonal,*
(2) *a basis for (G_e, h_α^*),*
(3) *unconditional,*
(4) *not boundedly complete,*
(5) *not strictly orthogonal.*

Proof. Since e is orthogonal b must be orthogonal and since b is orthogonal and $L(b)$ is dense in G_e and since $b_i \neq N(d_\alpha, h_\alpha)^*$ for each i, it follows that b is an unconditional basis for G_e. Since $[(d_\alpha, h_\alpha)^*, h_\alpha^*]$ is not separable there exists a point $y \in (d_\alpha, h_\alpha)^* - G_e$. Suppose $h_\alpha^*(y)=c$. Suppose further that n is a positive integer and $y^n=\sum_{i=1}^n y_i b_i$. Then $h_\alpha^*(y^n) \leqq c$. But $y \notin G_e$ so b is not boundedly complete. Suppose n is a positive integer and $\alpha^n=\sum_{i=1}^n \alpha_i b_i$. Let $x=(1/\alpha_1)e_1$. Then $x \in U(d_\alpha, h_\alpha)$ and $\alpha^n(x)=1$. Suppose $y=(y_1, y_2, \ldots)$ is a point of $U(d_\alpha, h_\alpha)$. Then $|\alpha^n(y)| = |\sum_{i=1}^n y_i \alpha_i| \leqq h_\alpha(y)=1$. So $h_\alpha^*(\alpha^n)=1$. Hence b is not strictly orthogonal.

COROLLARY 2.2. *Suppose $\alpha \in Z-Z_1$ and (S, g) is a normed linear complete space. Then (G_e, h_α^*) is not isomorphic to $[(S, g)^*, g^*]$.*

Proof. Singer has shown [7] that a normed linear complete space (S, g) with an unconditional basis, p, is isomorphic to the conjugate space of some normed linear space if and only if p is boundedly complete. Thus Corollary 2.2 follows.

COROLLARY 2.3. *If $\alpha \in Z$ then (d_α, h_α) is not reflexive.*

In case $\alpha \in Z_1$ the question whether or not (d_α, h_α) is congruent to the conjugate space of some normed linear space is answered by the following theorem.

THEOREM 2.5. *Suppose $\alpha \in Z_1$ and g_α is the norm on l_1 defined as follows: if $x \in l_1$ and $x=(x_1, x_2, \ldots)$ then*

$$g_\alpha(x) = \text{lub}\left\{\left|\sum_{i=1}^{\infty} y_i x_i\right| \mid y \in U(d_\alpha, h_\alpha), y = (y_1, y_2, \ldots)\right\}.$$

Then each of the following statements is true.

(1) $[(C_0, h_\alpha)^*, h_\alpha^*]$ *is congruent to* (l_1, g_α);

(2) g_α *is equivalent to* $|\cdot|_1$;

(3) $[(l_1, g_\alpha)^*, g_\alpha^*]$ *is congruent to* (d_α, h_α);

(4) $[((C_0, h_\alpha)^*, h_\alpha^*)^*, h_\alpha^{**}]$ *is congruent to* (d_α, h_α).

III. $d_\alpha = d_\beta$. W. J. Davis, in a private communication, has characterized the extreme points of $U(d_\alpha, h_\alpha)$ in the case $\alpha \in Z_0 - Z_1$.

THEOREM 3.1 (DAVIS). *Suppose* $\alpha \in Z_0 - Z_1$, $x \in d_\alpha$, $x = (x_1, x_2, \ldots)$ *and* $h_\alpha(x) = 1$. *Then* (1) *implies* (2).

(1) x *is an extreme point.*

(2) *There exists an integer* n *such that if* $i > n$ *then* $x_i = 0$ *and if* $x_j \neq 0$ *and* $x_k \neq 0$ *then* $|x_j| = |x_k|$.

This gives us the following result of Garling.

THEOREM 3.2 [2]. *Suppose* $\alpha \in Z_0 - Z_1$, $f \in (d_\alpha, h_\alpha)^*$ *and* $r \in \mathscr{P}$. *Suppose further that for each* i, $|f_{r(i)}| \geqq |f_{r(i+1)}|$. *Then*

$$h_\alpha^*(f) = \mathrm{lub}_n \sum_{i=1}^{n} |f_{r(i)}| \Big/ \sum_{i=1}^{n} \alpha_i.$$

Garling has also characterized G_e.

THEOREM 3.3 [2]. *Suppose* $\alpha \in Z_0 - Z_1$, $f \in (d_\alpha, h_\alpha)^*$ *and* $r \in \mathscr{P}$. *Suppose further that for each* i, $|f_{r(i)}| \geqq |f_{r(i+1)}|$. *Then* $f \in G_e$ *if and only if*

$$\lim_{n \to \infty} \sum_{i=1}^{n} |f_{r(i)}| \Big/ \sum_{i=1}^{n} \alpha_i = 0.$$

THEOREM 3.4 [5]. *Suppose each of* α *and* β *is in* Z *and for each positive integer* $S(\alpha, n) = \sum_{i=1}^{n} \alpha_i$. *Then* $d_\alpha = d_\beta$ *if and only if there exists a number* k_1 *and a number* k_2 *such that if* n *is a positive integer then* $S(\alpha, n) \leqq k_1 S(\beta, n)$ *and* $S(\beta, n) \leqq k_2 S(\alpha, n)$.

OBSERVATION. Whenever $d_\alpha = d_\beta$ then h_α is equivalent to h_β.

THEOREM 3.5. *Suppose* $\alpha \in Z - Z_1$ *and* $f \in (d_\alpha, h_\alpha)^*$. *Suppose further that* $\beta = (f_1, f_2, \ldots)$. *Then each two of the following statements are equivalent.*

(1) $d_\alpha = d_\beta$;

(2) $f \in (d_\alpha, h_\alpha)^* - G_e$;

(3) *if* $F \in (d_\alpha, h_\alpha)^*$ *such that for each* i, $F_i = \alpha_i$ *then* $F \in (d_\beta, h_\beta)^*$.

IV. A collection of manifolds in $(d_\alpha, h_\alpha)^*$.

DEFINITION 4.1. Suppose (S, g) is a normed linear space and H is a linear manifold in $(S, g)^*$. The statement that H is absolutely total means that if $x \in S$ then

$$g(x) = \mathrm{lub}\,\{|f(x)| \mid f \in H \text{ and } g^*(f) = 1\}.$$

DEFINITION 4.2. The statement that a normed linear space (S, g) has property t means that if H is an absolutely total linear manifold in $(S, g)^*$ then H is dense in $[(S, g)^*, g^*]$.

B. E. Wilder has shown [8] that each of $(c_0, |\cdot|_0)$ and $(c, |\cdot|_c)$, where $|\cdot|_0$ and $|\cdot|_c$ are the ordinary norms on c_0 and c respectively, is a nonreflexive space with property t. He has also shown that $(c_{0,1}, |\cdot|_{0,1})$, where $|\cdot|_{0,1}$ is the ordinary max. norm on $c_{0,1}$, does not have property t. It has been conjectured that the only nonreflexive spaces that have property t are isomorphic to $(c_0, |\cdot|_0)$. The following theorem settles this conjecture in the negative.

THEOREM 4.1. *Suppose* $\alpha \in Z - Z_1$ *and there exists a number* M *such that if* i *is a positive integer* $\alpha_i < (M+1)\alpha_{i+1}$. *Then* (G_e, h_α^*) *has property* t.

Proof. Suppose for convenience that $\alpha_1 = 1$. J^{G_e} is a congruence from (d_α, h_α) to $[(G_e, h_\alpha^*)^*, h_\alpha^{**}]$. Let T denote the inverse of J^{G_e}. Suppose that L is an absolutely total linear manifold in $(G_e, h_\alpha^*)^*$, and that n is a positive integer. Then $b_n \in G_e$ and if $\varepsilon > 0$ there exists a point $f \in L$ such that $h_\alpha^{**}(f) \leqq 1$ and $|h_\alpha^*(b_n) - f(b_n)| < \varepsilon/(M+2)$. Suppose $T(f) = (f_1, f_2, \ldots)$. Then since $h_\alpha^*(b_n) = 1/\alpha_1 = 1$ and $f(b_n) = b_n(T(f)) = f_n$, we have that $|1 - f_n| < \varepsilon/(M+2)$. $h_\alpha^{**}(f) = h_\alpha(\sum_{i=1}^{\infty} f_i b_i) \leqq 1$, so $|f_n| \leqq 1$ and $1 - |f_n| \leqq 1 - f_n = |1 - f_n| < \varepsilon/(M+2)$. Pick $r \in \mathscr{P}$ such that $r(1) = n$ and if $i \geqq 2$, $|f_{r(i)}| \geqq |f_{r(i+1)}|$. For each i, let $F_i = f_{r(i)}$. Then

$$|F_1| \cdot \alpha_1 + \sum_{i=2}^{\infty} |F_i| \alpha_i = \sum_{i=1}^{\infty} |F_i| \alpha_i \leqq h_\alpha^{**}(f) \leqq 1.$$

So

$$\sum_{i=2}^{\infty} |F_i| \alpha_i \leqq 1 - |f_n| < \frac{\varepsilon}{(M+2)}.$$

Let x and y be points of d_α defined by $x = (1 - f_n) \cdot e_n$ and $y = f_n e_n - \sum_{i=1}^{\infty} f_i e_i$. Suppose $p = p_1, p_2, \ldots$ is the point sequence in $(G_e, h_\alpha^*)^*$ that is biorthogonal to b. Then if i is a positive integer, $T(p_i) = e_i$. So

$$h_\alpha^{**}(p_n - f) = h_\alpha\left(e_n - \sum_{i=1}^{\infty} f_i e_i\right) = h_\alpha(x+y) \leqq h_\alpha(x) + h_\alpha(y)$$
$$= 1 - f_n + h_\alpha(y) < \frac{\varepsilon}{(M+2)} + h_\alpha(y).$$

Now

$$h_\alpha(y) = \sum_{i=1}^{\infty} |F_{i+1}| \alpha_i = \sum_{i=1}^{\infty} |F_{i+1}| \alpha_{i+1} + \sum_{i=1}^{\infty} |F_{i+1}| \cdot [\alpha_i - \alpha_{i+1}]$$
$$< \frac{\varepsilon}{(M+2)} + M \sum_{i=1}^{\infty} |F_{i+1}| \cdot \alpha_{i+1} < \frac{\varepsilon}{(M+2)} + \frac{M\varepsilon}{(M+2)} = \frac{\varepsilon(M+1)}{(M+2)}.$$

So

$$h_\alpha^{**}(p_n - f) = h_\alpha\left(e_n - \sum_{i=1}^{\infty} f_i e_i\right) < \frac{\varepsilon}{(M+2)} + \frac{\varepsilon(M+1)}{(M+2)} = \varepsilon.$$

Hence p_n is a point or a limit point of L.

Suppose s is a positive integer and each of $c_1, c_2, \ldots, c_s$ is a number. Suppose further that $x=c_1p_1+\cdots+c_np_n$ and for each j, $1\leqq j\leqq s$, let $y_1^j, y_2^j, \ldots$ be a point sequence in L converging to p_j. If $\varepsilon>0$ and if $j\leqq s$ is a positive integer such that $c_j\neq 0$, then there exists a number n_j such that if $i>n_j$ then $|y_i^j-p_j|<\varepsilon/(|c_j|\cdot s)$. For each positive integer j, $j\leqq s$, let $y_j=c_1y_j^1+\cdots+c_sy_j^s$. Then $y_j\in L$. Let $N=\max\{n_j\}$. Then if $i>N$,

$$\begin{aligned} h_\alpha^{**}(x-y_i) &= h_\alpha^{**}(c_1(p_1-y_i^1)+\cdots+c_s(p_s-y_i^s)) \\ &\leqq |c_1|\cdot h_\alpha^{**}(p_1-y_i^1)+\cdots+|c_s|\cdot h_\alpha^{**}(p_s-y_i^s) < \varepsilon. \end{aligned}$$

So x is a point or a limit point of L. Hence any point in $L(p)$ is a point or limit point of L and thus the closure of L contains $L(p)$. $L(e)$ is dense in (d_α, h_α) and J^{G_e} maps $L(e)$ onto $L(p)$ so $L(p)$ is dense in $[(G_e, h_\alpha^*)^*, h_\alpha^{**}]$. Thus L is dense in $[(G_e, h_\alpha^*)^*, h_\alpha^{**}]$.

Every α in $Z-Z_0$ has the property that there is a number M such that $\alpha_i<(M+1)\alpha_{i+1}$ for each i. Some of the sequences in Z_0-Z_1 have this property, for example $\alpha=(1, 1/2, \ldots, 1/i, \ldots)$, while some other sequences in Z_0-Z_1 do not. If $\alpha\in Z_0-Z_1$ then (d_α, h_α) is not isomorphic to $(l_1, |\cdot|_1)$ and thus (G_e, h_α^*) is not isomorphic to $(c_0, |\cdot|_0)$. Thus we have the following corollary.

COROLLARY 4.1. *There exists a nonreflexive normed linear space (S, g) that has property t but is not isomorphic to $(c_0, |\cdot|_0)$.*

CONJECTURE. If $\alpha\in Z_0-Z_1$ then (G_e, h_α^*) has property t.

V. **Regular functionals.**

DEFINITION 5.1. Suppose (S, g) is a normed linear space and $f\in(S, g)^*$. The statement that f is regular on (S, g) means that there exists a point $x\in S$ such that $g(x)=1$ and $f(x)=g^*(f)$.

$R(S, g)$ denotes the subset of $(S, g)^*$ to which the point f belongs only in the case that f is regular on (S, g).

THEOREM 5.1. *Suppose that (S, g) is a normed linear space and that $p=p_1, p_2, \ldots$ is a monotone basis for (S, g). Suppose further that $q=q_1, q_2, \ldots$ is the point sequence in $(S, g)^*$ that is biorthogonal to p. If $f\in L(q)$ then $f\in R(S, g)$.*

Proof. Suppose $x\in S$ and $x=\sum_{i=1}^\infty x_ip_i$. If n is a positive integer let $\bar{x}^n$ be the point of E_n defined by $\bar{x}^n=(x_1, x_2, \ldots, x_n)$. Let g_n denote the norm on E_n defined by $g(\bar{x}^n)=g(\sum_{i=1}^n x_ip_i)$. Suppose $y\in L(q)$ and $y=y_1q_1+\cdots+y_nq_n$. Then if $x\in S$ and $x=\sum_{i=1}^\infty x_ip_i$, $y(x)=\sum_{i=1}^n y_ix_i$. Let y' be the point of $[(E_n, g_n)^*, g_n^*]$ defined as follows: if $x\in E_n$ and $x=(x_1, x_2, \ldots, x_n)$ then $y'(x)=\sum_{i=1}^n y_ix_i$. y' is regular so there exists a point $z=(z_1, z_2, \ldots, z_n)$ in E_n such that $g_n(z)=1$ and $y'(z)=g_n^*(y')$. Examine the point x of S defined by $x=\sum_{i=1}^n z_ip_i$. $g(x)=g_n(z)=1$ and $y(x)=\sum_{i=1}^n y_iz_i=g_n^*(y')$. Now suppose $r=\sum_{i=1}^\infty r_ip_i$ and $g(r)=1$. Then $g_n(\bar{r}^n)\leqq 1$ and

$$|y(r)| = \left|\sum_{i=1}^n y_ir_i\right| = |y'(\bar{r}^n)| \leqq g_n^*(y').$$

So $g^*(y)=g_n^*(y')$ and $y\in R(S, g)$.

COROLLARY 5.1. *Suppose (S, g) is a normed linear complete space, $p=p_1, p_2, \ldots$ is a basis for (S, g) and $q=q_1, q_2, \ldots$ is the point sequence in $(S, g)^*$ that is biorthogonal to p. Then there exists a norm h on S such that h is equivalent to g and $L(q) \subseteq R(S, h)$.*

Proof. It is well known [1, p. 67] that there exists a norm h on S equivalent to g such that p is monotone in (S, h). Hence, by Theorem 5.1, $L(q) \subseteq R(S, h)$.

THEOREM 5.2. *Suppose $\alpha \in Z - Z_0$, e is the ordinary basis for (d_α, h_α) and $f \in (G_e, h_\alpha^*)^*$. Suppose further that $p=p_1, p_2, \ldots$ is the point sequence in $(G_e, h_\alpha^*)^*$ that is biorthogonal to the basis b in (G_e, h_α^*). Then $\mathscr{L}(p) = R(G_e, h_\alpha^*)$.*

Proof. Since b is an orthogonal basis for (G_e, h_α^*) then, by Theorem 5.1, $\mathscr{L}(p) \subseteq R(G_e, h_\alpha^*)$. Suppose $f \in (G_e, h_\alpha^*)^* - L(p)$. J^{G_e} is a congruence from (d_α, h_α) to $[(G_e, h_\alpha^*)^*, h_\alpha^{**}]$ and for each i, $J^{G_e}(e_i) = p_i$. Thus if T denotes the inverse of J^{G_e} and $T(f) = (f_1, f_2, \ldots)$, then $T(f) \in d_\alpha - \mathscr{L}(e)$. Pick $r \in \mathscr{P}$ such that for each i, $|f_{r(i)}| \geqq |f_{r(i+1)}|$ and let $F_i = |f_{r(i)}|$. Suppose n is a positive integer and $\alpha^n = \sum_{i=1}^n \alpha_i b_i$. Then if $T(F) = (F_1, F_2, \ldots)$,

$$F(\alpha^n) = \sum_{i=1}^{n} F_i \alpha_i < \sum_{i=1}^{n+1} F_i \alpha_i = F(\alpha^{n+1}).$$

Suppose $y \in U(G_e, h_\alpha^*)$. Then $\lim_{i\to\infty} y_i = 0$. Pick $s \in \mathscr{P}$ such that for each i, $|y_{s(i)}| \geqq |y_{s(i+1)}|$ and let $Y_i = |y_{s(i)}|$. Let $Y = \sum_{i=1}^\infty Y_i b_i$ and $c_1 = \operatorname{glb} \alpha_i$. Since $\lim_{i\to\infty} y_i = 0$ there exists a number n_1 such that if $i > n_1$ then $Y_i < c_1$. Let $c_2 = \sum_{i=1}^{n_1} F_i(\alpha_i - Y_i)$. Suppose $c_2 < 0$. Then $\sum_{i=1}^{n_1} F_i \alpha_i < \sum_{i=1}^{n_1} F_i Y_i$. There exists a number $t > 0$ such that the point z of (d_α, h_α) defined by $z = \sum_{i=1}^{n_1} t \cdot F_i b_i$ has norm 1. So

$$t \sum_{i=1}^{n_1} F_i \alpha_i = 1 < t \cdot \sum_{i=1}^{n_1} F_i Y_i = Y(z).$$

So $h_\alpha^*(Y) > 1$. But $h_\alpha^*(Y) = h_\alpha^*(y) = 1$ so $c_2 \geqq 0$, and

$$\sum_{i=1}^{n_1+1} F_i(\alpha_i - Y_i) = c_3 > c_2 \geqq 0.$$

Let n_2 be a positive integer such that $\sum_{i=n_2+1}^\infty F_i Y_i < c_3/2$. Suppose $N = \max\{n_1+1, n_2\}$. Then if $n > N$,

$$F(\alpha^n) - F(Y) = \sum_{i=1}^{n} F_i(\alpha_i - Y_i) - \sum_{i=n+1}^{\infty} F_i Y_i > c_3 - \frac{c_3}{2} = \frac{c_3}{2} > 0.$$

So $F(\alpha^n) > F(Y)$. $F(Y) \geqq F(y)$ and so F is not regular and f is not regular. Hence $R(G_e, h_\alpha^*) = L(e)$ and the theorem is proved.

It may be noted that if we define the norm $h_{\alpha,0}$ on c_0 by $h_{\alpha,0}(x) = h_\alpha^*(T_1^{-1}(x))$ for each $x \in c_0$, then T_1 restricted to G_e is a congruence from (G_e, h_α^*) to $(c_0, h_{\alpha,0})$ that maps the basis b in (G_e, h_α^*) onto the basis e in $(c_0, h_{\alpha,0})$. Thus Theorem 4.2 gives us, in case $\alpha_1 = 1$ and $\alpha_2 = 0$, the usual characterization of $R(c_0, |\cdot|_0)$.

THEOREM 5.3. *Suppose that $\alpha \in Z_1$ and e is the ordinary basis for (c_0, h_α). Then $L(b) = R(c_0, h_\alpha)$ if and only if $\alpha_2 = 0$.*

Proof. If $\alpha_2=0$ then (c_0, h_α) is congruent to $(c_0, |\cdot|_0)$ and $h_\alpha = \alpha_1 \cdot |\cdot|_0$. So $R(c_0, h_\alpha) = R(c_0, |\cdot|_0)$. But $R(c_0, |\cdot|_0) = L(b)$ so $R(c_0, h_\alpha) = L(b)$. Suppose $\alpha_2 \neq 0$. If $\alpha' = \sum_{i=1}^{\infty} \alpha_i b_i$, then $h_\alpha^*(\alpha') = 1$. Suppose $\alpha' \notin L(b)$ and $y = (y_1, y_2, \ldots)$ is the point of $U(c_0, h_\alpha)$ defined as follows: $y_1 = 1/\alpha_1$ and $y_i = 0$ if $i > 1$. Then $\alpha'(y) = 1$ so $\alpha' \in R(c_0, h_\alpha)$ and $L(b) \neq R(c_0, h_\alpha)$. Suppose now that $\alpha' \in L(b)$. Then there exists an integer n such that $\alpha_n \neq 0$ and $\alpha_{n+1} = 0$. Let f be the point of $(c_0, h_\alpha)^*$ defined as follows:

$$f_i = \alpha_i \quad \text{if } 1 \leqq i \leqq n-1 \quad \text{and} \quad f_i = \frac{\alpha_n}{2^{i-n+1}} \quad \text{if } i \geqq n.$$

Then $f \notin L(b)$ and it can be shown that f is regular on (c_0, h_α). Hence $L(b) \neq R(c_0, h_\alpha)$, and the theorem is proved.

DEFINITION 5.2. Suppose g is a norm on l_1. The statement that g has property r means that

(1) g is equivalent to $|\cdot|_1$; and

(2) if $x = (x_1, x_2, \ldots)$ is a point in l_1 and $s \in \mathscr{P}$ and if $y = (y_1, y_2, \ldots)$ is the point in l_1 such that for each i, $y_i = |x_{s(i)}|$, then $g(y) = g(x)$.

THEOREM 5.4. *Suppose that g is a norm on l_1 and g has property r. Suppose that $f \in (l_1, g)^*$ and that if j is a positive integer then $|f_j| < \text{lub}_i |f_i|$. Then $f \notin R(l_1, g)$.*

This result is well known in case $g = |\cdot|_1$ and a proof may be constructed similar to the proof of that case.

THEOREM 5.5. *Suppose that $\alpha \in Z - Z_0$. Then only one of the following statements is true.*

(1) *For each positive integer i, $\alpha_i = \alpha_1$.*

(2) *$R(d_\alpha, h_\alpha)$ is a proper subset of $R(l_1, |\cdot|_1)$.*

Proof. Suppose (1) is true. Then the transformation T from (d_α, h_α) to $(l_1, |\cdot|_1)$ defined by $T(x) = \alpha_1 x$, for each $x \in d_\alpha$, is a congruence and $R(d_\alpha, h_\alpha) = R(l_1, |\cdot|_1)$. So (2) is not true. It is well known that $f \in (l_1, |\cdot|_1)^*$ then $f \notin R(l_1, |\cdot|_1)$ if and only if for each positive integer j, $|f_j| < \text{lub}_i \{|f_i|\}$. Therefore, since h_α has property r, $R(d_\alpha, h_\alpha) \subseteq R(l_1, |\cdot|_1)$. Suppose (1) is not true. Let n be the least integer such that $\alpha_n > \alpha_{n+1}$. Let f be the point of $(d_\alpha, h_\alpha)^*$ defined as follows:

$$f_i = 1 \quad \text{if } 1 \leqq i \leqq n \quad \text{and} \quad f_i = \frac{i-n}{i-n+1} \quad \text{if } i > n; \qquad f \in R(l_1, |\cdot|_1).$$

However it can be shown that f is not in $R(d_\alpha, h_\alpha)$, so (2) is true.

Thus it is seen that, in case $\alpha \in Z - Z_0$, $R(d_\alpha, h_\alpha)$ is largest when (d_α, h_α) is congruent to $(l_1, |\cdot|_1)$.

DEFINITION 5.3. Suppose (S, g) is a normed linear space and H is a linear manifold in $(S, g)^*$. The statement that H is maximal regular in $(S, g)^*$ means that

(1) $H \subseteq R(S, g)$.

(2) If L is a linear manifold in $(S, g)^*$ and H is a proper subset of L, then there exists a point $f \in L - H$ such that f is not in $R(S, g)$.

DEFINITION 5.4. Suppose that (S, g) is a normed linear space. Then Q denotes the transformation from (S, g) to $[((S, g)^*, g^*)^*, g^{**}]$ defined as follows: if $x \in S$. and $f \in (S, g)^*$ then $Q_{(x)}(f) = f(x)$. $Q(S)$ denotes the image of Q.

THEOREM 5.6. *Suppose g is a norm on l_1 and g has property r. Suppose further that the ordinary basis e for (l_1, g) is orthogonal. Then G_e is maximal regular.*

Proof. By Theorem 5.1, $G_e \subseteq R(l_1, g)$. Suppose L is a linear manifold in $(l_1, g)^*$ and G is a proper subset of L. Suppose further that $f \in L - G_e$ and $T_1(f) = (f_1, f_2, \ldots)$. Consider the following two cases.

I. Suppose there exists a positive integer n such that $F_n = T_1^{-1}(f_{n+1}, f_{n+2}, \ldots)$ is not regular on (l_1, g). In this case let y be the point of G_e such that $y_i = -f_i$ if $1 \leqq i \leqq n$ and $y_i = 0$ if $i > n$. Then $y + f \in L$ and $y + f$ is not regular on (l_1, g).

II. Suppose that for each positive integer n, $F_n = T_1^{-1}(f_{n+1}, f_{n+2}, \ldots)$ is regular on (l_1, g). Let n_1 denote the least integer such that $|f_{n_1}| = |T_1(f)|_m$. Let n_2 denote the least integer such that $n_2 > n_1$ and $|f_{n_2}| = |T_1(F_{n_1})|_m$. If j is a positive integer, $j \geqq 2$, let n_j denote the least integer such that $n_j > n_{j-1}$ and $|f_{n_j}| = |T_1(F_{n_{j-1}})|_m$. Then $|f_{n_1}|, |f_{n_2}|, \ldots$ is a nonincreasing subsequence of $|f_1|, |f_2|, \ldots$ converging to a number $k > 0$. Let $f_{s_1}, f_{s_2}, \ldots$ be the subsequence of $f_1, f_2, \ldots$ to which the number f_j belongs only in the case that $|f_j| \geqq k$. For each positive integer i, let $d_i = |f_{s_i}| - k + k/2s_i$. Define $y = (y_1, y_2, \ldots)$ as follows:

$$
\begin{aligned}
y_i &= 0 && \text{if } i \neq s_j \text{ for every } j, \\
&= -d_j && \text{if } i = s_j \text{ for some } j \text{ and } f_{s_j} \geqq 0, \\
&= d_j && \text{if } i = s_j \text{ for some } j \text{ and } f_{s_j} < 0.
\end{aligned}
$$

It can be shown that $Y = T_1^{-1}(y)$ is in G_e so $Y + f \in L$ and $f + Y$ is not regular on (l_1, g).

COROLLARY 5.2. *Suppose $\alpha \in Z_1$. Then $Q(c_0)$ is maximal regular in*

$$[((c_0, h_\alpha)^*, h_\alpha^*), h_\alpha^{**}].$$

COROLLARY 5.3. *Suppose $\alpha \in Z - Z_0$ and e is the ordinary basis for (d_α, h_α). Then $Q(G_e)$ is maximal regular in $[((G_e, h_\alpha^*)^*, h_\alpha^{**})^*, h_\alpha^{***}]$.*

CONJECTURE. Suppose (S, g) is a normed linear space. Then $Q(S)$ is maximal regular in $[((S, g)^*, g^*)^*, g^{**}]$.

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