

ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF HYPERBOLIC INEQUALITIES⁽¹⁾

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Abstract. This paper discusses the asymptotic behavior of C^2 solutions $u = u(t, x_1, \dots, x_\nu)$ of the inequality (1) $|Lu| \leq k_1(t, x)|u| + k_2(t, x)\|\nabla u\|$, in domains in (t, x) -space which grow unbounded in x as $t \rightarrow \infty$. The operator L is a second order hyperbolic operator with variable coefficients. The main results establish the maximum rate of decay of nonzero solutions of (1). This rate depends on the asymptotic behavior of k_1, k_2 , and the time derivatives of the coefficients of L .

1. Introduction. Let L be defined on C^2 functions $u = u(t, x_1, \dots, x_\nu)$ by

$$Lu = Au - \partial^2 u / \partial t^2$$

where

$$A \equiv \sum_{i,j=1}^{\nu} \frac{\partial}{\partial x_i} a_{ij} \frac{\partial}{\partial x_j}$$

is a symmetric uniformly elliptic operator. Thus we assume that the $a_{ij} = a_{ij}(t, x)$ are C^1 functions on $\mathcal{H} \equiv \{(t, x) \in \mathbf{R} \times \mathbf{R}^\nu : t \geq 0\}$ with $a_{ij} = a_{ji}$ for $1 \leq i, j \leq \nu$. We also make the assumption

(A₀) There are positive constants m, M such that

$$m^2 \leq \sum_{i,j=1}^{\nu} a_{ij}(t, x) \xi_i \xi_j \leq M^2$$

whenever $t \geq 0$ and $\sum_{i=1}^{\nu} \xi_i^2 = 1$.

For $\varepsilon \geq 0$ and $R \geq 0$ let $D(\varepsilon, \infty, R)$ denote the set $\{(t, x) \in \mathbf{R} \times \mathbf{R}^\nu : \varepsilon \leq t; |x| \leq Mt + R\}$, and let $S(T, R)$ denote $\{(T, x) : |x| \leq MT + R\}$. Suppose u is a solution of

$$(1.1) \quad |Lu| \leq k_1(t, x)|u| + k_2(t, x)\|\nabla u\|$$

in some $D(\varepsilon, \infty, R)$. The decay rate of u is measured in terms of the energy integral

$$\mathcal{E}(u, T, R) \equiv \int_{S(T, R)} \{u^2 + \|\nabla u\|^2\} dx.$$

Received by the editors August 31, 1970.

AMS 1969 subject classifications. Primary 3519; Secondary 3536, 3552.

Key words and phrases. Hyperbolic partial differential inequality, maximum decay rate, a priori inequalities.

⁽¹⁾ Research supported in part by Air Force Office of Scientific Research Grant Number AF-AFOSR 68-1462.

As a special case of our main theorem we get the following results:

(I) Suppose that $|(a_{ij})_t| = O(t^{-1})$; $k_1(t, x) = O(t^{-2})$; and $k_2(t, x) = O(t^{-1})$. Then a nonzero solution of (1.1) in $D(\epsilon, \infty, R)$ cannot decay so fast that, for every positive α , $\lim_{T \rightarrow \infty} T^\alpha \mathcal{E}(u, T, R) = 0$.

(II) Suppose that $|(a_{ij})_t| = O(1)$; $k_1(t, x) = O(1)$; $k_2(t, x) = O(1)$. Then a nonzero solution of (1.1) in $D(\epsilon, \infty, R)$ cannot decay so fast that, for all $\alpha > 0$, $\lim_{T \rightarrow \infty} e^{\alpha T} \mathcal{E}(u, T, R) = 0$.

(III) Suppose there is a $\gamma > 1$ such that $|(a_{ij})_t| = O(t^{(\gamma-1)})$; $k_1(t, x) = O(t^{2(\gamma-1)})$; and $k_2(t, x) = O(t^{(\gamma-1)})$. Then a nonzero solution of (1.1) in $D(\epsilon, \infty, R)$ cannot decay so fast that, for all $\alpha > 0$, $\lim_{T \rightarrow \infty} e^{\alpha T^\gamma} \mathcal{E}(u, T, R) = 0$.

The methods and immediate motivation for this work are derived from Protter's treatment [5], [6] of the asymptotic behavior of solutions of hyperbolic inequalities in interior domains. The crux of the method lies in finding appropriate families of weighted L_2 estimates for a C^2 function v and its gradient in terms of Lv . The estimates of this paper differ from those in [5], [6] by requiring either no boundary conditions or weaker boundary conditions. The changes in the derivation of the crucial estimates are suggested by techniques of Hörmander [1] for determining the sign of certain quadratic forms.

My decay rate results are comparable to those of Protter [5], [6] and Ogawa [4] for interior domains. Related problems about decay rates of solutions of hyperbolic equations have been studied by Morawetz [3], Strauss [7], and Littman [2] using different methods usually involving transform techniques. In particular, the partial differential inequality (1.1) arises from the uniqueness problem for the nonlinear equation

$$Lu = F(t, x, u, \nabla u)$$

provided F satisfies a Lipschitz condition of the form

$$|F(t, x, a, \mathbf{a}) - F(t, x, b, \mathbf{b})| \leq k_1(t, x)|a - b| + k_2(t, x)\|\mathbf{a} - \mathbf{b}\|.$$

Let ν be a positive integer. We use the coordinates (t, x_1, \dots, x_ν) in $R \times R^\nu$. The first coordinate t is the time coordinate; the rest make up the space variable $x = (x_1, \dots, x_\nu)$. Let $r = (x_1^2 + \dots + x_\nu^2)^{1/2}$.

In general we use subscripts to denote derivatives; thus

$$u_t = \partial u / \partial t; \quad u_{x_i} = \partial u / \partial x_i \quad \text{for } i = 1, \dots, \nu.$$

The gradient ∇u denotes the $(\nu + 1)$ -tuple $(u_t, u_{x_1}, \dots, u_{x_\nu})$. However we take the Laplacian with respect to space coordinates only; $\Delta u = \sum_{i=1}^\nu u_{x_i x_i}$.

By a *domain* we mean the closure of an open set which has piecewise smooth boundary. Derivatives at boundary points of a domain are to be understood as the appropriate one-sided derivatives.

These results are excerpted from my Ph.D. thesis, University of California, Berkeley, 1970. I wish to thank my advisor, Professor M. H. Protter, for his assistance and encouragement.

2. **Basic lemmas.** We start with the formula for the integral of the expression $2Lv\lambda v_t$ over a bounded domain $D \subseteq \text{Int}(\mathcal{H})$ where $\lambda \in C^1(D)$ and $v \in C^2(D)$. Let $n = (n_0, n_1, \dots, n_\nu)$ be the outer unit normal along ∂D . Then integration by parts yields the formula [1], [7]

$$\begin{aligned}
 \int_D 2Lv\lambda v_t &= \int_D \left\{ \lambda_t \left(v_t^2 + \sum_{i,j=1}^\nu a_{ij} v_{x_i} v_{x_j} \right) - 2v_t \sum_{i,j=1}^\nu a_{ij} \lambda_{x_i} v_{x_j} \right\} \\
 (2.1) \qquad &+ \int_D \lambda \sum_{i,j=1}^\nu (a_{ij})_t v_{x_i} v_{x_j} \\
 &- \int_{\partial D} \lambda \left\{ n_0 \left(v_t^2 + \sum_{i,j=1}^\nu a_{ij} v_{x_i} v_{x_j} \right) - 2v_t \sum_{i,j=1}^\nu a_{ij} n_i v_{x_j} \right\}.
 \end{aligned}$$

To exploit this formula we consider the region \mathcal{H} as a Lorentz manifold with the Lorentzian metric

$$(2.2) \qquad ((b, c)) \equiv b_0 c_0 - \sum_{i,j=1}^\nu a_{ij}(t, x) b_i c_j.$$

For vector fields b and c on \mathcal{H} we define a quadratic form $P_{b,c}$ at each point of \mathcal{H} by

$$(2.3) \qquad P_{b,c}(\xi) \equiv 2((b, \xi))((c, \xi)) - ((b, c))((\xi, \xi)).$$

Throughout we use h to denote the unit vector $h = (1, 0, \dots, 0) \in \mathbf{R}^{\nu+1}$. Computing explicitly we find that

$$P_{h,c}(\xi) = c_0 \left\{ \xi_0^2 + \sum_{i,j=1}^\nu a_{ij} \xi_i \xi_j \right\} - 2\xi_0 \sum_{i,j=1}^\nu a_{ij} c_i \xi_j.$$

Thus (2.1) can be rewritten more concisely as

$$(2.4) \qquad \int_D 2Lv\lambda v_t = \int_D \left\{ P_{h,\nabla\lambda}(\nabla v) + \lambda \sum_{i,j=1}^\nu (a_{ij})_t v_{x_i} v_{x_j} \right\} - \int_{\partial D} \lambda P_{h,n}(\nabla v).$$

We now describe conditions on D and λ under which the integrands on the right-hand side of (2.4) have definite sign.

We first consider the boundary integrand, $\lambda P_{h,n}(\nabla v)$. Following Hörmander [1] we classify tangent vectors in terms of the Lorentz metric (2.2). Thus we say $b = (b_0, b_1, \dots, b_\nu)$ is *timelike* at (t, x) if $((b, b)) > 0$; *spacelike* if $((b, b)) < 0$; and *characteristic* if $((b, b)) = 0$. Furthermore, we call b *positive* if $b_0 > 0$, and *negative* if $b_0 < 0$. Hörmander has proved that if b and c are any two positive timelike vectors with respect to a Lorentz metric $((,))$, then the form $P_{b,c}$ defined by (2.3) is positive definite. Since any positive characteristic vector is a limit of positive timelike vectors, it follows that $P_{h,n}(\xi) \geq 0$ for all ξ provided that n is positive non-spacelike.

We call a domain D *convenient* if the outer unit normal n along ∂D is never spacelike, i.e. if $((n, n)) \geq 0$ along ∂D . The boundary of a convenient domain D can

be decomposed into two parts: the part $S^+(D)$ on which n is positive, and the part $S^-(D)$ on which n is negative. Since $P_{h,n}$ is linear in n , we have $P_{h,n}(\nabla v)$ nonnegative along $S^+(D)$ and nonpositive along $S^-(D)$. For a positive function λ we therefore have determined the sign of the integrand $\lambda P_{h,n}(\nabla v)$.

Next we turn to the integrand

$$\mathcal{J} = \left\{ P_{h,\nabla\lambda}(\nabla v) + \lambda \sum_{ij=1}^{\nu} (a_{ij})_t v_{x_i} v_{x_j} \right\}$$

in (2.4) and confine our attention to functions λ of the form

$$\lambda = \lambda(\alpha) = e^{\alpha f(t)}$$

where f is a smooth function of positive t and α is a positive parameter. Three specific functions f to which our results apply are

- (i) $f(t) = \ln(t)$,
- (ii) $f(t) = t$,
- (iii) $f(t) = t^\gamma$, for a constant $\gamma > 1$.

Let \mathcal{F} denote the set containing these specific functions. Rather than compute separately for each $f \in \mathcal{F}$, we do a single computation appealing to several technical properties (P₁) which are easily verified for each $f \in \mathcal{F}$. The first of these properties is

$$(P_1) \quad f \in C^3(\mathbf{R}^+); \quad f_t > 0; \quad \lim_{t \rightarrow \infty} f(t) = \infty.$$

If f satisfies (P₁) then each $\lambda(\alpha) = e^{\alpha f(t)}$ is a C^1 function such that

$$\lambda(\alpha) > 0; \quad \lambda_t(\alpha) = \alpha f_t e^{\alpha f}; \quad \nabla \lambda = \lambda_t h.$$

Then

$$P_{h,\nabla\lambda}(\nabla v) = \lambda_t P_{h,h}(\nabla v) = \alpha f_t \lambda(\alpha) P_{h,h}(\nabla v)$$

and

$$\mathcal{J} = \lambda(\alpha) \left\{ \alpha f_t P_{h,h}(\nabla v) + \sum_{ij=1}^{\nu} (a_{ij})_t v_{x_i} v_{x_j} \right\}.$$

In order to be sure that \mathcal{J} is a positive definite form in ∇v we make the following assumption about the growth of the time derivatives $(a_{ij})_t$:

(A₁) For each $\varepsilon > 0$ there is a $B(\varepsilon) > 0$ such that $|(a_{ij})_t| \leq B(\varepsilon) f_t$ at all points (t, x) where $t \geq \varepsilon$.

LEMMA 2.1. *Suppose L satisfies (A₁) with respect to an f satisfying (P₁). If $\frac{1}{2}\alpha m^2 \geq \nu B(\varepsilon)$ and $v \in C^2(\mathcal{H})$, then at all (t, x) with $t \geq \varepsilon$ we have*

$$(2.5) \quad \left\{ \alpha f_t P_{h,h}(\nabla v) + \sum_{ij=1}^{\nu} (a_{ij})_t v_{x_i} v_{x_j} \right\} \geq \frac{1}{2} \alpha f_t P_{h,h}(\nabla v) \geq 0.$$

Proof. From (2.3) and (A₀) it follows that

$$P_{h,h}(\nabla v) \equiv v_t^2 + \sum_{ij=1}^{\nu} a_{ij} v_{x_i} v_{x_j} \geq m^2 \sum_{i=1}^{\nu} v_{x_i}^2.$$

Because of (P₁) we get for all positive α

$$\alpha f_i P_{h,h}(\nabla v) \geq \frac{1}{2} \alpha f_i P_{h,h}(\nabla v) + \frac{1}{2} \alpha f_i m^2 \sum_{i=1}^{\nu} v_{x_i}^2.$$

On the other hand, whenever $t \geq \varepsilon$ we have

$$\begin{aligned} \left| \sum_{i,j=1}^{\nu} (a_{ij})_t v_{x_i} v_{x_j} \right| &\leq \sum_{i,j=1}^{\nu} |(a_{ij})_t| |v_{x_i}| |v_{x_j}| \\ &\leq B(\varepsilon) f_i \left(\sum_{i=1}^{\nu} |v_{x_i}| \right)^2 \leq \nu B(\varepsilon) f_i \sum_{i=1}^{\nu} v_{x_i}^2. \end{aligned}$$

So

$$\left\{ \alpha f_i P_{h,h}(\nabla v) + \sum_{i,j=1}^{\nu} (a_{ij})_t v_{x_i} v_{x_j} \right\} \geq \frac{1}{2} \alpha f_i P_{h,h}(\nabla v) + \frac{1}{2} \alpha m^2 f_i \sum_{i=1}^{\nu} v_{x_i}^2 - \nu B(\varepsilon) f_i \sum_{i=1}^{\nu} v_{x_i}^2.$$

Thus if $\frac{1}{2} \alpha m^2 \geq \nu B(\varepsilon)$, the inequality (2.5) follows.

Using this result we derive two important integral inequalities.

LEMMA 2.2. *Suppose (A₁) holds for some f satisfying (P₁). Let D be a bounded convenient domain in which $t \geq \varepsilon > 0$. If $v \in C^2(D)$ and $\frac{1}{2} \alpha m^2 \geq \nu B(\varepsilon)$, then*

$$(2.6) \quad \int_D 2Lv e^{\alpha f(t)} v_t \geq - \int_{S^+(D)} e^{\alpha f(t)} P_{h,n}(\nabla v)$$

and

$$(2.7) \quad \frac{1}{2} \alpha \int_D f_i e^{\alpha f(t)} P_{h,h}(\nabla v) \leq 2 \int_D e^{\alpha f(t)} |Lv v_t| + \int_{S^+(D)} e^{\alpha f(t)} P_{h,n}(\nabla v).$$

Proof. For each function $\lambda(\alpha) = e^{\alpha f(t)}$, $\alpha > 0$, we can apply (2.4) to obtain

$$(2.8) \quad \int_D 2Lv \lambda(\alpha) v_t = \int_D e^{\alpha f(t)} \left\{ \alpha f_i P_{h,h}(\nabla v) + \sum_{i,j=1}^{\nu} (a_{ij})_t v_{x_i} v_{x_j} \right\} - \int_{\partial D} e^{\alpha f(t)} P_{h,n}(\nabla v).$$

Since D is convenient, we have $P_{h,n}(\nabla v) \leq 0$ on $S^-(D)$, and thus

$$- \int_{\partial D} e^{\alpha f} P_{h,n}(\nabla v) \geq - \int_{S^+(D)} e^{\alpha f} P_{h,n}(\nabla v).$$

If $\frac{1}{2} \alpha m^2 \geq \nu B(\varepsilon)$, then by the preceding lemma

$$\int_D e^{\alpha f} \left\{ \alpha f_i P_{h,h}(\nabla v) + \sum_{i,j=1}^{\nu} (a_{ij})_t v_{x_i} v_{x_j} \right\} \geq \frac{1}{2} \alpha \int_D f_i e^{\alpha f} P_{h,h}(\nabla v) \geq 0.$$

Thus, for large enough α we have

$$\int_D 2Lv \lambda(\alpha) v_t \geq \frac{1}{2} \alpha \int_D f_i \lambda(\alpha) P_{h,h}(\nabla v) - \int_{S^+(D)} \lambda(\alpha) P_{h,n}(\nabla v).$$

The inequalities (2.6) and (2.7) now follow directly since $f_i \lambda(\alpha) P_{h,h}(\nabla v)$ is non-negative.

COROLLARY 2.3. *Let $m' = \min \{1, m^2\}$. Then for α sufficiently large*

$$(2.9) \quad \frac{1}{2}\alpha m' \int_D f_i \lambda(\alpha) \|\nabla v\|^2 \leq 2 \int_D \lambda(\alpha) |Lv v_i| + \int_{S^+(D)} \lambda(\alpha) P_{h,n}(\nabla v).$$

Proof. This follows from (2.7) since we have

$$P_{h,h}(\nabla v) = \left(v_i^2 + \sum_{i,j=1}^v a_{ij} v_{x_i} v_{x_j} \right) \geq v_i^2 + m^2 \sum_{i=1}^v v_{x_i}^2 \geq m' \left(v_i^2 + \sum_{i=1}^v v_{x_i}^2 \right).$$

3. Basic a priori estimates. Suppose v is a C^2 function. In this section we derive the crucial family of weighted L_2 estimates for v and ∇v in terms of Lv .

Let f satisfy (P_1) . We assume that L satisfies (A_1) relative to this f . In the course of the derivation we will impose additional technical conditions (P_i) on f , conditions which are verified for all $f \in \mathcal{F}$.

First we apply a variant of Protter's method [5], [6] to estimate v on a bounded convenient domain D in which $t \geq \epsilon > 0$. We employ the parametrized family of weight functions $\lambda = \lambda(\alpha) = e^{\alpha f(t)}$, $\alpha > 0$.

For an $\alpha > 0$, let z denote the auxiliary function $z = \lambda(\alpha)v$. Since $v = e^{-\alpha f}z$, computation shows that

$$(3.1) \quad \lambda(\alpha)Lv = Lz + 2\alpha f_i z_t + \alpha(f_{tt} - \alpha f_t^2)z.$$

Applying the elementary inequality $(A + B + C)^2 \geq 2(A + C)B$, we find

$$(3.2) \quad \lambda(2\alpha)(Lv)^2 \geq 2\{Lz + \alpha(f_{tt} - \alpha f_t^2)z\}\{2\alpha f_i z_t\}.$$

Expanding the right-hand side of (3.2) we obtain

$$(3.3) \quad \lambda(2\alpha)(Lv)^2 \geq 2\alpha\{2Lz f_i z_t\} + 2\alpha^2 f_t(f_{tt} - \alpha f_t^2) \frac{\partial}{\partial t}(z^2)/\partial t.$$

Integrating (3.3) over D does not yield a useful estimate. Instead we must first multiply (3.3) through by the positive quantity $f_i^{-1}\lambda(\beta)$ where β is chosen large enough that $\frac{1}{2}\beta m^2 \geq \nu B(\epsilon)$. This yields

$$(3.4) \quad \int_D f_i^{-1}\lambda(\beta + 2\alpha)(Lv)^2 \geq 2\alpha \int_D Lz\lambda(\beta)z_t + 2\alpha^2 \int_D \lambda(\beta)(f_{tt} - \alpha f_t^2) \frac{\partial}{\partial t}(z^2).$$

Now β was chosen so that Lemma 2.2 applies. Thus

$$(3.5) \quad \int_D Lz\lambda(\beta)z_t \geq - \int_{S^+(D)} \lambda(\beta)P_{h,n}(\nabla z).$$

Integration by parts yields

$$(3.6) \quad \begin{aligned} \int_D \lambda(\beta)(f_{tt} - \alpha f_t^2) \frac{\partial}{\partial t}(z^2) &= \int_D \frac{\partial}{\partial t} \{z^2 e^{\beta f}(f_{tt} - \alpha f_t^2)\} - \int_D z^2 \frac{\partial}{\partial t} \{e^{\beta f}(f_{tt} - \alpha f_t^2)\} \\ &= \int_{\partial D} n_0 z^2 e^{\beta f(t)}(f_{tt} - \alpha f_t^2) \\ &\quad + \int_D z^2 e^{\beta f} \{ \beta f_t (\alpha f_t^2 - f_{tt}) + (2\alpha f_t f_{tt} - f_{ttt}) \}. \end{aligned}$$

We add two more technical properties of f to improve (3.5):

(P₂) For all $t > 0$, $\alpha > 0$, $(f_{tt} - \alpha f_t^2) \geq -(\alpha + 1)f_t^2$. For each $\varepsilon > 0$, there is an $\alpha_1(\varepsilon) > 0$ such that $(f_{tt} - \alpha f_t^2)$ is negative whenever $t \geq \varepsilon$ and $\alpha \geq \alpha_1(\varepsilon)$.

(P₃) For each $\varepsilon > 0$ and $\beta \geq 3$, there is an $\alpha_2(\varepsilon, \beta)$ such that

$$\alpha \beta f_t^3 + (2\alpha - \beta)f_t f_{tt} - f_{ttt} \geq \alpha f_t^3$$

whenever $t \geq \varepsilon$ and $\alpha \geq \alpha_2(\varepsilon, \beta)$.

The property (P₂) simplifies the boundary integral in (3.6). We have assumed that $t \geq \varepsilon > 0$ in D and we know that $n_0 > 0$ on $S^+(D)$. Thus by (P₂) if $\alpha > \alpha_1(\varepsilon)$ we have

$$\begin{aligned} n_0 z^2 e^{\beta f} (f_{tt} - \alpha f_t^2) &\geq 0 \quad \text{on } S^-(D), \\ n_0 z^2 e^{\beta f} (f_{tt} - \alpha f_t^2) &\geq -(\alpha + 1)n_0 z^2 e^{\beta f} f_t^2 \quad \text{on } S^+(D), \end{aligned}$$

and therefore

$$\int_{\partial D} n_0 z^2 e^{\beta f} (f_{tt} - \alpha f_t^2) \geq -(\alpha + 1) \int_{S^+(D)} n_0 z^2 e^{\beta f} f_t^2.$$

The property (P₃) has the consequence that

$$\int_D z^2 \lambda(\beta) \{ \alpha \beta f_t^3 + (2\alpha - \beta)f_t f_{tt} - f_{ttt} \} \geq \alpha \int_D z^2 \lambda(\beta) f_t^3 = \alpha \int_D f_t^3 \lambda(\beta + 2\alpha) v^2$$

provided that $\beta \geq 3$ and $\alpha \geq \alpha_2(\varepsilon, \beta)$. Now if β is chosen so that both $\beta \geq 3$ and $\frac{1}{2}\beta m^2 \geq \nu B(\varepsilon)$, then with the help of (P₂) and (P₃) we strengthen (3.6) to obtain the inequality

$$(3.7) \quad \int_D \lambda(\beta) (f_{tt} - \alpha f_t^2) \frac{\partial z^2}{\partial t} \geq -(\alpha + 1) \int_{S^+(D)} n_0 z^2 \lambda(\beta) f_t^2 + \alpha \int_D f_t^3 \lambda(\beta + 2\alpha) v^2$$

for all $\alpha \geq \max \{ \alpha_1(\varepsilon), \alpha_2(\varepsilon, \beta) \}$. Combining (3.5) and (3.7) with (3.4) we obtain the estimate

$$(3.8) \quad \begin{aligned} 2\alpha^3 \int_D f_t^3 \lambda(\beta + 2\alpha) v^2 &\leq \int_D f_t^{-1} \lambda(\beta + 2\alpha) (Lv)^2 \\ &\quad + 2\alpha \int_{S^+(D)} \lambda(\beta) P_{h,n}(\nabla z) + 2\alpha^2(\alpha + 1) \int_{S^+(D)} n_0 \lambda(\beta) f_t^2 z^2 \end{aligned}$$

which is valid for all sufficiently large α .

With this indication of the purpose of (P₂) and (P₃) we can summarize the derivation of (3.8) in the following lemma:

LEMMA 3.1. *Suppose f satisfies (P_i) for $i = 1, 2, 3$ and L satisfies (A₁) relative to f . Let D be a bounded convenient domain in which $t \geq \varepsilon > 0$. Let β be a constant such that $\beta \geq 3$ and $\frac{1}{2}\beta m^2 \geq \nu B(\varepsilon)$. If $v \in C^2(D)$ then for all sufficiently large α*

$$(3.9) \quad 2\alpha^3 \int_D f_t^3 e^{(\beta + 2\alpha)f} v^2 \leq \int_D f_t^{-1} e^{(\beta + 2\alpha)f} (Lv)^2 + E_1$$

where, letting $z = e^{\alpha f} v$, E_1 denotes the quantity

$$E_1 = 2\alpha \int_{S^+(D)} e^{\beta f} P_{h,n}(\nabla z) + 2\alpha^2(\alpha + 1) \int_{S^+(D)} n_0 f_i^2 e^{\beta f} z^2.$$

Proof. Since we have defined $\lambda(\alpha) = e^{\alpha f}$ for all $\alpha > 0$, the inequality (3.9) is essentially a restatement of (3.8).

The next task is to derive a companion estimate for $\|\nabla v\|$ in terms of Lv , one with the same weight function multiplying $(Lv)^2$ as in (3.9). This is done using (2.8) from Corollary 2.3. Recall that $m' = \min\{1, m^2\}$.

LEMMA 3.2. *Suppose f satisfies (P_i) for $i = 1, 2, 3$ and L satisfies (A₁) relative to f . Let D be a bounded convenient domain in which $t \geq \varepsilon > 0$. Let β be a constant such that $m'\beta > 2$ and $\frac{1}{2}m^2\beta \geq \nu B(\varepsilon)$. If $v \in C^2(D)$, then for all $\alpha > 0$*

$$(3.10) \quad m'\alpha \int_D f_i e^{(\beta + 2\alpha)f} \|\nabla v\|^2 \leq \int_D f_i^{-1} e^{(\beta + 2\alpha)f} (Lv)^2 + E_2$$

where E_2 denotes the quantity $\int_{S^+(D)} e^{(\beta + 2\alpha)f} P_{h,n}(\nabla v)$.

Proof. If $\frac{1}{2}\delta m^2 \geq \nu B(\varepsilon)$, then by Corollary 2.3 we have the inequality

$$(2.8) \quad \frac{1}{2}\delta m' \int_D f_i e^{\delta f} \|\nabla v\|^2 \leq 2 \int_D e^{\delta f} |Lv v_i| + \int_{S^+(D)} e^{\delta f} P_{h,n}(\nabla v).$$

Since $f_i > 0$ everywhere we have

$$2|Lv v_i| \leq f_i^{-1} (Lv)^2 + f_i v_i^2 \leq f_i^{-1} (Lv)^2 + f_i \|\nabla v\|^2.$$

Thus for sufficiently large δ we get

$$(3.11) \quad (\frac{1}{2}\delta m' - 1) \int_D f_i e^{\delta f} \|\nabla v\|^2 \leq \int_D f_i^{-1} e^{\delta f} (Lv)^2 + \int_{S^+(D)} e^{\delta f} P_{h,n}(\nabla v).$$

If we set $\delta = 2\alpha + \beta$, then (2.8) and (3.11) hold for all $\alpha > 0$. Furthermore $\frac{1}{2}\delta m' - 1 = \alpha m' + \frac{1}{2}\beta m' - 1 \geq \alpha m'$, so (3.10) follows.

The boundary integral terms E_1 and E_2 are easily computed for the particular bounded convenient domains we employ. For any nonnegative ε and R we define the unbounded domain

$$D(\varepsilon, \infty, R) = \{(t, x) \in \mathbf{R} \times \mathbf{R}^v : \varepsilon \leq t < \infty; r \leq Mt + R\}.$$

If L is the wave operator, $L = \Delta - \partial^2/\partial t^2$, then $m = M = 1$ in (A₀) and $D(0, \infty, R)$ is the "domain of influence" of the sphere of radius R at time $t = 0$. In general the lateral boundary, $r = Mt + R$, will not be a characteristic hypersurface for L . The outer normal along $r = Mt + R$ is given by

$$n = (1 + M^2)^{1/2} (-M, x_1/r, \dots, x_v/r)$$

and thus

$$((n, n)) = (1 + M^2) \left\{ (-M)^2 - \sum_{ij=1}^v a_{ij} \frac{x_i}{r} \frac{x_j}{r} \right\}.$$

Because of (A₀)

$$((n, n)) \geq (1 + M^2)\{M^2 - M^2\} = 0$$

and therefore the lateral boundary of $D(\varepsilon, \infty, R)$ is never spacelike. The domains $D(0, \infty, R)$ are the smallest convenient conical regions containing the “domain of influence” of the initial sphere of radius R .

The bounded convenient domains for which we specialize the estimates (3.10) and (3.11) are the “cork-shaped” domains

$$D(\varepsilon, T, R) = \{(t, x) : \varepsilon \leq t \leq T; r \leq Mt + R\}$$

where $0 < \varepsilon < T$ and $0 < R$. The boundary $\partial D(\varepsilon, T, R)$ has three smooth parts:

$$\begin{aligned} S_{\text{lat}} &= \{(t, x) \in \partial D(\varepsilon, T, R) : r = Mt + R\}, \\ S(T, R) &= \{(t, x) \in \partial D(\varepsilon, T, R) : t = T\}, \\ S(\varepsilon, R) &= \{(t, x) \in \partial D(\varepsilon, T, R) : t = \varepsilon\}. \end{aligned}$$

On the lateral boundary S_{lat} we have seen that the outer unit normal n is negative nontimelike. On $S(\varepsilon, R)$ clearly $n = (-1, 0, \dots, 0)$. And on $S(T, R)$, $n = (1, 0, \dots, 0)$. Thus these $D(\varepsilon, T, R)$ are indeed convenient domains, and

$$\begin{aligned} S^+(D(\varepsilon, T, R)) &= S(T, R) \\ S^-(D(\varepsilon, T, R)) &= S_{\text{lat}} \cup S(\varepsilon, R). \end{aligned}$$

Suppose D is one of the $D(\varepsilon, T, R)$. The terms E_1 and E_2 of the Lemmas 3.1 and 3.2 can be estimated in terms of the energy integral

$$\mathcal{E}(v, T, R) = \int_{S(T, R)} \{v^2 + \|\nabla v\|^2\} dx.$$

Since t is constant on $S(T, R)$ we have

$$E_1 = 2\alpha e^{\beta f(T)} \int_{S(T, R)} P_{h,h}(\nabla z) + 2\alpha^2(\alpha + 1)f_i^2(T)e^{\beta f(T)} \int_{S(T, R)} z^2$$

and

$$E_2 = e^{(\beta + 2\alpha)f(T)} \int_{S(T, R)} P_{h,h}(\nabla v).$$

Let $M' = \max\{1, M^2\}$. So for any C^1 function w , it follows that

$$P_{h,h}(\nabla w) = w_i^2 + \sum_{i,j=1}^y a_{ij}w_{x_i}w_{x_j} \leq w_i^2 + M^2 \sum_{i=1}^y w_{x_i}^2 \leq M' \|\nabla w\|^2.$$

Since $z = e^{\alpha f}v$, we find that

$$\begin{aligned} \|\nabla z\|^2 &= (e^{\alpha f}v)_i^2 + \sum_{i=1}^y (e^{\alpha f}v)_{x_i}^2 \\ &\leq e^{2\alpha f} \left\{ (\alpha f_i v + v_i)^2 + \sum_{i=1}^y v_{x_i}^2 \right\} \\ &\leq 2e^{2\alpha f} \{ \alpha^2 f_i^2 v^2 + \|\nabla v\|^2 \}. \end{aligned}$$

Thus

$$E_1 \leq 2\alpha^2\{2M'\alpha + \alpha + 1\}f_t^2(T)e^{(\beta + 2\alpha)f(T)} \int_{S(T,R)} v^2 + 4\alpha M' e^{(\beta + 2\alpha)f(T)} \int_{S(T,R)} \|\nabla v\|^2$$

and

$$E_2 \leq M' e^{(\beta + 2\alpha)f(T)} \int_{S(T,R)} \|\nabla v\|^2.$$

At this point we introduce the following additional restriction on f :

(P₄) There is a constant $\mu \geq 1$ such that $f_t^2(T) \leq \mu e^{f(T)}$ whenever $T > 1$.

This property is easily verified for $f \in \mathcal{F}$; indeed for $f(t) = \ln(t)$ or $f(t) = t$ we can take $\mu = 1$, and if $f(t) = t^\gamma$ with $\gamma > 1$ then $\mu = 2\gamma^2$ will suffice.

If we now assume that f satisfies (P_i) for $i = 1, 2, 3, 4$, then for sufficiently large T

$$f_t^2(T) \leq \mu e^{f(T)}, \quad 1 < e^{f(T)} \leq \mu e^{f(T)},$$

and thus

$$(3.12) \quad E_1 + E_2 \leq 2\alpha^2\{2\alpha M' + \alpha + 1\}\mu e^{(1 + \beta + 2\alpha)f(T)} \int_{S(T,R)} v^2 + (4\alpha + 1)M'\mu e^{(1 + \beta + 2\alpha)f(T)} \int_{S(T,R)} \|\nabla v\|^2.$$

Letting

$$(3.13) \quad p(\alpha) = 2\alpha^2\{2\alpha M' + \alpha + 1\}\mu + (4\alpha + 1)M'\mu$$

we get the bound

$$(3.14) \quad E_1 + E_2 \leq p(\alpha)e^{(1 + \beta + 2\alpha)f(T)} \mathcal{E}(v, T, R)$$

for sufficiently large α and T .

The next result gives the crucial family of a priori inequalities from which our decay results follow.

THEOREM 3.3. *Suppose that f satisfies (P_i) for $i = 1, 2, 3, 4$ and that L satisfies (A₁) relative to f . Let ϵ and R be positive constants. For each $T > \epsilon$ let D_T denote $D(\epsilon, T, R)$ and let S_T denote $S(T, R)$. Choose β so that $\beta \geq 3$, $m^2\beta > 2\nu B(\epsilon)$, and $m'\beta > 2$. If $v \in C^2(D(\epsilon, \infty, R))$ and if α and T are sufficiently large, then*

$$(3.15) \quad 2\alpha^3 \int_{D_T} f_t^3 e^{(\beta + 2\alpha)f} v^2 + m'\alpha \int_{D_T} f_t e^{(\beta + 2\alpha)f} \|\nabla v\|^2 \leq 2 \int_{D_T} f_t^{-1} e^{(\beta + 2\alpha)f} (Lv)^2 + E_3$$

where, using the $p(\alpha)$ defined in (3.13),

$$E_3 = p(\alpha)e^{(1 + \beta + 2\alpha)f(T)} \mathcal{E}(v, T, R).$$

Proof. Each D_T is a bounded convenient domain in which $t \geq \epsilon > 0$. The constant β is chosen to meet the conditions of Lemmas 3.1 and 3.2. Thus there is a

constant α , depending on ε and β and R only, such that (3.9) and (3.10) hold when $\alpha \geq \alpha$ and $T > \varepsilon$. Adding (3.9) and (3.10) we get

$$2\alpha^3 \int_{D_T} f_t^3 e^{(\beta+2\alpha)f} v^2 + m' \alpha \int_{D_T} f_t e^{(\beta+2\alpha)f} \|\nabla v\|^2 \leq 2 \int_{D_T} f_t^{-1} e^{(\beta+2\alpha)f} (Lv)^2 + E_1 + E_2.$$

But with the help of (P₁) and (P₂) we have seen that (3.14) holds for sufficiently large α and T . The inequality (3.15) now follows.

4. Decay rate results. Let f be a function with the properties (P_{*i*}) for $i = 1, 2, 3, 4$. In this section we study the inequality

$$(1.1) \quad |Lu| \leq k_1(t, x)|u| + k_2(t, x)\|\nabla u\|$$

under the following assumptions about the coefficients:

(A₀) There are positive constants m, M such that

$$m^2 \leq \sum_{i,j=1}^v a_{ij}(t, x) \xi_i \xi_j \leq M^2$$

whenever $t \geq 0$ and $\sum_{i=1}^v \xi_i^2 = 1$.

(A₁) For each $\varepsilon > 0$ there is a $B(\varepsilon) > 0$ such that $|(a_{ij})_t| \leq B(\varepsilon)f_t$ whenever $t \geq \varepsilon$.

(A₂) In each $D(\varepsilon, \infty, R)$ with $\varepsilon > 0$ and $R > 0$,

$$k_1(t, x) = O(f_t^2) \quad \text{and} \quad k_2(t, x) = O(f_t).$$

Notice that (A₂) does not control the asymptotic behavior of k_1 and k_2 uniformly in x , but only on each truncated cone $D(\varepsilon, \infty, R)$. Actually (A₁) could also be given as $|(a_{ij})_t| = O(f_t)$ in each $D(\varepsilon, \infty, R)$ at the cost of more complicated technical requirements in the preceding sections.

The rate of decay of a solution u of (1.1) in some $D(\varepsilon, \infty, R)$ is described in terms of the energy integral

$$\mathcal{E}(u, T, R) = \int_{S(T, R)} \{u^2 + \|\nabla u\|^2\}.$$

Let $g(t)$ be a continuous function with $\lim_{t \rightarrow \infty} g(t) = 0$. We say that u decays faster than $g(t)$ in $D(\varepsilon, \infty, R)$ if

$$\lim_{T \rightarrow \infty} \left\{ \frac{\mathcal{E}(u, T, R)}{g(T)} \right\} = 0.$$

We say u does not decay as fast as $g(t)$ if the ratio of $\mathcal{E}(u, T, R)$ to $g(T)$ grows unbounded as $T \rightarrow \infty$.

THEOREM 4.1. *Let u be a C^2 solution of*

$$(1.1) \quad |Lu| \leq k_1(t, x)|u| + k_2(t, x)\|\nabla u\|$$

in some $D(\varepsilon, \infty, R)$. There is a positive constant ρ such that if u decays faster than $e^{-\rho f(T)}$ in $D(\varepsilon, \infty, R)$, then u must vanish identically in $D(\varepsilon, \infty, R)$.

Proof. For each $T > \varepsilon$ let D_T denote $D(\varepsilon, T, R)$. To show $u=0$ in $D(\varepsilon, \infty, R)$ it suffices to show that u vanishes in all D_T .

Since u is C^2 on $D(\varepsilon, \infty, R)$ we may apply Theorem 3.3. Thus there is a β depending on ε such that for all sufficiently large α and T

$$(3.15) \quad 2\alpha^3 \int_{D_T} f_i^3 e^{(\beta+2\alpha)f} u^2 + \alpha m' \int_{D_T} f_i e^{(\beta+2\alpha)f} \|\nabla u\|^2 \leq 2 \int_{D_T} f_i^{-1} e^{(\beta+2\alpha)f} (Lu)^2 + E_3$$

where $E_3 = p(\alpha)e^{(1+2\alpha+\beta)f(T)}\mathcal{E}(u, T, R)$.

Because u is a solution of (1.1) we have

$$(Lu)^2 \leq \{k_1|u| + k_2\|\nabla u\|\}^2 \leq 2\{k_1^2 u^2 + k_2^2 \|\nabla u\|^2\}.$$

By the assumption (A_2) we have constants K_1 and K_2 such that

$$k_1(t, x) \leq K_1 f_i^2(t); \quad k_2(t, x) \leq K_2 f_i(t)$$

at all point of $D(\varepsilon, \infty, R)$. Hence,

$$(Lu)^2 \leq 2K_1^2 f_i^4 u^2 + 2K_2^2 f_i^2 \|\nabla u\|^2.$$

Combining this with (3.15) we obtain the inequality

$$(4.1) \quad \begin{aligned} & 2\alpha^3 \int_{D_T} f_i^3 e^{(\beta+2\alpha)f} u^2 + \alpha m' \int_{D_T} f_i e^{(\beta+2\alpha)f} \|\nabla u\|^2 \\ & \leq 2 \int_{D_T} f_i^{-1} e^{(\beta+2\alpha)f} \{2K_1^2 f_i^4 u^2 + 2K_2^2 f_i^2 \|\nabla u\|^2\} + E_3 \\ & \leq 4K_1^2 \int_{D_T} f_i^3 e^{(\beta+2\alpha)f} u^2 + 4K_2^2 \int_{D_T} f_i e^{(\beta+2\alpha)f} \|\nabla u\|^2 + E_3. \end{aligned}$$

Hence we have

$$(4.2) \quad (2\alpha^3 - 4K_1^2) \int_{D_T} f_i^3 e^{(\beta+2\alpha)f} u^2 \leq (4K_2^2 - \alpha m') \int_{D_T} f_i e^{(\beta+2\alpha)f} \|\nabla u\|^2 + E_3.$$

The inequality (4.2) is true for all sufficiently large α and T . We now pick an α large enough that (4.2) holds and also so large that

$$(2\alpha^3 - 4K_1^2) \geq 1; \quad (4K_2^2 - \alpha m') \leq 0.$$

For this α and all sufficiently large T we now have

$$(4.3) \quad 0 \leq \int_{D_T} f_i^3 e^{(\beta+2\alpha)f} u^2 \leq E_3 = p(\alpha)e^{(1+\beta+2\alpha)f(T)}\mathcal{E}(u, T, R).$$

The integral over D_T in (4.3) is a nonnegative increasing function of T . Set $\rho = 1 + \beta + 2\alpha$. So, if

$$\lim_{T \rightarrow \infty} e^{\rho f(T)}\mathcal{E}(u, T, R) = 0$$

then it follows from (4.3) that

$$\int_{D_T} f_i^3 e^{(\beta+2\alpha)f} u^2 = 0, \quad \text{for all } T > \varepsilon,$$

and therefore that $u=0$ in all D_T .

This theorem gives a bound on the rate of decay of a nonzero solution u in $D(\varepsilon, \infty, R)$: its energy $\mathcal{E}(u, T, R)$ cannot decay faster than $e^{-\rho f(T)}$.

COROLLARY 4.2. *Suppose u satisfies (1.1) in $\mathcal{H} = \{(t, x) : t \geq 0\}$. If, for every $\varepsilon > 0$ and $R > 0$, u decays faster than all $e^{-\rho f(T)}$, $\rho > 0$, in $D(\varepsilon, \infty, R)$, then $u = 0$ in \mathcal{H} .*

Proof. For each positive ε and R the theorem applies to show that u vanishes in $D(\varepsilon, \infty, R)$. Thus $u(t, x) = 0$ wherever $t > 0$. By continuity $u = 0$ in \mathcal{H} .

Each of the functions $f \in \mathcal{F}$ satisfy all the properties (P_i), $1 \leq i \leq 4$. The interpretations of Theorem 4.1 for $f(t) = \ln(t)$, $f(t) = t$, and $f(t) = t^\gamma$ with $\gamma > 1$ respectively yield the results (I), (II), and (III) given in the introduction.

Corollary 4.2 is the best result possible if we add one more condition on f , namely

(P₅) For all $\alpha > \nu$, $\lim_{T \rightarrow \infty} T^\nu e^{-\alpha f(T)} = 0$.

This property is easily verified for the functions in \mathcal{F} .

Suppose now that f satisfies (P₅) as well as (P_i), $i = 1, \dots, 4$. For a fixed $\alpha > \nu$ set $w(t, x) = e^{-\alpha f(t)}$. Then by a straightforward computation

$$Lw = \alpha(f_u - \alpha f_t^2)w.$$

Set $k_1(t, x) = \alpha(\alpha + 1)f_t^2$ and $k_2(t, x) = 0$. These k_i satisfy (A₂). Because of (P₂) we have

$$|Lw| = \alpha|f_u - \alpha f_t^2| |w| \leq k_1(t, x)|w| + k_2(t, x)\|\nabla w\|.$$

So this w is a nonzero solution of an inequality of the form (1.1). Computation shows that

$$\mathcal{E}(w, T, R) = \int_{S(T, R)} e^{-2\alpha f} (1 + \alpha^2 f_t^2) dx.$$

Let C_ν be the measure of the unit ball in \mathbf{R}^ν . Using (P₄) we find

$$\mathcal{E}(w, T, R) \leq \mu(1 + \alpha^2)e^{(1-2\alpha)f(T)}C_\nu(MT + R)^\nu.$$

By using (P₅) we see that

$$\lim_{T \rightarrow \infty} (1 + \alpha^2)C_\nu e^{-\alpha f(T)}(MT + R)^\nu = 0$$

for any $R > 0$ and thus that w decays faster than $e^{-(\alpha-1)f(T)}$ in each $D(\varepsilon, \infty, R)$. Thus for any particular α we can find a nonzero solution of (1.1) which decays faster than $e^{-(\alpha-1)f(T)}$. So no rate of decay slower than that of Corollary 4.2 is sufficient to insure that a solution vanishes.

5. Decay rates outside a characteristic conoid. The maximal decay rate established in §4 holds for solutions in other domains provided appropriate boundary conditions are added. As examples we consider in this section solutions outside a characteristic conoid, and in the next section solutions outside a reflecting obstacle.

Throughout this section we assume that f satisfies (P_i) for $1 \leq i \leq 5$ and that L satisfies (A₁) relative to f .

The inequality (1.1) can be considered as describing the time course of a disturbance u in x -space. The form

$$d\rho^2 = \sum_{i,j=1}^{\nu} a_{ij}(t, x) dx_i dx_j$$

is a time varying metric in x -space. The characteristic conoid \mathcal{C} at the origin is the set of points (t, x) such that t is the ρ -distance between x and 0 at time t . It can be verified that \mathcal{C} is a characteristic hypersurface in $\mathbf{R} \times \mathbf{R}^\nu$. In the special case that L is the wave operator and $\nu=3$, the conoid \mathcal{C} is just the forward light cone. Saying that (t, x) lies outside \mathcal{C} means that the ρ -distance between x and 0 at time t is not less than t .

Notice that (A_0) assures us that \mathcal{C} lies in the region $\{(t, x) : r \leq Mt\}$. Thus for $\epsilon > 0$ and $R > 0$ it follows that the sets

$$P(\epsilon, \infty, R) \equiv \{(t, x) \in D(\epsilon, \infty, R) : (t, x) \text{ lies outside } \mathcal{C}\}$$

are unbounded domains. All the bounded domains

$$P(\epsilon, T, R) \equiv \{(t, x) \in P(\epsilon, \infty, R) : t \leq T\}$$

are convenient since the boundary $\partial P(\epsilon, T, R)$ is composed of smooth parts along the characteristic hypersurface \mathcal{C} , and along the nontimelike hypersurfaces $t=T$, $t=\epsilon$, and $r=Mt+R$.

The first step in adapting Theorem 4.1 to the case of solutions outside \mathcal{C} is to adapt Theorem 3.3.

THEOREM 5.1. *For fixed positive ϵ and R let P_T denote $P(\epsilon, T, R)$ for each $T > \epsilon$. Let S_T denote $\{(t, x) \in \partial P_T : t=T\}$. Suppose v is a C^2 function outside \mathcal{C} which vanishes along \mathcal{C} . There is a β such that for all sufficiently large α and T*

$$(5.1) \quad 2\alpha^3 \int_{P_T} f_t^3 e^{(\beta+2\alpha)f} v^2 + \alpha m' \int_{P_T} f_t e^{(\beta+2\alpha)f} \|\nabla v\|^2 \leq \int_{P_T} f_t^{-1} e^{(\beta+2\alpha)f} (Lv)^2 + E_4$$

where, for the polynomial p of (3.13), E_4 denotes the quantity

$$E_4 = p(\alpha) e^{(1+\beta+2\alpha)f(T)} \int_{S_T} \{v^2 + \|\nabla v\|^2\} dx.$$

Proof. Choose β so large that the Lemmas 3.1 and 3.2 hold for all P_T with $T > \epsilon$. This choice of β is independent of v . Then under the hypotheses of Lemma 3.1 we get the estimates

$$(3.9) \quad 2\alpha^3 \int_{P_T} f_t^3 e^{(\beta+2\alpha)f} v^2 \leq \int_{P_T} f_t^{-1} e^{(\beta+2\alpha)f} (Lv)^2 + E_1(T)$$

where, in terms of $z = e^{\beta f} v$, $E_1(T)$ denotes the quantity

$$E_1(T) = 2 \int_{S^+(P_T)} e^{\beta f} P_{h,n}(\nabla z) + 2\alpha^2(\alpha+1) \int_{S^+(P_T)} n_0 f_t^2 e^{\beta f} z^2.$$

The S^+ part of ∂P_T is composed of S_T and the portion of ∂P_T along \mathcal{C} . Since v vanishes along \mathcal{C} so does z . Thus we know that ∇z is a scalar multiple, ζn , of the outer unit normal along $\mathcal{C} \cap \partial P_T$. Since \mathcal{C} is characteristic we have $((n, n))=0$ on \mathcal{C} . Thus along \mathcal{C}

$$P_{h,n}(\nabla z) = \zeta^2 P_{h,n}(n) = \zeta^2 \{2((h, n))((n, n)) - ((h, n))((n, n))\} = 0.$$

So both integrands in $E_1(T)$ vanish off S_T , and $E_1(T)$ takes the form

$$E_1(T) = 2\alpha \int_{S_T} e^{\beta f} P_{h,n}(\nabla z) + 2\alpha^2(\alpha + 1) \int_{S_T} n_0 f_i^2 e^{\beta f} z^2.$$

Similarly, under the hypotheses for Lemma 3.2 we find that

$$\alpha m' \int_{P_T} f_i e^{(\beta + 2\alpha)f} \|\nabla v\|^2 \leq \int_{P_T} f_i^{-1} e^{(\beta + 2\alpha)f} (Lv)^2 + E_2(T)$$

where

$$E_2(T) = \int_{S_T} e^{(\beta + 2\alpha)f} P_{h,n}(\nabla v).$$

Now since $t=T$ along S_T , we can repeat the argument of §3 to obtain the estimate

$$E_1(T) + E_2(T) \leq p(\alpha) e^{(1 + \beta + 2\alpha)f(T)} \int_{S_T} \{v^2 + \|\nabla v\|^2\}$$

where $p(\alpha)$ is defined by (3.13).

With (5.1) established, we can mimic the proof of Theorem 4.1 exactly for solutions of (1.1) outside \mathcal{C} which vanish on \mathcal{C} .

THEOREM 5.2. *Suppose k_1 and k_2 satisfy the conditions*

$$(A_2) \quad k_1(t, x) = O(f_i^2); \quad k_2(t, x) = O(f_i),$$

in some $P(\varepsilon, \infty, R)$ with $\varepsilon > 0, R > 0$. Let u be a C^2 solution of

$$(1.1) \quad |Lu| \leq k_1(t, x)|u| + k_2(t, x)\|\nabla u\|$$

in $P(\varepsilon, \infty, R)$ which vanishes along \mathcal{C} . Then there is a positive ρ such that if

$$\lim_{T \rightarrow \infty} e^{\rho f(T)} \int_{S_T} \{u^2 + \|\nabla u\|^2\} = 0,$$

then u vanishes identically in $P(\varepsilon, \infty, R)$.

Proof. For a fixed large β and all large enough α and T we have the estimate (5.1). Because of (1.1) and (A_2) we find that

$$(Lu)^2 \leq 2K_1^2 f_i^4 u^2 + 2K_2^2 f_i^2 \|\nabla u\|^2$$

in $P(\varepsilon, \infty, R)$. Combining this with (5.1) we obtain

$$(5.2) \quad (2\alpha^3 - 4K_1^2) \int_{P_T} f_i^3 e^{(\beta + 2\alpha)f} u^2 \leq (4K_2^2 - \alpha m') \int_{P_T} f_i e^{(\beta + 2\alpha)f} \|\nabla u\|^2 + E_3(T)$$

for all large enough α and T . Now fix a value of α so large that (5.2) holds and that $(2\alpha^3 - 4K_1^2) \geq 1$ and $(4K_2^2 - \alpha m') \leq 0$. With this fixed α and sufficiently large T we now obtain

$$0 \leq \int_{P_T} f_t^3 e^{(\beta + 2\alpha)t} u^2 \leq p(\alpha) e^{(1 + 2\alpha + \beta)t(T)} \int_{S_T} \{u^2 + \|\nabla u\|^2\}.$$

Picking $\rho = 1 + 2\alpha + \beta$, the theorem follows.

6. Decay in exterior domains. Much attention has been given to the asymptotic behavior of solutions of the wave equation outside a reflecting obstacle. In this section we consider solutions of

$$(1.1) \quad |Lu| \leq k_1(t, x)|u| + k_2(t, x)\|\nabla u\|$$

in an exterior domain. We assume that L satisfies (A_0) and (A_1) with respect to some function f satisfying (P_i) for $1 \leq i \leq 5$.

Let \mathcal{O} be a bounded domain in x -space with smooth boundary and connected complement. Let $R_0 = \max\{r(x) : x \in \mathcal{O}\}$. We consider solutions outside \mathcal{O} , in other words, solutions in the region \mathcal{R} where

$$\mathcal{R} = \{(t, x) : t \geq 0; x \notin \text{Int}(\mathcal{O})\}.$$

We further restrict our attention to solutions in the class

$$\mathcal{U} = \{u \in C^2(\mathcal{R}) : u(t, x) = 0 \text{ if } x \in \partial\mathcal{O}\}.$$

The technique employed to establish bounds on the decay rate in this situation is a slight modification of that used in §§3 and 4.

The standard domains in the a priori estimates are the sets

$$\begin{aligned} Q(\varepsilon, T, R) &= \mathcal{R} \cap D(\varepsilon, T, R) \\ &= \{(t, x) : \varepsilon \leq t \leq T; r \leq Mt + R; x \notin \text{Int}(\mathcal{O})\} \end{aligned}$$

for $\varepsilon > 0$, $R > R_0$, and $\varepsilon < T < \infty$. The boundary $\partial Q(\varepsilon, T, R)$ is composed of three pieces:

$$\begin{aligned} S^+ &= \{(t, x) \in \partial Q(\varepsilon, T, R) : t = T\}, \\ S^- &= \{(t, x) \in \partial Q(\varepsilon, T, R) : t = \varepsilon \text{ or } r = Mt + R\}, \\ S^t &= \{(t, x) \in \partial Q(\varepsilon, T, R) : x \in \partial\mathcal{O}\}. \end{aligned}$$

On $S^+ \cup S^-$ the outer unit normal n satisfies $((n, n)) \geq 0$. But on S^t n is spacelike; indeed the time component n_0 is zero. So the domains $Q(\varepsilon, T, R)$ fail to be convenient. However the formulas of Lemma 2.2 are still valid if we insist that $v \in \mathcal{U}$.

LEMMA 6.1. *Let Q denote one of the domains $Q(\varepsilon, T, R)$. If $v \in \mathcal{U}$, and $\frac{1}{2}\alpha m^2 \geq \nu B(\varepsilon)$, then*

$$(6.1) \quad \int_Q 2Lve^{\alpha t} v_t \geq - \int_{S^+(\mathcal{Q})} e^{\alpha t} P_{h,n}(\nabla v)$$

and

$$(6.2) \quad \frac{1}{2}\alpha m' \int_Q f_t e^{\alpha f} \|\nabla v\|^2 \leq 2 \int_Q e^{\alpha f} |v_t L v| + \int_{S^+(Q)} e^{\alpha f} P_{h,n}(\nabla v).$$

Proof. We can apply (2.4) with $\lambda = e^{\alpha f}$ to obtain

$$\int_Q 2Lve^{\alpha f} v_t = \int_Q e^{\alpha f} \left\{ \alpha f_t P_{h,h}(\nabla v) + \sum_{ij=1}^v (a_{ij})_t v_{x_i} v_{x_j} \right\} - \int_{\partial Q} e^{\alpha f} P_{h,n}(\nabla v).$$

Because of Lemma 2.1 and the definition of m' , we get

$$\int_Q 2Lve^{\alpha f} v_t \geq \frac{1}{2}\alpha m' \int_Q f_t e^{\alpha f} \|\nabla v\|^2 - \int_{\partial Q} e^{\alpha f} P_{h,n}(\nabla v).$$

On $S^-(Q)$ we know that $P_{h,n}(\nabla v) \leq 0$. On S^t , we have $n_0 = 0$ and $v_t = 0$. Hence, on S^t ,

$$P_{h,n}(\nabla v) = n_0 P_{h,h}(\nabla v) - 2v_t \sum_{ij=1}^v a_{ij} n_i v_{x_j} = 0.$$

Therefore

$$\int_Q 2Lve^{\alpha f} v_t \geq \frac{1}{2}\alpha m' \int_Q f_t e^{\alpha f} \|\nabla v\|^2 - \int_{S^+(Q)} e^{\alpha f} P_{h,n}(\nabla v).$$

The inequalities (6.1) and (6.2) now follow directly.

Using this modified version of Lemma 2.2, the method of §3 adapts to establish the following estimates.

THEOREM 6.2. For fixed $\varepsilon > 0$, $R > R_0$, let Q_T denote $Q(\varepsilon, T, R)$ for each $T > \varepsilon$. There is a β such that if $v \in \mathcal{U}$ and if α and T are sufficiently large then

$$(6.3) \quad 3\alpha^3 \int_{Q_T} f_t^3 e^{(\beta+2\alpha)f} v^2 + \alpha m' \int_{Q_T} f_t e^{(\beta+2\alpha)f} \|\nabla v\|^2 \leq 2 \int_{Q_T} f_t^{-1} e^{(\beta+2\alpha)f} (Lv)^2 + E_5$$

where, for p defined in (3.13), E_5 denotes the quantity

$$E_5 = p(\alpha) e^{(1+\beta+2\alpha)f(T)} \int_{S^+(Q_T)} \{v^2 + \|\nabla v\|^2\}.$$

Letting $Q(\varepsilon, \infty, R)$ denote the unbounded domain $\mathcal{B} \cap D(\varepsilon, \infty, R)$ we make the following assumption about the coefficients k_i in (1.1):

(A₂) $k_1(t, x) = O(f_t^2)$; $k_2(t, x) = O(f_t)$ uniformly in each $Q(\varepsilon, \infty, R)$.

Suppose $u \in \mathcal{U}$ is a solution of (1.1) in some $Q(\varepsilon, \infty, R)$. Then following the method of proof of Theorem 4.1, we find an α so large that

$$(6.4) \quad 0 \leq \int_{Q_T} f_t^3 e^{(\beta+2\alpha)f} u^2 \leq p(\alpha) e^{(1+2\alpha+\beta)f(T)} \int_{S^+(Q_T)} \{u^2 + \|\nabla u\|^2\}$$

for all sufficiently large T .

THEOREM 6.3. Suppose $u \in \mathcal{U}$ is a solution of (1.1) in $Q(\varepsilon, \infty, R)$. Let S_T denote $S^+(Q_T) = \{(T, x) : x \notin \text{Int}(\mathcal{O}); r \leq Mt + R\}$. There is a $\rho > 0$ such that if

$$(6.5) \quad \lim_{T \rightarrow \infty} e^{\rho f(T)} \int_{S_T} \{u^2 + \|\nabla u\|^2\} = 0$$

then u vanishes identically in $Q(\varepsilon, \infty, R)$.

Proof. Set $\rho = 1 + 2\alpha + \beta$ for the α and β of (6.4). Since the integral over Q_T in (6.4) is an increasing nonnegative function of T , then (6.5) implies that u must vanish in all Q_T , and thus in $Q(\varepsilon, \infty, R)$.

REFERENCES

1. L. Hörmander, *Uniqueness theorems and estimates for normally hyperbolic partial differential equations of the second order*, Toltte Skand. Math. Kongr., Lund, 1953, pp. 105–115. MR 16, 483.
2. W. Littman, *Maximal rates of decay of solutions of partial differential equations*, Bull. Amer. Math. Soc. 75 (1969), 1273–1275. MR 39 #7270.
3. C. Morawetz, *Exponential decay of solutions of the wave equation*, Comm. Pure Appl. Math. 19 (1966), 439–444. MR 34 #4664.
4. H. Ogawa, *Lower bounds for solutions of hyperbolic inequalities*, Proc. Amer. Math. Soc. 16 (1965), 853–857. MR 33 #1596.
5. M. H. Protter, *Asymptotic behavior and uniqueness theorems for hyperbolic equations and inequalities*, Technical Report, Contract AF 49(638)–398, University of California, 1960.
6. ———, *Asymptotic behaviour and uniqueness theorems for hyperbolic operators*, Outlines Joint Sympos. Partial Differential Equations (Novosibirsk, 1963), Acad. Sci. USSR Siberian Branch, Moscow, 1963, pp. 348–353. MR 34 #1659.
7. W. Strauss, *Decay and asymptotics for $\square u = F(u)$* , J. Functional Analysis 2 (1968), 409–457. MR 38 #1385.

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