

A CLASS OF COMPLETE ORTHOGONAL SEQUENCES OF STEP FUNCTIONS⁽¹⁾

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Abstract. A class of orthogonal sets of step functions is defined and each member is shown to be complete in $L_2(0, 1)$. Pointwise convergence theorems are obtained for the Fourier expansions relative to these sets. The classical Haar orthogonal set is shown to be a set of this class and the class itself is seen to be a subclass of the "generalized Haar systems" defined recently by Price.

1. Introduction. Convergence theorems for the Fourier expansions relative to the classical set of Haar functions have interestingly weak hypotheses. If $f \in L_1(0, 1)$ is the derivative of its indefinite integral at $x \in [0, 1]$, the Haar expansion converges to $f(x)$ at this point, and, if f is continuous on $[0, 1]$, the convergence is uniform on this interval.

In this paper we prove that each sequence of points which is dense in $[0, 1]$ determines a complete orthonormal set of step functions whose associated Fourier expansion has convergence properties similar to the Haar expansion. In fact, the Haar set is a member of the class of sets defined in §2. This class, in turn, is seen to be a subclass of the class of "generalized Haar systems" defined by Price [3].

2. Definition of the sequences $\{\theta_n\}$. Suppose that $A = \{a_1, a_2, \dots\}$ is a sequence of distinct points in $(0, 1)$ which is dense in $[0, 1]$ and let $\{g_n\}$, $n = 0, 1, 2, \dots$, be the set of unit step functions defined by

$$g_0(x) \equiv 1 \quad \text{on } [0, 1]$$

and, for $n \geq 1$,

$$\begin{aligned} g_n(x) &= 0, & x \in [0, a_n), \\ &= 1, & x \in [a_n, 1]. \end{aligned}$$

Since no two of these functions have discontinuities at the same point, it is clear that the g_i are linearly independent on $[0, 1]$. Consequently, one can use the Gram-Schmidt process to construct an orthonormal sequence of functions $\{\theta_n(x)\}$

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such that each θ_n is a linear combination of the g_k , $k \leq n$. Because of the triangular nature of this construction, each g_i can also be expressed as a linear combination of the θ_k , $k \leq i$.

3. Completeness of $\{\theta_n\}$. To obtain convergence theorems for the Fourier expansions relative to the orthonormal sequences defined in §2 one needs a rather obvious property of the sequence A which is given in Lemma 1. In the statement of this lemma and throughout this paper the term "adjacent points" of a finite subset $A_N \subset A$ will be used to denote successive elements of this subset when the elements are arranged in order of magnitude; i.e. a_m and a_n are adjacent points of A_N if and only if there is no $a_k \in A_N$ such that $a_m < a_k < a_n$ or $a_n < a_k < a_m$.

LEMMA 1. *Let $\{a_1, a_2, \dots\}$ be a sequence of distinct points of $(0, 1)$ which is dense in $I = [0, 1]$. Then for each $\delta > 0$ there is an integer N_δ such that for each $N > N_\delta$,*

- (i) *any pair of adjacent points a_m and a_n in the subset $A_N = \{a_1, a_2, \dots, a_N\}$ satisfy $|a_m - a_n| < \delta$,*
- (ii) *$d(x, A_N) < \delta$ for all $x \in I$. ($d(x, A_N)$ is the distance from x to the set A_N defined in the usual manner.)*

THEOREM 1. *The orthonormal sequence of functions $\{\theta_n\}$ is complete in $L_2(0, 1)$.*

Proof. Suppose that $f \in L_2(0, 1)$ and $\int_0^1 f \theta_n dx = 0$ for all n . Then since each g_n is a linear combination of the θ_i , $i \leq n$,

$$\int_0^1 f g_n dx = \int_{a_n}^1 f dx = 0 \quad \text{for all } n,$$

and since A is dense in $[0, 1]$,

$$\int_x^1 f dx = 0 \quad \text{for all } x \in [0, 1];$$

i.e.

$$f \stackrel{\text{a.e.}}{=} 0 \quad \text{on } [0, 1].$$

4. Pointwise convergence of the Fourier $\{\theta_n\}$ expansion. Since the orthonormal sequence $\{\theta_n\}$ is complete in $L_2(0, 1)$, any function f in this space has the norm-convergent Fourier expansion

$$(1) \quad f(x) \sim \sum b_n \theta_n(x), \quad \text{where } b_n = \int_0^1 f \theta_n dx.$$

In this section we obtain sufficient conditions for the convergence of this expansion in the pointwise sense to $f(x)$. The main result may be stated as follows:

THEOREM 2. *The Fourier- θ_n expansion of a function $f \in L_1(0, 1)$ converges to $f(x)$ at each point $x \in [0, 1]$ at which f is the derivative of the indefinite integral F of f . This holds, in particular, (a) almost everywhere, (b) at every point of continuity of f .*

Proof. Fix $N > 0$ and consider the partial sum S_N of f . Let $0 < a_{i_1} < a_{i_2} < \cdots < a_{i_N} < 1$ be the points $0, a_1, a_2, \dots, a_N, 1$ arranged in increasing order. Since each $\theta_i, i \leq N$, is a linear combination of the $g_i, i \leq N$, S_N is a step function whose intervals of constancy are $[0, a_{i_1}), [a_{i_1}, a_{i_2}), \dots, [a_{i_N}, 1]$. We first show that for x in any such interval I , S_N is equal to the average of f over I ; i.e.

$$S_N(x, f) = \frac{1}{|I|} \int_I f dt \quad \text{if } x \in I.$$

Suppose first that $f \in L_2(0, 1)$ and let $K_0, K_1, \dots, K_N$ denote the characteristic functions of the intervals $[0, a_{i_1}), \dots, [a_{i_N}, 1]$. Then

$$S_N = \sum_0^N b_n \theta_n = \sum_0^N B_n K_n,$$

where the B 's are constants. Clearly B_i is the value S_N takes in the interval associated with K_i . Now if T_N is any linear combination of the $\theta_k, k \leq N$, it is well known that $\int_0^1 (f - T_N)^2 dx$ assumes its minimum value when T_N is the partial sum S_N . Thus the B 's must have values which minimize the integral $\int_0^1 (f - \sum_0^N B_n K_n)^2 dx$ and, equating the partial derivatives with respect to the B 's to 0, we obtain, for each $m = 0, 1, 2, \dots, N$,

$$\int_0^1 f K_m dx = B_m \int_0^1 K_m^2 dx.$$

If f is merely in $L_1(0, 1)$, there is a sequence $\{f_k\}$ of functions in $L_2(0, 1)$ such that $\|f - f_k\|_1 \rightarrow 0$. It is immediate that each Fourier coefficient of f is the limit of the corresponding coefficient of f_k and we have, for $x \in I$,

$$S_N(x, f) = \lim_k S_N(x, f_k) = \lim_k \frac{1}{|I|} \int_I f_k dx = \frac{1}{|I|} \int_I f dx.$$

Now if x_0 is in the interval $I = [a, b)$ where a and b are adjacent points of the subset A_N ,

$$\begin{aligned} |f(x_0) - S_N(x_0, f)| &= \left| f(x_0) - \frac{1}{|I|} \int_I f dx \right| \\ &= \left| \frac{x_0 - a}{|I|} \left[f(x_0) - \frac{1}{x_0 - a} \int_a^{x_0} f dx \right] + \frac{b - x_0}{|I|} \left[f(x_0) - \frac{1}{b - x_0} \int_{x_0}^b f dx \right] \right|. \end{aligned}$$

Since by Lemma 1 the length of I can be made arbitrarily small by taking N sufficiently large, the last equation implies that the Fourier expansion of f converges to $f(x_0)$ at any point $x_0 \in [0, 1]$ at which f is the derivative of its indefinite integral.

The expression for S_N obtained in the preceding proof and a theorem from [4, p. 32] can be used to establish the following corollaries to Theorem 2.

COROLLARY 1. If $f \in L_p(0, 1)$, then $\|f - S_N\|_p \rightarrow 0$ ($1 < p < \infty$).

COROLLARY 2. If $f \in L_p(0, 1)$, then $\|\text{Sup}_N S_N\| \leq A_p \|f\|_p$ ($1 < p \leq \infty$) where A_p is constant for each p .

THEOREM 3. The Fourier- θ_n expansion of a function f which is continuous on $[0, 1]$ converges uniformly to $f(x)$ on $[0, 1]$.

Proof. If $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(x_1) - f(x_2)| < \varepsilon$ when $x_1, x_2 \in [0, 1]$ and $|x_1 - x_2| < \delta$. By Lemma 1 an integer N_δ exists such that if $N > N_\delta$, the set A_N determines a partition of $[0, 1]$ into subintervals of length less than δ . Thus if x_0 is in any subinterval $I = [a, b]$ of this partition we have

$$\begin{aligned} |f(x_0) - S_N(x_0, f)| &= \left| f(x_0) - \frac{1}{|I|} \int_I f(x) dx \right| \\ &= |f(x_0) - f(\xi)| \quad \text{where } \xi \in [a, b] \\ &< \varepsilon. \end{aligned}$$

5. **Behavior of the Fourier- θ_n expansion at a point of discontinuity of f .** Suppose f has an isolated finite discontinuity at a point a_i of the set A . Since the step function g_i has a unit jump at a_i , a function G which is continuous at a_i can be constructed by adding a constant multiple of g_i to f . By Theorem 2, the Fourier expansion of G converges to $G(a_i)$ and since the θ_n expansion of g_i is a finite sum, it follows that the Fourier expansion of f must converge at a_i . In fact, one can readily prove the following

THEOREM 4. If $f \in L_2(0, 1)$ has a finite discontinuity at a point a_i of the sequence A (which determines $\{\theta_n\}$) the Fourier- θ_n expansion of f converges to $f(a_i^+)$ at this point.

The fact that the expansion converges to $f(a_i^+)$ rather than $\frac{1}{2}[f(a_i^+) + f(a_i^-)]$ or some other value between $f(a_i^+)$ and $f(a_i^-)$ is not significant since this value is determined solely by the definition of $g_i(a_i)$.

No general statement can be made concerning the convergence of (1) at a point of discontinuity of f that is not in A . To show this we consider two expansions corresponding to different A sequences (hence to different θ_n sets) of a simple step function with a discontinuity at a point $c \in (0, 1)$ which is not in A . Let

$$\begin{aligned} g_c(x) &= 0, & x \in [0, c), \\ &= 1, & x \in [c, 1], \end{aligned}$$

where $c \notin A$. Suppose that N is an integer sufficiently large for the subset A_N to contain points in both $(0, c)$ and $(c, 1)$ and let S_N denote the partial sum of the Fourier expansion of g_c . If $a_l(N)$ and $a_r(N)$ are the points of A_N adjacent to c on the left and right, respectively, we find

$$(2) \quad S_N(c, g_c) = \frac{1}{a_r - a_l} \int_{a_l}^{a_r} g_c dx = \frac{a_r(N) - c}{a_r(N) - a_l(N)}.$$

Equation (2) suggests that the convergence of the Fourier expansion at the point c depends on the sequence A . To see that this is actually the case, construct a sequence A , by choosing $a_1 = c/2$ and $a_2 = (c+1)/2$, the respective midpoints of $(0, c)$ and $(c, 1)$. These two points along with c determine four successive subintervals of $(0, 1)$; let their midpoints (from left to right) be a_3, a_4, a_5, a_6 . Continue this subdivision process (with 2^n new midpoints at the n th stage) to obtain A . In (2), $a_r(N) - c$ is the distance from c to the closest point of A_N on the right and has the form $(1-c)/2^k$ for some integer k . On the other hand, the distance from c to the closest point of A_N on the left will either be $c/2^k$ or $c/2^{k+1}$ depending on N . In the first case (2) gives $S_N = 1 - c$, and in the second $S_N = 2(1-c)/(2-c)$. Since these expressions are independent of N , we see that the sequence $\{S_n(c, g_c)\}$ consists of two distinct constant subsequences and cannot converge.

As a second example we construct the sequence A as follows: $a_1 = c/2$, $a_2 = (1+c)/2$, $a_3 = c/3$, $a_4 = 2c/3$, $a_5 = (1+2c)/3$, $a_6 = (2+c)/3, \dots$; i.e. the elements of A are the distinct points which divide the intervals $(0, c)$ and $(c, 1)$ into two equal parts, three equal parts, four equal parts, etc. For this sequence, the distance from c to the closest point of A_N on the right has the form $(1-c)/k(N)$ where $k(N)$ is an integer that depends on N ; and the distance from c to $a_i(N)$ is either $c/k(N)$ or $c/[k(N)+1]$. In the first case (2) gives $S_N = 1 - c$ and in the second

$$S_N = \frac{1-c}{1-c/(k(N)+1)}.$$

Since $k(N)$ approaches infinity with N , the limit of the subsequence given by the second equation is also $1 - c$. Thus $\lim S_n(c)$ exists for this sequence A ; i.e. the Fourier expansion of g_c determined by this particular sequence converges at c to $1 - c$.

6. The structure of $\{\theta_n\}$. The orthonormal sequence $\{\theta_n\}$ defined in §2 is obtained by applying the Gram-Schmidt orthogonalization process to the linearly independent sequence $\{g_i\}$. Since this process gives θ_n as a linear combination of the $g_i, i \leq n$, it is clear that θ_n is a step function which is constant on the subintervals $[0, a_1), [a_1, a_2), \dots, [a_n, 1]$ of $[0, 1]$ determined by the successive points of A_n . We shall now see that a precise expression for θ_n in terms of the points of the subset A_n can be obtained by induction.

Let $\theta_0 = g_0$ and assume that $\theta_{n-1}, n \geq 1$, has been determined. Suppose a_n falls in the interval (a, b) whose endpoints are successive points of the partition of $[0, 1]$ determined by A_{n-1} , and let f_n be given by

$$\begin{aligned} f_n(x) &= 1/(a_n - a), & x \in [a, a_n), \\ &= -1/(b - a_n), & x \in [a_n, b), \\ &= 0, & \text{otherwise.} \end{aligned}$$

Since a and b are either in A_{n-1} or are endpoints of $[0, 1]$, it is obvious that f_n is a linear combination of the g_i , $i \leq n$. Furthermore if $k < n$, θ_k is constant on (a, b) so

$$(\theta_k, f_n) = \int_0^1 \theta_k f_n dx = c \int_a^b f_n dx = 0$$

and we see that f_n is orthogonal to each θ_k , $k < n$. It follows, of course, that the function θ_n given by the Gram-Schmidt process must be $\pm f_n / \|f_n\|$.

7. Examples and remarks. (A) The set of rationals in $(0, 1)$ is countable and different enumerations of this set lead to an infinity of sequences of the type A described in §2. For example, one can take $a_1 = 1/2$, $a_2 = 1/3$, $a_3 = 2/3$, $a_4 = 1/4$, $a_5 = 3/4$, ... (where the irreducible fractions with denominator 2, 3, 4, ... are used successively in blocks) and the corresponding $\{g_i\}$ will be the set of all unit step functions with jumps at the rational points of $(0, 1)$. The orthonormal set $\{\theta_n\}$, in this case, consists of step function with discontinuities at the rationals.

(B) If p is any prime, the set of all numbers of the form k/p^m where m and k are integers, $k < p^m$ and $k \not\equiv 0 \pmod{p}$, is countable since it is a subset of the rationals and is obviously dense in I . Any such set with an appropriate specific enumeration could therefore be used for the sequence A . In fact, if A is a particular sequence of this type with $p=2$, the corresponding g_i can be orthonormalized to obtain the familiar Haar functions $\{\chi_j^n\}$. Thus if $a_1 = 1/2$, $a_2 = 1/4$, $a_3 = 3/4$, $a_4 = 1/8$, $a_5 = 3/8$, ..., $a_n = (2n+1-2^k)/2^k$, ..., where k is the smallest integer such that $2^k > n$, the corresponding θ_i are the classical Haar functions.

Price [3] has defined a class of orthonormal sets of step functions of a more general type. Price's definition is not sufficiently restrictive to imply the completeness of his sets. It is readily verified that the class of sequences $\{\theta_n\}$ defined in this paper is a subclass of Price's and that our requirement that A be dense in $[0, 1]$ (which insures completeness) is the essential difference.

Franklin [1] constructed a complete orthonormal sequence of linear functions related to the Haar functions. It can be readily generalized. Let $\{g_n\}$ be defined as in §2 and construct a sequence $\{h_n\}$ on $[0, 1]$ with

$$h_0(x) \equiv 1 \quad \text{and} \quad h_n(x) = \int_0^x g_{n-1}(t) dt, \quad n \geq 1.$$

The sequence $\{h_n\}$ is linearly independent on $[0, 1]$ and the Gram-Schmidt process yields a complete orthonormal sequence of continuous polygonal functions.

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