

ON THE EXISTENCE OF TRIVIAL INTERSECTION SUBGROUPS

BY

MARK P. HALE, JR.

Abstract. Let G be a transitive nonregular permutation group acting on a set X , and let H be the subgroup of G fixing some element of X . Suppose each nonidentity element of H fixes exactly b elements of X . If $b=1$, G is a Frobenius group, and it is well known that H has only trivial intersection with its conjugates. If $b>1$, it is shown that this conclusion still holds, provided H satisfies certain natural conditions. Applications to the study of Hall subgroups and certain simple groups related to Zassenhaus groups are given.

1. A transitive permutation group G in which every nonidentity element fixes either one or no letters is called a Frobenius group. If H is the stabilizer of a single letter in such a group, it is easy to see that H is a trivial intersection (t.i.) subgroup of G , that is $H \cap H^x = 1$ unless $H = H^x$. Theorem A gives a condition for the existence of t.i. subgroups in permutation groups which are generalizations of Frobenius groups.

If π is a set of primes, x is a π -element of a group G if $o(x)$, the order of x , is divisible only by primes in π . A subgroup H of G is a π -subgroup if every element of H is a π -element; if also the order $|H|$ of H is prime to the index $|G:H|$ of H in G , H is a π -Hall subgroup. Let π' denote the collection of all primes not in π . A group G is called π -isolated if no π -element commutes with a π' -element.

Our main theorem is

THEOREM A. *Let G be transitive on X . Suppose that the subgroup H of G of all elements fixing one letter in X is π -Hall in G , and G is π -isolated. Then if every nonidentity element of H fixes exactly b letters, H is a trivial intersection subgroup of G . If $b>1$, H is also nilpotent, and $|N_G(H):H| = b$.*

The proof of this result depends on the description of partitioned groups, given in §2. In §3, the proof of Theorem A is given, and we conclude with some applications of this result.

All groups considered are finite. Basic notation, and statements of theorems quoted by name, may be found in [7]. We note that a group G is a Frobenius group if and only if G contains a proper trivial intersection subgroup H which is its own normalizer in G . A classical theorem of Frobenius' asserts that in this case, G

Presented to the Society, April 18, 1970; received by the editors April 27, 1970.

AMS 1969 subject classifications. Primary 2020; Secondary 2029, 2043.

Key words and phrases. Trivial intersection subgroup, Frobenius group, partitioned groups, Hall subgroup, centralizer of each of its nonidentity elements, exceptional characters.

Copyright © 1971, American Mathematical Society

contains a normal subgroup K , with the properties $K \cap H = 1$, $KH = G$. We will call K the Frobenius kernel and H the Frobenius complement of G .

The author wishes to express his deep gratitude to Professor Michio Suzuki for many helpful conversations. A preliminary version of parts of this paper was submitted as part of a doctoral dissertation at the University of Illinois, Urbana.

2. A partition of a group G is a collection σ of subgroups of G such that every nonidentity element of G occurs in one and only one subgroup in the set σ . A partition σ is called nontrivial if it does not consist of $\{G\}$ alone; σ is normal if, for each U in σ , U^x is in σ , for all x in G . If σ, τ are partitions of G and g is in G , then $\sigma^g = \{U^g : U \in \sigma\}$ and $\sigma \cap \tau = \{U \cap V : U \in \sigma, V \in \tau\}$ are also partitions of G . It is thus evident that any group admitting a nontrivial partition admits a nontrivial normal partition. If U is contained in σ , a partition of G , we call U a component of σ . The components of a normal partition are t.i. subgroups.

Combining results due to Baer, Kegel, and Suzuki, we may give the following description of groups admitting nontrivial normal partitions.

THEOREM 1. *Let G be a finite group having a nontrivial normal partition. Then one of the following cases holds:*

- (a) G is isomorphic to a Suzuki group $Sz(q)$ or a linear fractional group $PSL(2, q)$.
- (b) G is isomorphic to S_4 , the symmetric group on four letters.
- (c) G is a Frobenius group.
- (d) G is a p -group.
- (e) G contains a nilpotent normal subgroup K of prime index p in G , every element of G not in K has order p and is centralized only by p -elements, and p divides $|K|$.

Proof. Let N be the maximal nilpotent normal subgroup of G . If $N = 1$, G is nonsolvable; then by Suzuki [13], case (a) holds. If N is nontrivial but contains no Sylow subgroup of G , Theorem B and Theorem A of Baer [2] give case (b). Finally, if N contains a Sylow subgroup of G , then one of the remaining cases must hold, by Baer [1, Satz 5.1].

We will also need the following lemmas from [1].

LEMMA 1. *If G is a Frobenius group, then a Frobenius complement is a component in any nontrivial normal partition of G .*

LEMMA 2. *If σ is a partition of G , and x and y are commuting elements of G of distinct orders, then x, y are in the same component of σ .*

Proof. We may assume by symmetry that $o(x) < o(y)$; let $m = o(x)$. Then $(xy)^m = y^m \neq 1$. Since xy and y have a common power $y^m \neq 1$, xy, y and hence x lie in the same component of σ .

3. The following theorem, due to Suzuki, is needed for the proof of Theorem A.

THEOREM 2. *Let P be a p -Sylow subgroup of G . Suppose that for any $x \neq 1$ in P , $C_G(x)$ is a p -group, and $x^p = 1$; then P is a t.i. subgroup of G .*

Proof. If P is abelian, P is t.i. since for each $x \neq 1$ in P , $P = C_G(x)$. For $p=2$, $x^2=1$ for all x implies P is abelian, so we need only consider p odd, and P nonabelian.

If P contains any nonidentity element conjugate to its inverse, P is abelian, by Theorem (4D) of Brauer and Fowler [4]. Let D be maximal among all nontrivial intersections of P with one of its distinct conjugates P' , and define $M = N_G(D)$. Since P is nonabelian, M has odd order, and is solvable by the Theorem of Feit and Thompson [6]. Since the p' -elements of M centralize no nonidentity element of D , the q -Sylow subgroups of M are cyclic for $q \neq p$ [7, p. 200]. Theorem B of Hall and Higman [8] now applies to M ; thus there is a normal p' -subgroup K of M such that M/K has a normal p -Sylow subgroup. Since K must then centralize D , $K=1$, and M has a normal p -Sylow subgroup. However, P and P' can be chosen so that $P \cap M$ and $P' \cap M$ are p -Sylow subgroups of M . Thus $P \cap M = P' \cap M = D$, and $D = N_P(D)$, contrary to $D \neq P$. Thus P is a t.i. subgroup of G .

Throughout the rest of this section we will assume the following hypothesis:

(H1) G is a transitive permutation group acting on the set X . The subgroup H of all elements fixing some element α of X is a π -Hall subgroup of G , and G is π -isolated. Every element $x \neq 1$ of H fixes exactly b elements of X , $b > 1$.

LEMMA 3. *If (H1) holds for G , b is a π' -number, and every π -element of G fixes b letters of X .*

Proof. Let p be a prime in π , and let x be an element of order p in H . The set X contains $|G:H|$ elements, and x fixes b elements and permutes the rest in orbits of length p . Thus p divides $|G:H| - b$. Since p is relatively prime to $|G:H|$, p is relatively prime to b .

For the second part, suppose x is a π -element fixing no points in X . Then x is not of prime-power order, for then a conjugate of x would lie in a Sylow subgroup of H . Thus there are commuting π -elements u, v of coprime order with $x = uv$. By induction on $o(x)$, we may assume u and v both fix points of X . If u has no fixed points in common with v , u permutes the fixed points of v in orbits of length $o(u)$, and thus $o(u)$ divides b , a contradiction. Thus u and v have a common fixed point, which is a fixed point of $x = uv$, proving the lemma.

The next lemma allows us to apply §2 to our problem.

LEMMA 4. *Suppose G satisfies (H1). If H is not a t.i. subgroup of G , G contains a collection of t.i. subgroups which induce a nontrivial normal partition of H .*

Proof. Let $x \neq 1$ be any π -element of G , and let D_x denote the subgroup of all elements of G fixing the fixed points of x . Let x , and y be π -elements of G . Then if $t \neq 1$ is in both D_x and D_y , x and y have the same b fixed points, so $D_x = D_y$. Thus the subgroups D_x are t.i. subgroups of G . If $H \neq D_x$ for some x , H is nontrivially partitioned by the subgroups D_h , $h \neq 1$ in H .

Proof of Theorem A. If $b=1$, the theorem is the classical case; so we assume $b > 1$. If $H=1$, H is a t.i. subgroup; the last paragraph of the proof below applies.

We adopt (H1) on G ; suppose H is not a t.i. subgroup of G . Then H is partitioned, and one of the cases of Theorem 1 describes H . We will show that none of the cases (a)–(e) can occur.

(i) H is not $Sz(q)$, $PSL(2, q)$, or S_4 . The possible partitions of these groups are given in Suzuki [13] and Baer [2]. If H is one of these groups, a cyclic Hall subgroup K of odd order is a component of the partition of H , and $N_H(K)$ is a dihedral group. By a theorem of Jordan [15, p. 6], $N_G(K)$ is transitive on the b fixed points of K , so $b = |N_G(K) : N_H(K)|$. Since the automorphism group of an odd-order cyclic group is abelian, and $b \neq 1$, this contradicts the π -isolation of G .

(ii) H is not a Frobenius group. If H is a Frobenius group, a Frobenius complement C is a component of the partition of H , by Lemma 1. Let K be the Frobenius kernel of H ; by a theorem of Thompson [14], K is nilpotent. The partition of H induces a partition of K . We will show that there is a unique component U of the partition containing $Z(K)$. Let $x \neq 1$ be contained in $Z(K)$. If there is an element $y \neq 1$ in K with $\alpha(x) \neq \alpha(y)$, x and y lie in the same component U by Lemma 2. Then since U contains two elements of unequal order, $Z(K) \leq U$. Otherwise every element in K has order p , and K is a p -Sylow subgroup of H and hence of G . If any p' -element centralized a p -element, K would contain p' -elements. Thus the hypotheses of Theorem 2 hold for K , and K is a t.i. subgroup of G . Now choose $k \neq 1$ in K , and regarding the points of X as right cosets of H , let Ht be a point fixed by k . Then $Htk = Ht$, so $tk t^{-1} \in H$. Since K is the unique p -Sylow subgroup of H , $tk t^{-1} \in K$, and since K is a t.i. subgroup, $tK t^{-1} = K$. It now is clear that $HtK = Ht$, so any fixed point of k is a fixed point for K , that is $K = U$ is a component of H containing $Z(K)$.

Now consider the (unique) component U containing $Z(K)$. Since $Z(K)$ is a characteristic subgroup of K , U is normal in H . Thus U is normalized by the complement C of H . By the previously-cited theorem of Jordan, $M = N_G(U)$ is transitive on the fixed points of U , so $H < M$, and M contains π' -elements. If $C = N_M(C)$, then M is a Frobenius group with complement C , since C is a t.i. subgroup of G . The Frobenius kernel of M is nilpotent by Thompson's Theorem, so there are π' -elements commuting with π -elements in G , contrary to π -isolation. However, if there is a π' -element x of M normalizing C , x normalizes CU , and since x centralizes no π -element, CU is nilpotent, contrary to hypothesis. Finally suppose $N_M(C)$ contains a π -element x of prime power order. Then x permutes the b fixed points of C , and by Lemma 3, must fix one of them. Thus $\langle x, C \rangle$ is isomorphic to a subgroup of H ; since $N_H(C) = C$, we have x in C . Thus $N_M(C) \neq C$ is impossible; this disposes with the present case.

(iii) H is a t.i. subgroup. We have shown that if this is false, H is partitioned, and only cases (d) and (e) of Theorem 1 can possibly occur. In both these cases, $Z(H) \neq 1$. Let x be an element of prime order p in $Z(H)$. By π -isolation, $H = C(x)$, so we may regard X as the set of all conjugates x^t of x , t ranging over G . The permutation condition in (H1) now becomes

(H2) Every element $y \neq 1$ of H centralizes b conjugates of x .

In particular, x satisfies (H2), so H contains b conjugates of x . Let D be the component of the partition of H containing x . We will show that H is partitioned by conjugates of D . Let $y \neq 1$ be any element of H . If $o(y) \neq p$, then y is in D , by Lemma 2. Suppose $o(y) = p$. If $C_G(y)$ is not a p -group, there is a p' -element z with $yz = zy$. By π -isolation, z is a π -element, and hence yz has fixed points, which must be the same as those of y and z . Thus z centralizes x , and y, z , and x all lie in the same component D . If $o(y) = p$, and $C_G(y)$ is a p -group, $C_G(y)$ lies in some p -Sylow subgroup P of G . Since H is π -Hall, for some t , $C_G(y) \leq P \leq H^t$. By (H2), $X \cap C_G(y)$ and $X \cap H^t$ both contain b conjugates of x . Thus y and x^t have the same fixed points, and y lies in D^t .

Since H is partitioned by isomorphic subgroups, from Theorem 1 and Lemma 3 we see that H is a p -group which satisfies $z^p = 1$ for all z in H . Now H satisfies the hypothesis of Theorem 2, so H is a t.i. subgroup of G .

We have proven that H is a t.i. subgroup. It then follows that all elements of H fix the same b letters, so by the theorem of Jordan, $b = |N_G(H) : H|$. As $b \neq 1$, and H is a π -Hall subgroup, H is normalized by π' -elements, which centralize no elements of H . Thus $N_G(H)$ is a Frobenius group, and H is nilpotent by Thompson's Theorem. This completes the proof of Theorem A.

The hypotheses of Theorem A cannot be weakened significantly without invalidating the conclusion. For the following examples, let G be the group $PSL(2, 11)$, the structure of which can be found in [10, II.8]. Consider the transitive representation of G on right cosets $\{Hx\}$ of a 2-Sylow subgroup H . Each element of H fixes 9 cosets, but H is not a t.i. subgroup, since G contains a subgroup D isomorphic to a dihedral group of order 12.

For $\pi = \{2, 3\}$, D is a π -Hall subgroup and G is π -isolated, but D is not a t.i. subgroup. Even the requirement that G is π -isolated, contains a π -Hall subgroup, and contains subgroups partitioning the π -elements is not enough. The subgroups B isomorphic to $PSL(2, 5)$ contained in $PSL(2, 11)$ illustrate this. In spite of these examples, the use of §2 in studying permutation groups in which any element $x \neq 1$ fixing one letter fixes b letters might lead to a partial description of these groups.

4. As an application of our previous results, we give a criterion for the existence of abelian Hall subgroups.

THEOREM B. *If G is a group in which the centralizer of every nonidentity π -element is a π -Hall subgroup, G has an abelian π -Hall subgroup which is the centralizer of each of its nonidentity elements.*

Proof. It is enough to show G has an abelian π -Hall subgroup, as the remainder of the conclusion follows easily. Suppose first that π contains more than one prime. Let x be a π -element of prime order p , and let Q be any q -Sylow subgroup of $C(x)$, $q \neq p$. If $y \neq 1$ is any element in Q , $C(xy) = C(x) = C(y)$, since x and y are powers of xy . Thus $Q \leq Z(C(x))$, and the conclusion follows, by interchanging p

and q . If $\pi = \{p\}$, every p -element of G is central in some p -Sylow subgroup of G . If b is the number of conjugates of some p -element $x \neq 1$ occurring in any p -Sylow subgroup P , then every nonidentity p -element centralizes b conjugates of x . Thus Theorem A applies, and P is a t.i. subgroup of G . Now P centralizes each of its nonidentity elements, and the conclusion follows.

Theorem A also has applications in the study of certain simple groups. We call a subgroup A of a group G a special subgroup if A is the centralizer of each of its nonidentity elements, and $|N_G(A):A| = 2$.

Groups having special subgroups have been studied in [5], [9], and [11]; the known simple groups with this property are the groups $Sz(q)$, $PSL(3, 4)$, and $PSL(2, q)$. Feit and Thompson determined those simple groups having a special subgroup of order 3. We will describe our results here, referring the reader to [9] for the necessary preliminaries.

Let G be a simple group containing a special subgroup of order $a \geq 5$. Applying the method of exceptional characters to G , we obtain $(a-1)/2$ "exceptional" irreducible characters, and one "distinguished" irreducible character θ of G .

Employing knowledge of these characters and an idea of Suzuki's [12], a formula for $|G|$ may be given. This formula is $|G| = a\theta(1)(\theta(1) + \varepsilon)r^2$, where $\varepsilon = 1$ or -1 , and r is an integer with $r^2 \equiv 1$ modulo a . If $r = 1$ (this is the case in the known simple groups listed above), then much more information can be obtained on the conjugacy classes of G . In particular, every element has order dividing a , $\theta(1)$ or $\theta(1) + \varepsilon$. In this context, we offer the following generalizations of Theorem 1 of [9].

THEOREM C. *Let G be a simple group having a special subgroup of order $a \geq 5$. If in the group order formula $|G| = a\theta(1)(\theta(1) + \varepsilon)r^2$, $r = \varepsilon = 1$, and G has a subgroup of order $\theta(1)$, then G is isomorphic to either $Sz(q)$ or $PSL(2, q)$, with $q = \theta(1)$.*

REMARK. If $\theta(1)$ is even, or $\theta(1)$ is odd and $\theta(1) - \varepsilon \leq 22a$, it is easy to show that G contains an element x with $|C_G(x)| = \theta(1)$, using the information on conjugacy classes given in [9].

Description of proof. Let D be the subgroup of order $\theta(1)$ of G . Employing the known values of the exceptional characters of G , it can be seen that G satisfies the hypotheses of Theorem A with respect to the permutation representation of G on cosets of D , with $b = |A|$. It is then easy to show that the permutation character of G acting on right cosets of $N_G(D)$ is $1 + \theta$, so that G is doubly transitive. As the values of θ are known, we can show that in this representation, G is a Zassenhaus group. The theorem then follows from the classification of these groups.

THEOREM D. *If G is a simple group with a special subgroup of order a , and for some involution x in G , $|C_G(x)| \leq 9a/2$, then G is isomorphic to $PSL(2, q)$, for some q .*

Description of proof. Using an idea of Brauer's [3], we can show that $|C_G(x)| \leq 9a/2$ forces $r = 1$ in the group order formula, aside from cases involving small values of a , which can be eliminated by separate arguments. Then by considering

the distribution of conjugacy classes, we can see that every element of order dividing $\theta(1)$ has a centralizer of order $\theta(1)$, and that Theorem B applies to G . By considering elementary divisibility properties, we see that $|G| = a(2a + \varepsilon)(2a + 2\varepsilon)$, and the conclusion follows from Theorem 1 of [9].

BIBLIOGRAPHY

1. R. Baer, *Partitionen endlicher Gruppen*, Math. Z. **75** (1960/61), 333–372. MR **27** #1492.
2. ———, *Einfache Partitionen endlicher Gruppen mit nichttrivialer Fittingscher Untergruppe*, Arch. Math. **12** (1961), 81–89. MR **25** #115.
3. R. Brauer, *Some applications of the theory of blocks of characters of finite groups*. III, J. Algebra **3** (1966), 225–255. MR **34** #2716.
4. R. Brauer and K. Fowler, *On groups of even order*, Ann. of Math. (2) **62** (1955), 565–583. MR **17**, 580.
5. W. Feit and J. G. Thompson, *Finite groups which contain a self-centralizing subgroup of order 3*, Nagoya Math. J. **21** (1962), 185–197. MR **26** #192.
6. ———, *Solvability of groups of odd order*, Pacific J. Math. **13** (1963), 775–1029. MR **29** #3538.
7. D. Gorenstein, *Finite groups*, Harper & Row, New York, 1968. MR **38** #229.
8. P. Hall, and G. Higman, *On the p -length of p -soluble groups and reduction theorems for Burnside's problem*, Proc. London Math. Soc. (3) **6** (1956), 1–42. MR **17**, 344.
9. K. Harada, *A characterization of the groups LF (2, q)*, Illinois J. Math. **11** (1967), 647–659. MR **36** #1529.
10. B. Huppert, *Endliche Gruppen*. I, Die Grundlehren der math. Wissenschaften, Band 134, Springer-Verlag, Berlin and New York, 1967. MR **37** #302.
11. W. B. Stewart, *Strongly self centralizing 3-centralizers* (to appear).
12. M. Suzuki, *Applications of group characters*, Proc. Sympos. Pure Math., vol. 6, Amer. Math. Soc., Providence, R. I., 1962, pp. 101–105. MR **24** #A3196.
13. ———, *On a finite group with a partition*, Arch. Math. **12** (1961), 241–254. MR **25** #113.
14. J. G. Thompson, *Normal p -complements for finite groups*, J. Algebra **1** (1964), 43–46. MR **29** #4793.
15. H. Wielandt, *Finite permutation groups*, Lectures, University of Tübingen, 1954/55; English transl., Academic Press, New York, 1964. MR **32** #1252.

SOUTHERN ILLINOIS UNIVERSITY,
CARBONDALE, ILLINOIS 62901