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MAXIMA AND HIGH LEVEL EXCURSIONS OF STATIONARY GAUSSIAN PROCESSES(1)

BY
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Abstract. Let $X(t)$, $t \geqq 0$, be a stationary Gaussian process with mean 0, variance 1 and covariance function $r(t)$. The sample functions are assumed to be continuous on every interval. Let $r(t)$ be continuous and nonperiodic. Suppose that there exists α, $0<\alpha\leqq 2$, and a continuous, increasing function $g(t)$, $t\geqq 0$, satisfying

(0.1) $$\lim_{t\to 0}\frac{g(ct)}{g(t)} = 1, \quad \textit{for every } c>0,$$

such that

(0.2) $$1-r(t) \sim g(|t|)|t|^{\alpha}, \qquad t\to 0.$$

For $u>0$, let v be defined (in terms of u) as the unique solution of

(0.3) $$u^2 g(1/v)v^{-\alpha} = 1.$$

Let I_A be the indicator of the event A; then

$$\int_0^T I_{[X(s)>u]}\,ds$$

represents the time spent above u by $X(s)$, $0\leqq s\leqq T$. It is shown that the conditional distribution of

(0.4) $$v\int_0^T I_{[X(s)>u]}\,ds,$$

given that it is positive, converges for fixed T and $u\to\infty$ to a limiting distribution Ψ_α, which depends only on α but not on T or g.

Let $F(\lambda)$ be the spectral distribution function corresponding to $r(t)$. Let $F^{(p)}(\lambda)$ be the iterated p-fold convolution of $F(\lambda)$. If, in addition to (0.2), it is assumed that

(0.5) $$F^{(p)} \textit{ is absolutely continuous for some } p>0,$$

then $\max(X(s) : 0\leqq s\leqq t)$, properly normalized, has, for $t\to\infty$, the limiting extreme value distribution $\exp(-e^{-x})$.

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If, in addition to (0.2), it is assumed that

$$F(\lambda) \text{ is absolutely continuous with the derivative } f(\lambda), \tag{0.6}$$

and

$$\lim_{h\to 0} \log h \int_{-\infty}^{\infty} |f(\lambda+h)-f(\lambda)|\, d\lambda = 0, \tag{0.7}$$

then (0.4) has, for $u \to \infty$ and $T \to \infty$, a limiting distribution whose Laplace-Stieltjes transform is

$$\exp\left[\text{constant} \int_0^\infty (e^{-\lambda x}-1)\, d\Psi_\alpha(x)\right], \qquad \lambda > 0. \tag{0.8}$$

1. **Discussion of the results.** The conditional limiting distribution of (0.4) for fixed T was obtained for $\alpha=2$ and $g=$constant in [3], and for $\alpha=1$ and $g=$constant in [5]. We remark that when $\alpha=2$ then the only nontrivial case is $g\equiv$constant; indeed, $2t^{-2}(1-r(t))$ converges for $t\to 0$ to $\int_{-\infty}^{\infty} \lambda^2\, dF(\lambda)$ which is either 0, positive or infinite. If $g(0)=0$, then the first case arises and $X(t)\equiv X(0)$, so that $r(t)\equiv 1$; therefore, the only interesting case is $g(0)>0$. If $\alpha<2$, then $g(0)=0$ is possible; for example, if the spectral density $f(\lambda)$ exists and

$$f(\lambda) = |\lambda|^{-1-\alpha} g(|\lambda|^{-1}) \quad \textit{for large } |\lambda|,$$

then (0.2) is satisfied for some constant multiple of $g(|t|)$.

The extreme value limit distribution has been obtained for $\alpha=2$ in several works under increasingly general conditions (Volkonskiĭ and Rozanov [14], Cramér and Leadbetter [6], Beljaev [1], Qualls [11], and Berman [4]). For $\alpha=1$ and g constant, it was studied by Pickands [10]. A sufficient condition for (0.5) is

$$\int_{-\infty}^{\infty} |r(s)|^p\, ds < \infty \quad \textit{for some } p > 0; \tag{1.1}$$

indeed, if p is an integer, $r^p(s)$ is the Fourier-Stieltjes transform of $F^{(p)}$. (1.1) is implied by several of the conditions used in earlier works, namely, $r(t)=O(t^{-\varepsilon})$, $t\to\infty$ for some $\varepsilon>0$ (see [6, p. 257]) and the integrability of r^2 (see [4] and [10]).

The only previous investigation of the limiting distribution of (0.4) for $u\to\infty$ and $T\to\infty$ is that of Volkonskiĭ and Rozanov [14] in the case $\alpha=2$. In addition to (0.2) they also assumed that $1-r(t)-g(0)t^2 \sim \lambda_4 t^4/4!$; and, in place of the mild conditions (0.6) and (0.7), assumed that the process satisfies the "strong mixing condition." Their proof is based on the fact that the "horizontal-window" conditional limiting distribution of the excursion above a high level is Ψ_2, the Rayleigh distribution; and that the upcrossings tend to a limiting Poisson process.

Our condition (0.6) implies that $r(t)\to 0$ for $t\to\infty$ (Riemann-Lebesgue Theorem). It also implies that $\int_{-\infty}^{\infty} |f(\lambda+h)-f(\lambda)|\, d\lambda$ tends to 0 with h. The condition (0.7) prescribes a *rate* for such convergence. It is well known that this rate is related to the rate of convergence of $r(t)$ to 0 for $t\to\infty$ (see [13]).

2. **Limiting conditional distribution of a high level excursion.** Let $U(t)$, $t \geqq 0$, be a Gaussian process with continuous sample functions, and with the specific moment functions

$$EU(t) \equiv 0, \quad EU^2(0) = 0, \quad E|U(t)-U(s)|^2 = 2|t-s|^\alpha.$$

Put $\phi(u) = (2\pi)^{-1/2}e^{-x^2/2}$. Let $X(t)$ be a general process satisfying the conditions leading to and including (0.2).

THEOREM 2.1. *Under condition* (0.2), *for every* $T>0$ *and* $m \geqq 1$,

$$\lim_{u\to\infty} \frac{E[v \int_0^T I_{[X(s)>u]}\,ds]^m}{Tv\phi(u)/u} = m\int_0^\infty E\left\{\int_0^\infty I_{[U(s)>s^\alpha - z]}\,ds\right\}^{m-1} e^{-z}\,dz. \tag{2.1}$$

Proof. The expectation in the numerator of (2.1) is equal to

$$E\left\{\int_0^{Tv} I_{[X(s/v)>u]}\,ds\right\}^m = m\int_0^{Tv}\int_0^{s_m}\cdots\int_0^{s_m} P\left\{X\left(\frac{s_i}{v}\right) > u,\, i = 1,\ldots,m\right\} ds_1\cdots ds_m,$$

which, by stationarity, is equal to

$$m\int_0^{Tv}\int_0^{s_m}\cdots\int_0^{s_m} P\left\{X\left(\frac{s_m - s_i}{v}\right) > u,\, i = 1,\ldots,m-1,\, X(0) > u\right\} ds_1\cdots ds_m.$$

This is equal to

$$m\int_0^{Tv}\int_0^{s_m}\cdots\int_0^{s_m} P\left\{X\left(\frac{s_i}{v}\right) > u,\, i = 1,\ldots,m-1,\, X(0) > u\right\} ds_1\cdots ds_m \tag{2.2}$$

because, for any function f,

$$\int_0^t f(t-s)\,ds = \int_0^t f(s)\,ds.$$

By the total probability formula, the integrand in (2.2) is equal to

$$\int_u^\infty P\left\{X\left(\frac{s_i}{v}\right) > u,\, i = 1,\ldots,m-1 \,\middle|\, X(0) = y\right\}\phi(y)\,dy,$$

which, by the substitution $y = u + z/u$ and the identity

$$\phi(u+z/u) = \phi(u)e^{-z}\exp(-z^2/2u^2),$$

is equal to

$$\frac{\phi(u)}{u}\int_0^\infty P\left\{X\left(\frac{s_i}{v}\right) > u,\, i = 1,\ldots,m-1 \,\middle|\, X(0) = u + \frac{z}{u}\right\} e^{-z}\exp(-z^2/2u^2)\,dz. \tag{2.3}$$

We evaluate the conditional probability in (2.3). The process $u[X(s/v)-u]$ is conditionally Gaussian, given $X(0) = u + z/u$, with conditional mean

$$u^2[r(s/v)-1] + zr(s/v) \tag{2.4}$$

and conditional covariance (for $0 < s < t$)

$$u^2[r((t-s)/v) - r(s/v)r(t/v)]. \tag{2.5}$$

By (0.1), (0.2) and (0.3), the conditional mean converges to $-s^\alpha+z$ and the conditional covariance to $s^\alpha+t^\alpha-(t-s)^\alpha$. These are the mean and covariance, respectively, of the process $U(s)-s^\alpha+z$; thus, the process $u[X(s/v)-u]$ converges in conditional distribution to the process $U(s)-s^\alpha+z$. It follows that the integrand in (2.3) converges everywhere to

$$P\{U(s_i) > s_i^\alpha-z,\ i = 1,\dots,m-1\}e^{-z}. \tag{2.6}$$

We substitute (2.3) for the integrand in (2.2), divide by $Tv\phi(u)/u$, and formally pass to the limit under the sign of integration to get

$$m\int_0^\infty\cdots\int_0^\infty P\{U(s_i) > s_i^\alpha-z,\ i = 1,\dots,m-1\}e^{-z}\,dz\,ds_1\cdots ds_{m-1},$$

which is equal to the right-hand side of (2.1).

The inversion of the limit operation has to be justified. The integrand in (2.2) does not depend on the variable s_m; therefore, the integral is equal to

$$m\int_0^{Tv}(Tv-s_{m-1})\int_0^{s_{m-1}}\cdots\int_0^{s_{m-1}}P\left\{X\left(\frac{s_i}{v}\right) > u,\ i = 1,\dots,m-1,\ X(0) > u\right\}ds_1\cdots ds_{m-1},$$

which, when divided by $Tv\phi(u)/u$ and transformed as was (2.3), is

$$m\int_0^{Tv}\left(1-\frac{s_{m-1}}{Tv}\right)\int_0^{s_{m-1}}\cdots\int_0^{s_{m-1}}\int_0^\infty P\left\{X\left(\frac{s_i}{v}\right)>u, i=1,\dots,m-1\ \middle|\ X(0)=u+\frac{z}{u}\right\}\cdot e^{-z}\exp(-z^2/2u^2)\,dz\,ds_1\cdots ds_{m-1}. \tag{2.7}$$

If Y has a Gaussian distribution with mean μ and variance σ^2, then

$$P\{Y > u\} \leqq e^{-cu}Ee^{cY} = \exp(-cu+c\mu+\tfrac{1}{2}c^2\sigma^2), \qquad c > 0;$$

furthermore, if $X(s_i/v)>u$ for $i=1,\dots,m-1$, then

$$\frac{u}{m-1}\sum_{i=1}^{m-1}[X(s_i/v)-u] > 0;$$

thus, by (2.4) and (2.5),

$$\begin{aligned} &P\{X(s_i/v) > u,\ i=1,\dots,m-1 \mid X(0) = u+z/u\}\\ &\leqq P\left\{\frac{u}{m-1}\sum_{i=1}^{m-1}\left[X\left(\frac{s_i}{v}\right)-u\right] > 0\ \middle|\ X(0) = u+\frac{z}{u}\right\}\\ &\leqq \exp\left\{cu^2\left[\frac{1}{m-1}\sum_{i=1}^{m-1}\left(r\left(\frac{s_i}{v}\right)-1\right)+\frac{cz}{m-1}\sum_{i=1}^{m-1}r\left(\frac{s_i}{v}\right)\right]\right\}\\ &\quad\cdot\exp\left\{\frac{c^2}{2}\cdot\mathrm{Var}\left[\frac{u}{m-1}\sum_{i=1}^{m-1}\left(X\left(\frac{s_i}{v}\right)-u\right)\ \middle|\ X(0)\right]\right\}. \end{aligned} \tag{2.8}$$

Since the variance is unchanged by an additive constant, and since the conditional variance never exceeds the unconditional, we have

$$\operatorname{Var}\left\{\frac{u}{m-1}\sum_{i=1}^{m-1}\left(X\left(\frac{s_i}{v}\right)-u\right)\,\middle|\, X(0)\right\}$$

$$= \operatorname{Var}\left\{\frac{u}{m-1}\sum_{i=1}^{m-1}\left(X\left(\frac{s_i}{v}\right)-X(0)\right)\,\middle|\, X(0)\right\}$$

$$\leqq E\left\{\frac{u}{m-1}\sum_{i=1}^{m-1}\left(X\left(\frac{s_i}{v}\right)-X(0)\right)\right\}^2$$

$$\leqq E\left\{\frac{u^2}{m-1}\sum_{i=1}^{m-1}\left(X\left(\frac{s_i}{v}\right)-X(0)\right)^2\right\} = \frac{2u^2}{m-1}\sum_{i=1}^{m-1}\left[1-r\left(\frac{s_i}{v}\right)\right].$$

It follows that the last member of (2.8) is at most equal to

$$\exp\left[cz-c(1-c)\frac{u^2}{m-1}\sum_{i=1}^{m-1}\left[1-r\left(\frac{s_i}{v}\right)\right]\right] \tag{2.9}$$

for $0<c<1$. Under (0.1) and (0.2) there exists, for $T>0$, a constant $B>0$ such that

$$1-r(t) \geqq Bg(t)|t|^{\alpha}, \qquad 0 \leqq t \leqq T; \tag{2.10}$$

therefore, by (0.3), (2.9) is at most

$$\exp\left[cz-c(1-c)\frac{B}{m-1}\sum_{i=1}^{m-1}\frac{g(s_i/v)s_i^{\alpha}}{g(1/v)}\right].$$

Since g is monotonic, this is dominated by

$$\exp\left[cz-\frac{c(1-c)B}{m-1}\sum_{i=1}^{m-1}A(s_i)\right], \tag{2.11}$$

where $A(s)=s^{\alpha}$ or 0 accordingly as $s>1$ or $s\leqq 1$. When multiplied by e^{-z}, the function (2.11) is integrable over $z, s_1, \ldots, s_{m-1}$ and dominates the integrand in (2.7). This justifies the passage to the limit under the integral sign. The proof is complete.

The following is a preliminary result which will later be improved.

LEMMA 2.1. *Under the condition* (0.2), *for every* $T>0$,

$$\limsup_{u\to\infty}\frac{P\{\max(X(s): 0 \leqq s \leqq T) > u\}}{Tv\phi(u)/u} < \infty.$$

Proof. The interval $[0, T]$ is divisible into approximately Tv intervals, of each length $1/v$; thus, by Boole's inequality and stationarity, it suffices to show that

$$\limsup_{u\to\infty}\frac{P\{\max(X(s): 0 \leqq s \leqq 1/v) > u\}}{\phi(u)/u} < \infty.$$

Since

$$P\{\max X(s) > u\} \leqq P\{X(0) > u\} + P\{\max X(s) > u,\ X(0) \leqq u\},$$

it suffices, by the well-known estimate

$$\phi(u)\left(\frac{1}{u}-\frac{1}{u^3}\right) \leqq \int_u^\infty \phi(x)\,dy \leqq \frac{\phi(u)}{u}, \tag{2.12}$$

to show that the lim sup of

$$P\{\max\,(X(s/v) : 0 \leqq s \leqq 1) > u,\ X(0) \leqq u\}/(\phi(u)/u)$$

is finite. By the total probability formula this is equal to

$$\frac{u}{\phi(u)}\int_{-\infty}^{u} P\left\{\max\left(X\left(\frac{s}{v}\right) : 0 \leqq s \leqq 1\right) > u \,\middle|\, X(0) = y\right\}\phi(y)\,dy,$$

which, as in the argument leading to (2.3), is equal to

$$\int_0^\infty P\left\{\max_{0\leqq s\leqq 1} u\left[X\left(\frac{s}{v}\right)-u\right] > 0 \,\middle|\, X(0) = u-\frac{z}{u}\right\}\exp\,[z-z^2/2u^2]\,dz. \tag{2.13}$$

By formula (2.4), with $-z$ in place of z, the conditional mean of $u[X(s/v)-u]$ is

$$u^2[r(s/v)-1]-r(s/v)z. \tag{2.14}$$

Put

$$X_u(s) = u^2[X(s/v)-u]-u^2[r(s/v)-1]+r(s/v)z.$$

Now $r(s/v) \to 1$ as $u \to \infty$ uniformly for $0 \leqq s \leqq 1$; furthermore, we are concerned only with large values of u in proving the lemma; thus we may suppose that the conditional mean (2.14) satisfies

$$u^2[r(s/v)-1]-r(s/v)z < -\tfrac{1}{2}z, \quad \text{for } 0 \leqq s \leqq 1, \text{ and } z \geqq 0.$$

It follows that the conditional probability in (2.13) is at most

$$P\left\{\max_{0\leqq s\leqq 1} X_u(s) > \tfrac{1}{2}z \,\middle|\, X(0) = u-\frac{z}{u}\right\}. \tag{2.15}$$

By (2.14) and the definition of X_u,

$$E[X_u(s) \mid X(0) = u-z/u] \equiv 0;$$

$$\begin{aligned} E\{|X_u(t)-X_u(s)|^2 \mid X(0) = u-z/u\} &= \operatorname{Var}\{[X_u(t)-X_u(s)] \mid X(0) = u-z/u\} \\ &\leqq \operatorname{Var}(X_u(t)-X_u(s)) \\ &= u^2E|X(t/v)-X(s/v)|^2 \\ &= 2u^2[1-r((t-s)/v)], \qquad 0 < s < t. \end{aligned}$$

By (0.1) and (0.2) there exists a constant $K>0$ such that

$$2u^2[1-r((t-s)/v)] \leqq K|t-s|^\alpha, \qquad 0 < s < t \leqq 1,$$

and where K does not depend on u. We now apply Fernique's inequality [7] (for the proof, see [9]) to the conditioned process X_u: we find that (2.15) is at most

$$\text{constant} \times \int_{z/\text{constant}}^{\infty} \phi(y)\, dy.$$

Put this bound in (2.13), and use the estimate (2.12); then (2.13) is bounded.

LEMMA 2.2. *Let $U(t)$, $t \geqq 0$, be the process defined before Theorem* 2.1. *For every $z>0$ and every τ,*

$$E\left\{\exp\left(\tau \int_0^\infty I_{[U(t)>t^\alpha-z]}\, dt\right)\right\} \leqq \int_0^\infty \exp\,[\tau(\sqrt{2}y+\sqrt{z})^{\alpha/2}]\phi(y)\, dy. \tag{2.16}$$

The latter integral is finite.

Proof. Let Y be a random variable with a standard normal distribution. The stochastic process $\sqrt{2}Yt^{\alpha/2}$, $t \geqq 0$, is Gaussian with mean 0 and covariance function $2(st)^{\alpha/2}$. We have

$$\operatorname{Var}(\sqrt{2}Yt^{\alpha/2}) = \operatorname{Var} U(t) = 2t^\alpha$$

and

$$\begin{aligned} EU(s)U(t) &= s^\alpha + t^\alpha - (t-s)^\alpha \\ &= (t^{\alpha/2}-s^{\alpha/2})^2 + 2(st)^{\alpha/2} - (t-s)^\alpha \\ &\leqq 2(st)^{\alpha/2}, \qquad 0 < s < t \end{aligned}$$

(because $t^{\alpha/2} \leqq s^{\alpha/2}+(t-s)^{\alpha/2}$ for $0<s<t$); thus,

$$EU(s)U(t) \leqq E[(\sqrt{2}Ys^{\alpha/2})(\sqrt{2}Yt^{\alpha/2})], \qquad s \neq t.$$

We apply the well-known inequality of Slepian [12]: If $t_1, \ldots, t_m$ and z are arbitrary real numbers, then

$$P\{U(t_i) > t_i^\alpha - z, i = 1, \ldots, m\} \leqq P\{\sqrt{2}Yt_i^{\alpha/2} > t_i^\alpha - z, i = 1, \ldots, m\}.$$

Integrate over $t_i \geqq 0$, $i=1, \ldots, m$; then

$$E\left\{\int_0^\infty I_{[U(t)>t^\alpha-z]}\, dt\right\}^m \leqq E\left\{\int_0^\infty I_{[\sqrt{2}Yt^{\alpha/2}>t^\alpha-z]}\, dt\right\}^m.$$

The latter may be written as

$$E\left\{\int_0^\infty I_{[\sqrt{2}Yt>t^2-z]}\, d(t^{\alpha/2})\right\}^m. \tag{2.17}$$

If $t > \sqrt{2}Y^{+} + \sqrt{z}$, then $\sqrt{2}Yt < t^2 - z$ because $\sqrt{2}Y^{+} + \sqrt{z}$ is larger than the roots of the equation $t^2 - \sqrt{2}Yt - z = 0$; therefore,

$$\int_0^\infty I_{[\sqrt{2}Yt > t^2 - z]}\, d(t^{\alpha/2}) \leqq (\sqrt{2}Y^{+} + \sqrt{z})^{\alpha/2};$$

consequently, the expectation (2.17) is not more than $E(\sqrt{2}Y^{+} + \sqrt{z})^{m\alpha/2}$. Now multiply by $\tau^m/m!$, and sum over $m \geqq 0$; (2.16) follows.

THEOREM 2.2. *There exists a unique (up to an additive constant) bounded non-decreasing function $H_\alpha(x)$ on $(0, \infty)$ such that*

$$\int_0^\infty x^m\, dH_\alpha(x) = m \int_0^\infty E\left[\int_0^\infty I_{[U(t) > t^\alpha - z]}\, dt\right]^{m-1} e^{-z}\, dz, \tag{2.18}$$

for $m \geqq 1$, and

$$\int_0^\infty e^{tx}\, dH_\alpha(x) < \infty, \qquad t > 0, \tag{2.19}$$

and

$$H_\alpha(\infty) - H_\alpha(0+) > 0. \tag{2.20}$$

Under (0.2) *there also exists a positive constant V_α such that*

$$V_\alpha = \lim_{u \to \infty} \frac{P\{\max (X(s) : 0 \leqq s \leqq T) > u\}}{Tv\phi(u)/u}. \tag{2.21}$$

Proof. Let $F_u(x)$, $x \geqq 0$, be the distribution function of $v \int_0^T I_{[X(s) > u]}\, ds$. This random variable assumes a positive value if and only if $\max (X(s) : 0 \leqq s \leqq T) > u$; therefore,

$$F_u(\infty) - F_u(0+) = P\left\{\max_{0 \leqq s \leqq T} X(s) > u\right\}. \tag{2.22}$$

By Lemma 2.1, $(F_u(\infty) - F_u(0+))/(Tv\phi(u)/u)$ is bounded for $u \geqq 1$. By Theorem 2.1 the mth moment of the function $(F_u(x) - F_u(0+))/(Tv\phi(u)/u)$ converges to the right-hand side of (2.1). The moment generating function of the limiting moment sequence is finite everywhere: when multiplied by $t^m/m!$ and summed over $m \geqq 1$, the right side of (2.1) becomes

$$t \int_0^\infty E\left\{\exp\left(t \int_0^\infty I_{[U(s) > s^\alpha - z]}\, ds\right)\right\} e^{-z}\, dz, \tag{2.23}$$

which, by Lemma 2.2, is finite. The moment convergence theorem now implies that there exists a unique H_α such that

$$\frac{\int_0^\infty x^m\, dF_u(x)}{Tv\phi(u)/u} \to \int_0^\infty x^m\, dH_\alpha(x). \tag{2.24}$$

In the particular case $m = 1$, we have $\int_0^\infty x\, dH_\alpha(x) = 1$. This affirms (2.18), (2.19), and (2.20).

To prove (2.21) we show that

$$\lim_{u\to\infty} \frac{F_u(\infty)-F_u(0+)}{Tv\phi(u)/u} = H_\alpha(\infty)-H_\alpha(0+). \tag{2.25}$$

The relation (2.24) implies the weak convergence of $dF_u/Tv\phi(u)/u$ to dH_α on the open interval $(0, \infty)$, and also complete convergence on $(0, \infty]$:

$$\lim_{u\to\infty} \frac{F_u(\infty)-F_u(a)}{Tv\phi(u)/u} = H_\alpha(\infty)-H_\alpha(a) \quad \text{for } a > 0. \tag{2.26}$$

This implies "half" of (2.25):

$$\liminf_{u\to\infty} \frac{F_u(\infty)-F_u(0+)}{Tv\phi(u)/u} \geqq H_\alpha(\infty)-H_\alpha(0+). \tag{2.27}$$

For arbitrary $x>0$ replace u by $u+x/u$; then v is still the "right" function of u for large u (cf. opening paragraph of §3 below), and so

$$\limsup_{u\to\infty} \frac{F_{u+x/u}(\infty)-F_{u+x/u}(0+)}{Tv\phi(u+x/u)/(u+x/u)} = \limsup_{u\to\infty} \frac{F_u(\infty)-F_u(0+)}{Tv\phi(u)/u}.$$

From this and the relation $\phi(u+x/u)\sim\phi(u)e^{-x}$, we get

$$\limsup_{u\to\infty} \frac{F_{u+x/u}(\infty)-F_{u+x/u}(0+)}{Tv\phi(u)/u} = e^{-x} \limsup_{u\to\infty} \frac{F_u(\infty)-F_u(0+)}{Tv\phi(u)/u}. \tag{2.28}$$

Note the identity

$$F_{u+x/u}(\infty)-F_{u+x/u}(0+)$$
$$= P\left\{\max_{[0,T]} X > u+x/u,\ vL(u) \leqq \varepsilon\right\}+P\left\{\max_{[0,T]} X > u+x/u,\ vL(u) > \varepsilon\right\},$$

and (2.26) and (2.28); then

$$\begin{aligned} &\limsup_{u\to\infty} e^{-x} \frac{F_u(\infty)-F_u(0+)}{Tv\phi(u)/u} \\ &\quad = \limsup_{u\to\infty} \frac{P\{\max_{[0,T]} X > u+x/u,\ vL(u) \leqq \varepsilon\}}{Tv\phi(u)/u} + H_\alpha(\infty)-H_\alpha(\varepsilon). \end{aligned} \tag{2.29}$$

We shall show that the first term on the right-hand side tends to 0 with ε for every $x>0$; hence, since x is arbitrary, it will follow that

$$\limsup_{u\to\infty} \frac{F_u(\infty)-F_u(0+)}{Tv\phi(u)/u} \leqq H_\alpha(\infty)-H_\alpha(0+).$$

Combining this with (2.27) we get (2.25).

To complete the proof we estimate the first term on the right-hand side of (2.29). For simplicity put $T=1$. The unit interval is divisible into approximately v subintervals of length $1/v$; furthermore, it is evident that

$$v\int_A I_{[X(s)>u]}\,ds \leqq vL(u)$$

for every subinterval A of [0, 1]; thus, by the argument in the proof of Lemma 2.1 (Boole's inequality and stationarity) we find that

$$P\left\{\max_{[0,1]} X > u+x/u,\ vL(u) \leqq \varepsilon\right\}$$

is at most equal to

$$vP\left\{\max_{0\leqq s\leqq 1} u(X(s/v)-u) > x \int_0^1 I_{[u(X(s/v)-u)>0]}\,ds \leqq \varepsilon\right\}. \tag{2.30}$$

Now write the probability as the integral of the conditional probability given $X(0)=u-z/u$. By the standard weak convergence methods ([15], those based on the estimates of the conditional moments of the increments of the process, as in the proof of Lemma 2.1) it can be shown that the conditioned process $u(X(s/v)-u)$, $0\leqq s\leqq 1$, converges not only in distribution to the process $U(s)-s^{\alpha}-z$, as in Theorem 2.1, but also weakly over the function space $C[0, 1]$; therefore, the joint distribution of the functionals in (2.30) (the maximum and the occupation time) converges to the joint distribution under the limiting process. Divide (2.30) by $\phi(u)/u$ and pass to the limit under the conditional expectation integral, as in Lemma 2.1; the ratio converges to

$$\int_0^{\infty} P\left\{\max_{[0,1]} U(s)-s^{\alpha} > x+z,\ \int_0^1 I_{[U(s)-s^{\alpha}>z]}\,ds \leqq \varepsilon\right\} e^{z}\,dz.$$

The integrand converges pointwise to 0 as $\varepsilon\to 0$; indeed if the process $U(s)-s^{\alpha}$ spends little time above z, then it is very unlikely to exceed $z+x$. Passage to the limit under the sign of integration is again permitted by Fernique's inequality. The proof is complete.

An immediate consequence of Theorems 2.1 and 2.2 is:

THEOREM 2.3. *Let Ψ_{α} be the distribution function with the moment sequence*

$$V_{\alpha}^{-1}\int_0^{\infty} x^m\,dH_{\alpha}(x), \qquad m = 1,$$

that is, the distribution defined as

$$\begin{array}{rl} 0 & \text{for } x \leqq 0, \\ (H_{\alpha}(x)-H_{\alpha}(0+))/V_{\alpha} & \text{for } x > 0. \end{array}$$

Under (0.2), *the conditional distribution of $v\int_0^T I_{[X(s)>u]}\,ds$, given that it is positive, converges to Ψ_{α}.*

Proof. For any nonnegative random variable Y, $E(Y^m \mid Y>0)=EY^m/P\{Y>0\}$; thus, by (2.22),

$$E\left\{\left[v\int_0^T I_{[X(s)>u]}\,ds\right]^m \,\middle|\, v\int_0^T I_{[X(s)>u]}\,ds > 0\right\} = \frac{E\{v\int_0^T I_{[X(s)>u]}\,ds\}^m}{P\{\max(X(s) : 0 \leqq s \leqq T) > u\}}.$$

By Theorems 2.1 and 2.2 this converges, for $m\geqq 1$, to the right-hand side of (2.1), divided by V_{α}. This is the mth moment of the distribution function Ψ_{α}. Since H_{α}

has a moment generating function which is everywhere finite, so does Ψ_α; thus, Ψ_α is uniquely determined by its moments. The assertion of the theorem now follows by application of the moment convergence theorem.

3. **Limiting distribution of the maximum.** We begin with some remarks about the construction of the function $v=v(u)$ in (0.3). This condition was used in the proofs in §2 *only* to show that $u^2[1-r(s/v)]$ converges to s^α for $u\to\infty$. If $w(u)$ is an increasing function of u such that $w(u)/u\to 1$ for $u\to\infty$, and if $v'=v'(w)$ is defined as

$$w^2 g(1/v')(v')^{-\alpha} = 1,$$

then $u^2[1-r(s/v')]\sim w^2[1-r(s/v')]\to s^\alpha$. So v' may be interchanged with v; in other words, v may be defined by (0.3) not only in terms of u but also in terms of any function asymptotic to u.

In this section we consider the limiting distribution of $\max(X(s):0\leqq s\leqq t)$ for $t\to\infty$. For $t>0$, let v be defined in terms of t as the unique solution of

$$2\log t\, g(1/v)v^{-\alpha} = 1. \tag{3.1}$$

Our result is

THEOREM 3.1. *If* (0.2) *and* (0.5) *hold, then the random variable*

$$\sqrt{(2\log t)}\left[\max_{0\leqq s\leqq t} X(s)-\sqrt{(2\log t)}-\frac{\log(vV_\alpha/2\sqrt{(\pi\log t)})}{\sqrt{(2\log t)}}\right]$$

has, for $t\to\infty$, the limiting distribution $\exp(-e^{-x})$.

(The constant V_α is defined in Theorem 2.2.)

Proof. The main idea of the proof is, as in previous studies, that the distribution of the maximum of the process over a large time interval is close to the distribution of the maximum of certain independent random variables. Suppose t is a positive integer; then

$$\max_{0\leqq s\leqq t} X(s) = \max_{1\leqq j\leqq t}\left(\max_{j-1\leqq s\leqq j} X(s)\right).$$

If $\max(X(s):j-1\leqq s\leqq j)$, $j=1,\ldots,t$, were independent, then $\max(X(s):0\leqq s\leqq t)$ would be the maximum of t independent random variables with the common distribution function

$$G(x) = P\left\{\max_{0\leqq s\leqq 1} X(s)\leqq x\right\};$$

thus, $\max(X(s):0\leqq s\leqq t)$ would have the distribution function $G^t(x)$.

For fixed x, put

$$u = \sqrt{(2\log t)}+\frac{(\log(vV_\alpha/2\sqrt{(\pi\log t)})+x)}{\sqrt{(2\log t)}}. \tag{3.2}$$

By the definition (3.1) of v, and the increasing character of g, we have $v^\alpha/2 \log t \to g(0)$ (*positive or zero*); therefore,

$$(\log (vV_\alpha/2\sqrt{(\pi \log t)}))/\sqrt{(2 \log t)} \to 0,$$

and so, by (3.2),

$$u \sim \sqrt{(2 \log t)}, \qquad t \to \infty. \tag{3.3}$$

The opening remarks of this section imply: If v is defined by (3.1) and u by (3.2), then the relations between u and v employed in §2 may also be used under these new definitions.

It follows from (2.21) (with $T=1$), (3.2) and (3.3) that $1-G(u)\sim e^{-x}t^{-1}$, $t \to \infty$; therefore,

$$G^t(u) \to \exp(-e^{-x}); \tag{3.4}$$

thus, the limiting distribution of the maximum of t independent random variables with the common distribution function G is $\exp(-e^{-x})$.

The rest of the proof consists of showing that the submaxima over the various intervals may actually be assumed to be asymptotically independent. The proof follows a familiar pattern; we shall, wherever possible, refer to previous work for details.

We break the interval $[0, t]$ into approximately $[t]$ intervals of unit length. (There is a small piece left over when t is not an integer; however, the proof for such t is reducible to that for integral t.) For arbitrary ε, $0<\varepsilon<1$, we clip an open segment of length ε from the right endpoint of each interval. The remaining intervals are $I_j=[j-1, j-\varepsilon]$, $j=1, \ldots, [t]$. We have

$$\limsup_{t\to\infty} \left| P\left\{\max_{0\leqq s\leqq t} X(s) > u\right\} - P\left\{\max_{I_1\cup\cdots\cup I_{[t]}} X(s) > u\right\}\right| \leqq \varepsilon e^{-x}; \tag{3.5}$$

in fact, by Boole's inequality and stationarity, the difference of probabilities in (3.5) is not more than $tP\{\max(X(s) : 0\leqq s\leqq \varepsilon)>u\}$, which, by (2.21) and (3.2), converges to εe^{-x}.

Let M_t be the set of integer multiples of $(\log\ t)^{-3/\alpha}$; then, by the argument in [10],

$$\lim_{t\to 0} \sqrt{(2 \log t)} \left| \max_{I_1\cup\cdots\cup I_{[t]}} X(s) - \max_{M_t\cap(I_1\cup\cdots\cup I_{[t]})} X(s) \right| = 0 \tag{3.6}$$

in probability; thus, the limiting distributions of the two maxima are the same.

Put

$$n = \textit{integral part of } t(\log t)^{3/\alpha}, \tag{3.7}$$

and let $\phi(u_1, u_2; \rho)$ be the standard bivariate normal density with correlation coefficient ρ.

We shall show that in calculating the limiting distribution of the maximum over $(I_1 \cup \cdots \cup I_{[t]}) \cap M_t$, we may assume that the pieces of the process corresponding to different intervals I_j are mutually independent. As in [4], it suffices to show that

$$n \sum_{n\varepsilon/t \leq j \leq n} \int_0^{r(jt/n)} \phi(u, u; y)\, dy \tag{3.8}$$

converges to 0 as $t \to \infty$.

By (0.5), the remark following (1.1), and the Riemann-Lebesgue lemma, it follows that $r^p(t) \to 0$ for $t \to \infty$; therefore, $r(t) \to 0$. By the same argument as in [4], it suffices to show: there exists q, $0 < q < 1$, such that

$$nt^{q-2} \sum_{j=1}^{n} |r(jt/n)| \to 0 \quad \textit{for } t \to \infty. \tag{3.9}$$

By the Hölder inequality,

$$\sum_{j=1}^{n} |r(jt/n)| \leq n^{1-(1/2p)} \left(\sum_{j=1}^{n} r^{2p}(jt/n) \right)^{1/2p}. \tag{3.10}$$

Let $f^{(p)}(\lambda)$ be the Radon-Nikodým derivative of $F^{(p)}(\lambda)$; then, as is well known, the convolution $f^{(2p)}(\lambda)$ is a *continuous* function, and

$$r^{2p}(jt/n) = 2 \int_0^\infty \cos(\lambda jt/n) f^{(2p)}(\lambda)\, d\lambda.$$

Substitute this in (3.10); then, by the well-known cosine summation formula, the right-hand side of (3.10) is at most

$$n^{1-(1/2p)} \left(2 \int_0^\infty \frac{\sin \lambda t(1+1/2n)}{\sin(\lambda t/2n)} f^{(2p)}(\lambda)\, d\lambda \right)^{1/2p}.$$

By a standard argument and the relation $t/n \to 0$, the last expression is asymptotic to

$$n \left(4 \int_0^\infty \frac{\sin \lambda t}{\lambda t} f^{(2p)}(\lambda)\, d\lambda \right)^{1/2p}.$$

For arbitrary β, $0 < \beta < 1$, this is at most (by $|\sin \lambda t| \leq |\lambda t|^\beta$)

$$n \left(4 t^{\beta-1} \int_0^\infty \lambda^{\beta-1} f^{(2p)}(\lambda)\, d\lambda \right)^{1/2p}.$$

The integral is finite because $f^{(2p)}$ is continuous; thus, the expression above is of the order $nt^{(\beta-1)/p}$; therefore, the quantity on the left side of (3.9) is of the order $n^2 t^{q-2+(\beta-1)/p}$. By the definition (3.7) of n, this converges to 0 if q is chosen so that $0 < q < (1-\beta)/p$; therefore, (3.9) holds.

By the remarks preceding (3.8) and (3.9), the distribution of the maximum on $M_t \cap (I_1 \cup \cdots \cup I_{[t]})$ is asymptotically the same as that of $[t]$ independent random variables with the common distribution function

$$\bar{G}(x) = P\left\{ \max_{M_t \cap I_1} X(s) \leq x \right\}.$$

By the same argument as for (3.6), we may replace $M_t \cap I_j$ by the full set I_j, $j=1,\ldots,[t]$; thus, the maximum is asymptotically the same as the maximum of $[t]$ independent random variables with the common distribution function

$$G_\varepsilon(x) = P\left\{\max_{0\leqq s\leqq 1-\varepsilon} X(s) \leqq x\right\}.$$

By the reasoning leading to (3.4), with $T=1-\varepsilon$, we find that the limiting distribution of max $X(s)$ over $I_1 \cup \cdots \cup I_{[t]}$ is $\exp(-(1-\varepsilon)e^{-x})$. Since $\varepsilon>0$ is arbitrary, we put $\varepsilon=0$ here and in (3.5). This completes the proof of the theorem.

As in [4], we have

COROLLARY 3.1. *For arbitrary $b>0$, and with u given by* (3.2),

$$\lim_{t\to\infty} P\left\{\max_{0\leqq s\leqq tb} X(s) \leqq u\right\} = \exp(-be^{-x}).$$

Another result is

COROLLARY 3.2. *If $I_1, I_2, \ldots$ are the intervals defined above, then*

$$\sum_{i,j=1, i\neq j}^{[t]} \left| P\left\{\max_{I_i} X(s) > u, \max_{I_j} X(s) > u\right\} - P\left\{\max_{I_i} X(s) > u\right\}\cdot P\left\{\max_{I_j} X(s) > u\right\}\right|$$

converges to 0 *as* $t\to\infty$.

Proof. By stationarity, the inequalities on u may be reversed, and the summation simplified:

$$\sum_{j=2}^{[t]} ([t]-j)\left| P\left\{\max_{I_1\cup I_j} X(s) \leqq u\right\} - P^2\left\{\max_{I_1} X(s) \leqq u\right\}\right|. \tag{3.11}$$

Let M_t^* be the set of integral multiples of $(\log t)^{-6/\alpha}$. By the same reasoning as for (3.6), the sets I_j may be replaced by $I_j \cap M_t^*$. Put

$$n = \textit{integral part of } t(\log t)^{6/\alpha},$$

then, as in [4], the sum (3.11), with $I_j \cap M_t^*$ in place of I_j, $j=1,\ldots,[t]$, is at most

$$\sum_{j=2}^{[t]} ([t]-j)\cdot\frac{2n}{t} \sum_{n\varepsilon/t\leqq k\leqq 2n/t} \int_0^{r(kt/n+j-1)} \phi(u,u;y)\,dy,$$

which is at most

$$2n \sum_{n\varepsilon/t\leqq j\leqq 2n} \int_0^{r(jt/n)} \phi(u,u;y)\,dy.$$

Like (3.8), it converges to 0.

4. **Preliminary estimates of the limiting distribution of the high level excursion.** Throughout this section we assume that conditions (0.2) and (0.5) are satisfied.

v and u are defined by (3.1) and (3.2), respectively. For fixed ε, $0<\varepsilon<1$, I_j is the interval $[j-1, j-\varepsilon]$, as defined in the previous section.

For fixed $b>0$, let N be the number of intervals I_j for which $\max(X(s) : s \in I_j) > u$, $j=1,\ldots,[tb]$. As the sum of $[tb]$ indicator random variables, N has the expected value

$$[tb]P\{\max(X(s) : 0 \leqq s \leqq 1-\varepsilon) > u\},$$

which, by Theorem 2.2, is asymptotic to $V_\alpha tb(1-\varepsilon)v\phi(u)/u$, which by (3.1) and (3.2), converges to $b(1-\varepsilon)e^{-x}$; hence

$$EN \to b(1-\varepsilon)e^{-x}, \qquad t \to \infty. \tag{4.1}$$

By definition, $P\{N \geqq 1\} = P\{\max(X(s) : s \in I_1 \cup \cdots \cup I_{[t]}) > u\}$. By the proof of Theorem 3.1 and by Corollary 3.1, this implies

$$P\{N \geqq 1\} \to 1-\exp[-b(1-\varepsilon)e^{-x}]. \tag{4.2}$$

For any nonnegative integer valued random variable N, $P\{N>1\} \leqq EN - P\{N \geqq 1\}$; thus, by (4.1) and (4.2),

$$\limsup_{t\to\infty} P\{N > 1\} \leqq b(1-\varepsilon)e^{-x} - 1 + \exp[-b(1-\varepsilon)e^{-x}];$$

therefore, from the inequality $e^{-w}-1+w \leqq \frac{1}{2}w^2$, $w>0$, it follows that

$$\limsup_{t\to\infty} P\{N > 1\} \leqq \tfrac{1}{2}[b(1-\varepsilon)e^{-x}]^2 \leqq \tfrac{1}{2}b^2e^{-2x}. \tag{4.3}$$

LEMMA 4.1. *Let Ψ_α be defined as in Theorem* 2.3; *then, for any $y>0$,*

$$\limsup_{t\to\infty} P\left\{0 < v\sum_{j=1}^{[tb]}\int_{I_j} I_{[X(s)>u]}\,ds \leqq y\right\} \leqq b\Psi_\alpha(y)e^{-x}+\tfrac{1}{2}b^2e^{-2x}. \tag{4.4}$$

Proof. By the decomposition of the event into disjoint subsets, the probability in (4.4) is not more than

$$P\left\{0 < v\sum_{j=1}^{[tb]}\int_{I_j} I_{[X(s)>u]}\,ds \leqq y,\ N=1\right\}+P\{N>1\}. \tag{4.5}$$

The events $\{v\int_{I_j} I_{[X(s)>u]}\,ds>0,\ N=1\}$, $j=1,\ldots,[tb]$, are disjoint; therefore, (4.5) is not more than

$$\sum_{j=1}^{[tb]} P\left\{0 < v\int_{I_j} I_{[X(s)>u]}\,ds \leqq y,\ N=1\right\}+P\{N>1\},$$

which, by stationarity, is at most

$$tbP\left\{0 < v\int_0^{1-\varepsilon} I_{[X(s)>u]}\,ds \leqq y\right\}+P\{N>1\}.$$

Now pass to the limit and apply Theorems 2.2 and 2.3, Corollary 3.1, and (4.3):

$$tbP\left\{0 < v\int_0^{1-\varepsilon} I_{[X(s)>u]}\,ds \leqq y\right\}$$

$$= tbP\left\{\max_{0\leqq s\leqq 1-\varepsilon} X(s) > u\right\}\cdot P\left\{v\int_0^{1-\varepsilon} I_{[X(s)>u]}\,ds \leqq y \;\middle|\; \max_{0\leqq s\leqq 1-\varepsilon} X(s) > u\right\}$$

$$\sim be^{-x}(1-\varepsilon)\Psi_\alpha(y).$$

LEMMA 4.2. *For $b>0$ and $y>0$,*

$$\liminf_{t\to\infty} P\left\{0 < v\sum_{j=1}^{[tb]}\int_{I_j} I_{[X(s)>u]}\,ds \leqq y\right\} \geqq b(1-\varepsilon)e^{-x}\Psi_\alpha(y)-b^2e^{-2x}.$$

Proof. The probability above is at least

$$P\left\{0 < v\sum_{j=1}^{[tb]}\int_{I_j} I_{[X(s)>u]}\,ds \leqq y, N = 1\right\},$$

which, as in the previous proof, is equal to

$$\sum_{j=1}^{[tb]} P\left\{0 < v\int_{I_j} I_{[X(s)>u]}\,ds \leqq y, N = 1\right\}.$$

This is equal to

$$\sum_{j=1}^{[tb]} P\left\{0 < v\int_{I_j} I_{[X(s)>u]}\,ds \leqq y, \max\left(X(s) : s\in \bigcup_{k\neq j} I_k\right) \leqq u\right\},$$

which is also equal to

$$\sum_{j=1}^{[tb]} P\left\{0 < v\int_{I_j} I_{[X(s)>u]}\,ds \leqq y\right\}$$
$$(4.6)\qquad -\sum_{j=1}^{[tb]} P\left\{0 < v\int_{I_j} I_{[X(s)>u]}\,ds \leqq y, \max\left(X(s) : s\in \bigcup_{k\neq j} I_k\right) > u\right\}.$$

By stationarity, Theorems 2.2 and 2.3, and (3.1) and (3.2), the first sum in (4.6) is asymptotic to

$$tbP\left\{0 < v\int_0^{1-\varepsilon} I_{[X(s)>u]}\,ds \leqq y\right\} \to b(1-\varepsilon)e^{-x}\Psi_\alpha(y).$$

The second sum in (4.6) is at most

$$2\sum_{j=1}^{[tb]}\sum_{k=j+1}^{[tb]} P\left\{\max_{I_j} X(s) > u, \max_{I_k} X(s) > u\right\},$$

which, by Corollary 3.2, is asymptotic to $[tbP\{\max(X(s) : 0\leqq s\leqq 1-\varepsilon) > u\}]^2$, which, by Theorem 2.2, converges to $[be^{-x}(1-\varepsilon)]^2 < b^2e^{-2x}$.

LEMMA 4.3. *For $\lambda>0$, the Laplace-Stieltjes transform*

$$E\left[\exp\left(-\lambda v\sum_{j=1}^{[tb]}\int_{I_j} I_{[X(s)>u]}\,ds\right)\right]$$

has a lim sup ($t \to \infty$) *not exceeding*

$$\exp\left[-b(1-\varepsilon)e^{-x}\right]+be^{-x}\int_0^\infty e^{-\lambda y}\,d\Psi_\alpha(y)+\frac{b^2e^{-2x}}{2\lambda}$$

and a lim inf *at least equal to*

$$\exp\left[-b(1-\varepsilon)e^{-x}\right]+be^{-x}(1-\varepsilon)\int_0^\infty e^{-\lambda y}\,d\Psi_\alpha(y)-\frac{b^2e^{-2x}}{\lambda}.$$

Proof. Let $G(y)$ be the distribution function of

$$v\sum_{j=1}^{[tb]}\int_{I_j} I_{[X(s)>u]}\,ds;$$

then, by the reasoning leading to (4.2),

$$G(0+) = P\left\{\max\left(X(s) : s\in\bigcup_{j=1}^{[tb]} I_j\right) \leqq u\right\}\to \exp\left[-b(1-\varepsilon)e^{-x}\right].$$

This and Lemmas 4.1 and 4.2 furnish bounds for the lim sup and lim inf of $G(y) = [G(y)-G(0+)]+G(0+)$. The assertion of the lemma now follows by use of the identity

$$\int_0^\infty e^{-\lambda y}\,dG(y) = 1-\lambda\int_0^\infty e^{-\lambda y}[1-G(y)]\,dy.$$

5. **Limiting distribution of the time spent above u.** If $X(t)$ satisfies (0.6) then, as is well known, it has the stochastic integral representation

$$X(s) = \int_{-\infty}^{\infty} e^{i\lambda s}\sqrt{f(\lambda)}\,\xi(d\lambda), \tag{5.1}$$

where ξ is the standard Brownian motion process.

The function $r_1(s)=(1-|s|)^+$ is a correlation function. Its spectral density is $(1-\cos\lambda)/\pi\lambda^2$. Put

$$\rho(s) = \frac{\int_{-\infty}^{\infty} r_1(s-\tau)r_1(\tau)\,d\tau}{\int_{-\infty}^{\infty} r_1^2(\tau)\,d\tau}.$$

This is also a correlation function. Its spectral density is, by the convolution relation, equal to

$$\varphi(\lambda) = \frac{\lambda^{-4}(1-\cos\lambda)^2}{\int_{-\infty}^{\infty}\lambda^{-4}(1-\cos\lambda)^2\,d\lambda}.$$

Since $r_1(s)$ vanishes outside $[-1, 1]$, $\rho(s)$ vanishes outside $[-2, 2]$.

For $T>0$ we define a process $X_T(s)$ on the same probability space as the process $X(s)$: Let ξ be the Brownian motion in the representation (5.1), and put

$$(\varphi_T f)(\lambda) = \int_{-\infty}^{\infty} f(\lambda+y/T)\varphi(y)\,dy;$$

then define X_T as

$$X_T(s) = \int_{-\infty}^{\infty} e^{i\lambda s}\sqrt{((\varphi_T f)(\lambda))}\xi(d\lambda). \tag{5.2}$$

This is also Gaussian and stationary with mean 0, variance 1 and correlation function

$$EX_T(0)X_T(s) = \int_{-\infty}^{\infty} e^{i\lambda s}(\varphi_T f)(\lambda)\, d\lambda = r(s)\rho(s/T), \tag{5.3}$$

where $r(s)$ is the covariance function of $X(t)$ (cf. [2]).

LEMMA 5.1. *If* (0.7) *holds, then*

$$\lim_{T\to\infty} \log T[1-EX(0)X_T(0)] = 0.$$

Proof. It follows from (5.1) and (5.2) that

$$EX(0)X_T(0) = \int_{-\infty}^{\infty} [f(\lambda)\cdot(\varphi_T f)(\lambda)]^{1/2}\, d\lambda;$$

therefore

$$\begin{aligned} 1-EX(0)X_T(0) &= \int_{-\infty}^{\infty} \sqrt{f}[\sqrt{f}-\sqrt{(\varphi_T f)}]\, d\lambda \\ &= \int_{-\infty}^{\infty} \frac{\sqrt{f}}{\sqrt{f}+\sqrt{(\varphi_T f)}}(f-(\varphi_T f))\, d\lambda \\ &\leqq \int_{-\infty}^{\infty} |f-(\varphi_T f)|\, d\lambda \leqq \int_{-\infty}^{\infty} \varphi(y)\left(\int_{-\infty}^{\infty} \left|f\left(\lambda+\frac{y}{T}\right)-f(\lambda)\right|\, d\lambda\right) dy. \end{aligned}$$

The last integral is evaluated by splitting the domain of integration with respect to y:

$$\int_{|y|<\log T} \varphi(y)\left(\int_{-\infty}^{\infty} \left|f\left(\lambda+\frac{y}{T}\right)-f(\lambda)\right|\, d\lambda\right) \leqq \sup_{y\leqq(\log T)/T} \int_{-\infty}^{\infty} |f(\lambda+y)-f(\lambda)|\, d\lambda.$$

This is of smaller order than $(\log T)^{-1}$ for $T\to\infty$; indeed, if $h=(1/T)\log T$, then $\log T\sim -\log h$, and so (0.7) implies

$$-\log h\cdot\sup_{y\leqq h}\int_{-\infty}^{\infty} |f(\lambda+y)-f(\lambda)|\, d\lambda \leqq \sup_{y\leqq h}(-\log y)\int_{-\infty}^{\infty} |f(\lambda+y)-f(\lambda)|\, d\lambda \to 0.$$

The integral over the complementary domain is also $o((\log T)^{-1})$:

$$\begin{aligned} &\int_{|y|>\log T} \varphi(y)\left(\int_{-\infty}^{\infty} \left|f\left(\lambda+\frac{y}{T}\right)-f(\lambda)\right|\, d\lambda\right) dy \\ &\qquad \leqq \int_{|y|>\log T} \varphi(y)\left[2\int_{-\infty}^{\infty} f(\lambda)\, d\lambda\right] dy = 2\int_{|y|>\log T} \varphi(y)\, dy. \end{aligned}$$

Since $\varphi(y)=O(\lambda^{-4})$, the last integral, multiplied by $\log T$, converges to 0. This completes the proof.

Let m be an arbitrary positive integer and δ an arbitrary real number, $0<\delta<1$. For $t>0$ decompose $[0, t]$ into m equal subintervals of length t/m. Then clip off an open segment of length $t\delta/m$ from the right endpoint of each subinterval to form m closed subintervals

$$J_k = [(k-1)t/m, (k-\delta)t/m], \qquad k = 1, \dots, m.$$

Let I_j be the intervals defined earlier and put $I=\bigcup_j I_j$.

LEMMA 5.2. *Under* (0.7) *the random variables*

$$v\int_{J_k\cap I} I_{[X(s)>u]}\,ds, \qquad k = 1, \dots, m,$$

are asymptotically independent in the sense that the difference between their joint distribution and the product of their marginals converges to 0 *as* $t\to\infty$.

Proof. Put

$$T = t\delta/2m; \tag{5.4}$$

then, by (5.3), $EX_T(s)X_T(s')=0$ if $|s-s'|>t\delta/m$ because ρ vanishes outside $[-2, 2]$; therefore, the random variables $v\int_{J_k\cap I} I_{[X_T(s)>u]}\,ds$, $k=1,\dots,m$, are mutually independent. For the proof of the lemma it is sufficient to show that

$$\lim_{t\to\infty} E\left| v\int_{J_k\cap I} I_{[X(s)>u]}\,ds - v\int_{J_k\cap I} I_{[X_T(s)>u]}\,ds\right| = 0, \qquad k = 1, \dots, m.$$

The expected absolute difference above is at most equal to

$$2v\int_{J_k\cap I} P\{X(s) > u,\ X_T(s) \leqq u\}\,ds,$$

which, by stationarity, is

$$2vP\{X(0) > u,\ X_T(0) \leqq u\}\cdot\text{measure } (J_k\cap I)$$

which is at most

$$2vP\{X(0) > u,\ X_T(0) \leqq u\}t/m. \tag{5.5}$$

Put $\eta=EX(0)X_T(0)$; then, by an adaptation of the method in [6, p. 27],

$$P\{X(0) > u,\ X_T(0) \leqq u\} = \int_\eta^1 \phi(u, u; y)\,dy.$$

By the identity

$$\phi(u, u; y) = \phi(u)\phi\left(u\sqrt{\left(\frac{1-y}{1+y}\right)}\right)(1-y^2)^{-1/2}$$

and the change of variable from y to $u^2(1-y)$, we have

$$\int_\eta^1 \phi(u, u; y)\,dy = \frac{\phi(u)}{u}\int_0^{u^2(1-\eta)} \phi\left(\sqrt{\left(\frac{y}{2-yu^{-2}}\right)}\right)[y(2-yu^{-2})]^{-1/2}\,dy.$$

By (3.2) this is asymptotic ($t \to \infty$) to

$$\frac{e^{-x}}{vtV_\alpha}\int_0^{(2\log t)(1-\eta)} \phi(\sqrt{(y/2)})\frac{dy}{\sqrt{y}}.$$

It follows that (5.5) is asymptotic to

$$\frac{2e^{-x}}{V_\alpha m}\int_0^{(2\log t)(1-\eta)} \phi(\sqrt{(y/2)})\frac{dy}{\sqrt{y}},$$

which, by (5.4) and Lemma 5.1, converges to 0 as $t \to \infty$.

Our final result is

THEOREM 5.1. *Under the assumptions* (0.2), (0.6) *and* (0.7) *the random variable*

$$v\int_0^t I_{[X(s)>u]}\,ds, \tag{5.6}$$

where u and v are given by (3.2) *and* (3.1), *has a limiting distribution whose Laplace-Stieltjes transform is*

$$\exp\left\{e^{-x}\int_0^\infty (e^{-\lambda y}-1)\,d\Psi_\alpha(y)\right\}. \tag{5.7}$$

Proof. Put $b=(1-\delta)/m$; then

$$v\int_{J_1\cap I} I_{[X(s)>u]}\,ds \quad \text{and} \quad v\sum_{j=1}^{[tb]}\int_{I_j} I_{[X(s)>u]}\,ds$$

have the same limiting distributions because the difference between the two random variables is at most $v\int_{[tb]}^{tb} I_{[X[s]>u]}\,ds$, of which the expectation is at most $vP\{X(0)>u\}$, which, by (2.12), (3.1) and (3.2), converges to 0. By Lemmas 4.3 and 5.2, the Laplace-Stieltjes transform of the sum

$$\sum_{k=1}^m v\int_{J_k\cap I} I_{[X(s)>u]}\,ds \tag{5.8}$$

has a lim sup at most equal to

$$\left\{\exp\left[-e^{-x}(1-\delta)(1-\varepsilon)/m\right]+\frac{1-\delta}{m}e^{-x}\int_0^\infty e^{-\lambda y}\,d\Psi_\alpha(y)+\frac{(1-\delta)^2}{m^2}\cdot\frac{e^{-2x}}{2\lambda}\right\}^m \tag{5.9}$$

and a lim inf at least equal to

(5.10)

$$\left\{\exp\left[-e^{-x}(1-\delta)(1-\varepsilon)/m\right]+\frac{1-\delta}{m}e^{-x}(1-\varepsilon)\int_0^\infty e^{-\lambda y}\,d\Psi_\alpha(y)-\frac{(1-\delta)^2e^{-2x}}{\lambda m^2}\right\}^m.$$

Since $\bigcup_{k=1}^m J_k\cap I\subset[0,t]$, it follows from the relations (2.12), (3.1), and $|e^{-x}-e^{-x-h}|\leqq h$, for $x, h>0$, that

$$E\left|\exp\left[-\lambda v\sum_{k=1}^m\int_{J_k\cap I} I_{[X(s)>u]}\,ds\right]-\exp\left[-\lambda v\int_0^t I_{[X(s)>u]}\,ds\right]\right|$$

$$\leqq E\left|\lambda\sum_{k=1}^m v\int_{(k-\delta)t/m}^{kt/m} I_{[X(s)>u]}\,ds\right| \leqq \lambda tv\delta P\{X(0)>u\}\sim\lambda\delta/V_\alpha.$$

It follows that the transform of (5.6) has a lim sup at most equal to *expression* $(5.9)+\lambda\delta/V_\alpha$ and a lim inf at least *expression* $(5.10)-\lambda\delta/V_\alpha$.

Since δ and ε are arbitrary we put $\delta=\varepsilon=0$. Since m is an arbitrary positive integer, we let $m\to\infty$: The expressions (5.9) and (5.10), with $\delta=\varepsilon=0$, have the common limit (5.7).

REMARK. The transform (5.7) represents the distribution of the sum of a random number of independent random variables, where the summands have the common distribution Ψ_α, and where the random number has a Poisson distribution with mean e^{-x}.

Added in proof. The validity of (2.21) under a weaker condition on g was recently announced by C. R. Qualls and H. Watanabe in Notices Amer. Math. Soc. **18** (1971), 532. Abstract #684-F4.

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