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# MAPPING CYLINDER NEIGHBORHOODS OF ONE-COMPLEXES IN FOUR-SPACE

BY
J. L. BRYANT(1) AND R. C. LACHER(2)

**Abstract.** We prove the following theorem: *If $K$ is a 1-complex topologically embedded in $S^4$, and if $K$ has mapping cylinder neighborhoods in $S^4$ at almost all of its points, then $K$ is tame.* The proof uses engulfing and the theory of proper, one-acyclic mappings of 3-manifolds onto the real line.

Suppose $M^m$ is a (topological) manifold (without boundary) in the manifold $N^n$, $x \in M^m$. We say that *$M^m$ has a mapping cylinder neighborhood in $N^n$ at $x$* if there exist

(i) an open neighborhood $V$ of $x$ in $M^m$;

(ii) an open $(n-1)$-manifold $U$;

(iii) a proper mapping $\phi$ of $U$ onto $V$; and

(iv) a homeomorphism $\psi$ of $Z_\phi$ onto a neighborhood of $V$ in $N$ such that $\psi(v)=v$ for each $v \in V$.

The requirement that the image of $\psi$ be a neighborhood of $V$ forces $U$ to have two components when $m=n-1$ (and $V$ is small).

In the above definition, $Z_\phi$ denotes the mapping cylinder of $\phi$, with $U$ and $V$ identified as subsets of $Z_\phi$ as is customary. A proper map is one under which inverse images of compact sets are compact. We use $\boldsymbol{R}^n$ to denote euclidean $n$-space, $B^n$ is the closed unit ball centered at 0 in $\boldsymbol{R}^n$, and $S^n = \text{Bd } B^{n+1}$. The symbol "$\approx$" means "is homeomorphic to".

It has been conjectured that an $m$-manifold $M^m$ is locally flat in an $n$-manifold $N^n$ if and only if $M^m$ has mapping cylinder neighborhoods in $N^n$ at each point. When $n \leqq 3$, this conjecture has been proved by Nicholson in [17]. The case $n=4$, $m=3$ is proved in [14]. The main purpose of the present paper is to prove the conjecture in case $n=4$, $m=1$ (see Corollary 4.2 below). The cases $n-m=2$, $n \geqq 4$ are false (because there are nonlocally flat piecewise linear embeddings). When $n \geqq 5$, the cases $n-m \neq 2$ remain open.

The truth of the above mapping cylinder conjecture in case $n=k+l$, $m=l-1$ implies the truth of the following well-known conjecture: If the $l$-fold suspension

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of a homology $k$-sphere $H^k$ is a manifold, then $H^k$ is simply connected. Thus, for $n \geqq 5$, the cases $n-m=1$ might seem the more tractable of the mapping cylinder conjectures. See [7] for the latest on the homology sphere conjecture.

## 1. Acyclicity of certain maps.

DEFINITION. Let $Z$ be a space, $Y$ a closed subspace, $G$ an abelian group. We say that $Z$ *is* $\mathrm{lc}^k(G)$ mod $Y$ if, for any point $y \in Y$ and any neighborhood $U$ of $y$ in $Z$, there exists a neighborhood $V$ of $y$ in $U$ such that the inclusion-induced map $\tilde{H}_i(V-Y; G) \to \tilde{H}_i(U-Y; G)$ is zero (on reduced singular homology) for $0 \leqq i \leqq k$.

The following result is well known. ($\boldsymbol{Z}$ is the group of integers.)

THEOREM 1.1. *Suppose that $M^m$ is an m-manifold topologically embedded in the interior of the n-manifold $N^n$. Then $N^n$ is* $\mathrm{lc}^{n-m-2}(\boldsymbol{Z})$ mod $M^m$.

**Proof.** Let $x \in M$, and let $U$ be an open $n$-cell neighborhood of $x$ in $N$. Consider the reduced homology sequence of the pair $(U, U-M)$:

$$\cdots \to H_{i+1}(U, U-M) \to \tilde{H}_i(U-M) \to \tilde{H}_i(U) \to \cdots.$$

We have $\tilde{H}_i(U)=0$ for all $i$ and $H_{i+1}(U, U-M) \simeq H_c^{n-i-1}(U \cap M)=0$ for $n-i-1 > m$. Therefore $\tilde{H}_i(U-M)=0$ for $i \leqq n-m-2$.

DEFINITION. A compact set $X$ in the ANR $M$ is said to have *property* $\mathrm{uv}^k(G)$ (or be *strongly k-acyclic over* $G$) if for any neighborhood $U$ of $X$ in $M$ there exists a neighborhood $V$ of $X$ in $U$ such that $\tilde{H}_i(V; G) \to \tilde{H}_i(U; G)$ is zero for $0 \leqq i \leqq k$.

COROLLARY 1.2. *Suppose that $f$: $U^{n-1} \to V^m$ is a proper, surjective map between manifolds. If $Z_f$ embeds (locally) in euclidean n-space then $f^{-1}(y)$ has property* $\mathrm{uv}^{n-m-2}(\boldsymbol{Z})$ *for each $y \in V^m$.*

**Proof.** It follows immediately from (1.1) that $Z_f$ is $\mathrm{lc}^{n-m-2}(\boldsymbol{Z})$ mod $V$. The result follows from Theorem 2.2 of [13].

## 2. One-acyclic maps of 3-manifolds onto $\boldsymbol{R}$.

Let $\boldsymbol{Z}_*$ be either the integers or the integers modulo two, $\boldsymbol{R}$ the real line.

THEOREM 2.1. *Let $W^3$ be an open 3-manifold, and suppose that $f$: $W^3 \to \boldsymbol{R}$ is a proper, surjective map. Suppose further that $f^{-1}(t)$ has property $\mathrm{uv}^1(\boldsymbol{Z}_*)$ for each $t \in \boldsymbol{R}$. Then there exists a locally finite subset $F$ of $\boldsymbol{R}$ such that if $J$ is an open interval of $\boldsymbol{R}$ which contains no point of $F$ then $f^{-1}(J) \approx S^2 \times \boldsymbol{R}$.*

Before proving (2.1) we need a definition and a preliminary result.

DEFINITION. A compact set $X$ in the ANR $M$ is said to have *property* 1-UV if each neighborhood $U$ of $X$ in $M$ contains a neighborhood $V$ of $X$ such that every loop in $V$ is null-homotopic in $U$.

THEOREM 2.2. *Let $f$ be a proper, monotone map of the open 3-manifold $W^3$ onto $\boldsymbol{R}$. Suppose that, for each $t \in \boldsymbol{R}$, $f^{-1}(t)$ has a neighborhood which contains no fake cubes and $f^{-1}(t)$ has property 1-UV. Then $W^3 \approx S^2 \times \boldsymbol{R}$.*

**Proof.** Since $f|f^{-1}(U)$ induces an isomorphism $\pi_1(f^{-1}(U)) \to \pi_1(U)$ for any connected open set $U \subset \boldsymbol{R}$ (see [1] or [12]) we have that $f^{-1}(J)$ is simply connected for any open interval $J \subset \boldsymbol{R}$, and, moreover, $f^{-1}(J)$ has two simply connected ends.

Choose $J$ such that $f^{-1}(J)$ contains no fake cubes. Applying [9], we see that either end of $f^{-1}(J)$ has a collar neighborhood; thus, there exists an embedding

$$h: S^2 \times ((-\infty, -1) \cup (1, +\infty)) \to f^{-1}(J)$$

such that the complement of the image of $h$ is compact. Let $W_- = h(S^2 \times (-\infty, -2])$, $W_+ = h(S^2 \times [2, +\infty))$, and $W_0 = f^{-1}(J) - W_-^\circ - W_+^\circ$. $W_0$ is a compact simply connected 3-manifold whose boundary is the union of two 2-spheres. Moreover, $W_0$ contains no fake cubes. Such a 3-manifold must be homeomorphic to $S^2 \times I$; for, by boring a hole from one boundary component to the other, we can write $W_0$ as the union of a 3-cell and a homotopy 3-cell meeting along an annulus common to their boundaries. Thus, $f^{-1}(J) \approx S^2 \times \boldsymbol{R}$.

Now, let $\cdots t_{i-1} < t_i < t_{i+1} < \cdots$ be a locally finite set in $\boldsymbol{R}$ such that $f^{-1}(t_i, t_{i+2}) \approx S^2 \times \boldsymbol{R}$ for each $i$. We can engulf any compact set in $W$ with $f^{-1}(t_{-1}, t_{+1})$ using the structure of $f^{-1}(t_i, t_{i+2})$ to pull $f^{-1}(t_{-1}, t_{+1})$ along. Thus, $W$ contains no fake cubes, and the result follows from the argument in the preceding paragraph.

**Proof of (2.1).** Let $f: W^3 \to \boldsymbol{R}$ be given as in the hypothesis of (2.1). Define

$$F' = \{t \in \boldsymbol{R} \mid f^{-1}(t) \text{ does not have property 1-UV}\},$$
$$F'' = \{t \in \boldsymbol{R} \mid \text{every neighborhood of } f^{-1}(t) \text{ contains a fake cube}\},$$
$$F = F' \cup F''.$$

By (2.2), we need only show that $F$ is locally finite. A theorem of Kneser [11] shows that $F''$ is locally finite. The local finiteness of $F'$ follows from work of McMillan [15] (see also [16]) and Wright [18]. Let $F_0' = \{t \in \boldsymbol{R} \mid f^{-1}(t)$ does not have arbitrarily small neighborhoods with free fundamental group$\}$. It follows from Theorem 2 of [15] that $F_0'$ is locally finite (for otherwise a compact submanifold of $W$ has nonfinitely generated fundamental group by Grushko's theorem). But the arguments [18, Theorem 1] show that $F_0' \supset F'$. (If $t \notin F_0'$, then $f^{-1}(t)$ has small neighborhoods $V \subset U$, where $H_1(V; \boldsymbol{Z}_2) = 0$ and $\pi_1(U)$ is free. Let $S_0 G = G$, $S_{n+1}G =$ subgroup of $S_n G$ generated by all squares of elements of $S_n G$, and $S_\omega G = \bigcap_{n=1}^\infty S_n G$. It follows that $S_\omega \pi_1(V) = \pi_1(V)$ while $S_\omega \pi_1(U) = \{1\}$, so that every homomorphism $\pi_1(V) \to \pi_1(U)$ is trivial.)

REMARK. Examples show that $F$ in (2.1) may well be nonvoid. Using connected sum, stick homology spheres onto $S^2 \times \boldsymbol{R}$ to form $W^3$; the map $f$ is to shrink spines of the homology spheres to points and then project to $\boldsymbol{R}$.

## 3. A special case of the main theorem.

THEOREM 3.1. *Suppose that $\phi: S^2 \times \boldsymbol{R} \to \boldsymbol{R}$ is a proper, surjective map such that $\phi^{-1}(J) \approx S^2 \times \boldsymbol{R}$ for any open interval $J \subset \boldsymbol{R}$. Suppose further that $\psi: Z_\phi \to \boldsymbol{R}^4$ is an embedding of $Z_\phi$ onto a neighborhood of $\psi(\boldsymbol{R})$ in $\boldsymbol{R}^4$. Then $\psi(\boldsymbol{R})$ is locally flat in $\boldsymbol{R}^4$.*

Throughout the remainder of this section we shall adhere to the following notation:

(a) $\phi: S^2 \times \boldsymbol{R} \rightarrow \boldsymbol{R}$ and

(b) $\psi: Z_\phi \rightarrow \boldsymbol{R}^4$

are as in the hypothesis of Theorem 3.1.

(c) $S_0 = \psi(Z_\phi)$.

(d) For $0 \leqq t < 1$, $S_t$ denotes the portion of $\psi(Z_\phi)$ between the levels $t$ and 1, the range of $\phi$ being the 1-level of the mapping cylinder $Z_\phi$.

We may assume that $\psi(S^2 \times \boldsymbol{R}) = \text{Bd } S_0$ is locally flat in $\boldsymbol{R}^4$, since $S_0$ may be replaced by $S_t$ $(t > 0)$.

LEMMA 3.2. *Given a finite subset* $\{s_1, s_2, \ldots, s_n\}$ *of* $\boldsymbol{R}$ *with* $s_1 < s_2 < \cdots < s_n$, *there exists an embedding* $g: B^3 \times [1, n]$ *into* $S_0$ *such that* $g^{-1}(\text{Bd } S_0) = S^2 \times [1, n]$ *and* $g(0, k) = \psi(s_k)$ *for* $k = 1, \ldots, n$.

**Proof.** We first show how to find a 3-cell $B$ in $S_0$. Let $\alpha: S^2 \times (-1, 1) \rightarrow \phi^{-1}(-1, 1)$ be a homeomorphism. Let $\Sigma_t$ be $\alpha(S^2 \times \{t\}) \times \{t\}$, copied in $Z_\phi$ $(0 \leqq t < 1)$. Clearly $B = \psi(\bigcup_{0 \leqq t < 1} \Sigma_t \cup \{1\})$ is a 3-cell in $S_0$ that is locally flat in $S_0$ except possibly at $\psi(1)$. A result of Kirby [10] then implies that $B$ is locally flat. From the construction of $B$ it is also clear that $B$ meets Bd $S_0$ "nicely".

Now, for $k = 1, \ldots, n$, let $B_k$ be a 3-cell in $\psi(Z_{\phi | \phi^{-1}(s_{k-1}, s_{k+1})})$ as constructed above with $B_i \cap B_j = \varnothing$ if $i \neq j$ and $B_k \cap \psi(\boldsymbol{R}) = \psi(s_k)$. Let $A_k$ be the annulus in Bd $S_0$ bounded by Bd $B_k$ and Bd $B_{k+1}$ $(k = 1, \ldots, n-1)$. Then $B_k \cup A_k \cup B_{k+1}$ is a locally flat 3-sphere in $S_0$ that bounds a 4-cell $C_k$ in $S_0$ by the generalized Schoenflies Theorem [2]. Let $g_k: B^3 \times [k, k+1] \rightarrow C_k$ be a homeomorphism that takes $B^3 \times \{k\}$ onto $B_k$ and $B^3 \times \{k+1\}$ onto $B_{k+1}$. Construct the $g_k$'s inductively so that $g_k | B^3 \times \{k+1\} = g_{k+1} | B^3 \times \{k+1\}$ $(k = 1, \ldots, n-1)$. Defining $g: B^3 \times [1, n] \rightarrow S_0$ by $g | B^3 \times [k, k+1] = g_k$, we complete the proof of the lemma.

REMARK 1. It is clear that in the statement of Lemma 3.2 we could have replaced $S_0$ by $S_t$ for any $t \in [0, 1)$.

REMARK 2. The proof of Theorem 3.1 could now be completed by appealing to the technique of [3] as explained in [4]. The reason this is so is that for $t$ sufficiently close to 1 and for suitably chosen $s_1 < 0 < s_2 < \cdots < s_{n-1} < 1 < s_n$, we can get a small push of $\psi([0, 1])$ off of any 2-complex $K$ by first placing $g(\{0\} \times [1, n])$ and $K$ in general position. We have a complete, elementary proof, however, that uses a construction similar to one found in [6].

LEMMA 3.3. *Suppose that* $\{s_1', \ldots, s_m'\}$ *and* $\{s_1'', \ldots, s_n''\}$ *are subsets of* $\boldsymbol{R}$ *with* $s_1' < \cdots < s_m'$, $s_1'' < \cdots < s_n''$, *and* $s_i' \neq s_j''$ (*for any* $i, j$), *that* $t', t'' \in [0, 1)$, *and that* $g': B^3 \times [1, m] \rightarrow S_{t'}$ *and* $g'': B^3 \times [1, n] \rightarrow S_{t''}$ *are the associated embeddings obtained from Lemma* 3.2. *Let*

$$\{s_1, \ldots, s_{m+n}\} = \{s_1', \ldots, s_m'\} \cup \{s_1'', \ldots, s_n''\}$$

*arranged so that* $s_1 < \cdots < s_{m+n}$. *Then there exist* $t \in [0, 1)$ *and an embedding* $g\colon B^3 \times [1, m+n] \to S_t$ *such that*

$$g(B^3 \times \{k\}) = g'(B^3 \times \{i\}) \cap S_t \quad \text{if } s_k = s'_i,$$
$$= g''(B^3 \times \{j\}) \cap S_t \quad \text{if } s_k = s''_j.$$

**Proof.** Observe that the 3-cell $B$ constructed at the beginning of the proof of Lemma 3.2 separates each $S_t$ in the same manner in which it separates $S_0$. Furthermore, given $g'\colon B^3 \times [1, m] \to S_{t'}$ and $g''\colon B^3 \times [1, n] \to S_{t''}$, there exists $t \geqq \max\{t', t''\}$, $t < 1$, so that $g'(B^3 \times \{i\}) \cap g''(B^3 \times \{j\}) \cap S_t = \varnothing$ for any $i, j$. Thus the proof of Lemma 3.2 can be carried out using the 3-cells $g'(B^3 \times \{i\}) \cap S_t$ $(i = 1, \ldots, m)$ and $g''(B^3 \times \{j\}) \cap S_t$ $(j = 1, \ldots, n)$ in the construction of $g$.

**Proof of (3.1).** We are going to construct a homeomorphism $H$ of $\boldsymbol{R}^4$ that carries $\psi[0, 1]$ onto a locally flat arc. Since, in effect, $\psi[0, 1]$ is an arbitrary subinterval in $\psi(\boldsymbol{R})$, this will prove that $\psi(\boldsymbol{R})$ is locally flat.

Let $\varepsilon_1, \varepsilon_2, \ldots$ be a sequence of positive numbers such that $\sum_{i=1}^{\infty} \varepsilon_i < \infty$.

Inductively we can use (3.2) and (3.3) to find numbers $0 \leqq t'_1 < t'_2 < \cdots < 1$ and embeddings $g'_k\colon B^3 \times [-1, 2^k + 1] \to S_{t'_k}$ such that $g'_k(0, i) = \psi(i/2^k)$ and

$$g'_{k+1}(B^3 \times \{2i\}) = g'_k(B^3 \times \{i\}) \cap S_{t'_{k+1}}.$$

Let $g_1, g_2, \ldots$ be a subsequence of $g'_1, g'_2, \ldots$ (with corresponding subsequences $t_1, t_2, \ldots$ of $t'_1, t'_2, \ldots$ and $n_1, n_2, \ldots$ of $2^1+1, 2^2+1, \ldots$) chosen so that

$$\operatorname{diam} g_i(B^3 \times [j, j+1]) < \varepsilon_i \qquad (-1 \leqq j < n_i).$$

Set $J_i = g_i(\{0\} \times [0, n_i - 1])$, $B_{ij} = g_i(B^3 \times [j, j+1])$, and $B_i = \bigcup_{j=-1}^{n_i-1} B_{ij}$. Observe that whenever $k > i$ and $0 \leqq j \leqq n_i - 2$, $J_i \cap B_{ij}$, $J_k \cap B_{ij}$, and $\psi([0, 1]) \cap B_{ij}$ are properly embedded arcs in $B_{ij}$ sharing common endpoints. Let $h''_i\colon J_i \to \psi([0, 1])$ be a homeomorphism taking $J_i \cap B_{ij}$ onto $\psi([0, 1]) \cap B_{ij}$ that fixes the endpoints of $J_i \cap B_{ij}$, and let $h'_i$ be a homeomorphism of $\boldsymbol{R}^4$ such that $h'_i|\operatorname{Bd} B_{ij}$ and $h'_i|\boldsymbol{R}^4 - B_i$ are the identity and $h'_i|J_i = (h''_{i+1})^{-1}h''_i$. Thus $h'_i$ is an $\varepsilon_i$-homeomorphism and $h'_i(J_i) = J_{i+1}$.

Observe also that if $U$ is any neighborhood of $J_i$ $(i \geqq 2)$, then there exists an $\varepsilon_i$-homeomorphism $\theta_i$ of $\boldsymbol{R}^4$ such that $\theta_i|(\boldsymbol{R}^4 - B_{i-1}) \cup J_i = \text{identity}$ and $\theta_i(U) \supset B_i$.

For $0 < r \leqq 1$ let $B^3_r = \{rx \mid x \in B^3\}$ and let $C_i = g_1(B^3_{1/i} \times [-1/i, n_1 - 1 + 1/i])$.

Let $h_1 = h'_1$. Assuming $h_k\colon \boldsymbol{R}^4 \to \boldsymbol{R}^4$ is defined such that $h_k(J_1) = J_{k+1}$, let $\theta_{k+1}$ be an $\varepsilon_{k+1}$-homeomorphism of $\boldsymbol{R}^4$ such that $\theta_{k+1} \mid (\boldsymbol{R}^4 - B_k) \cup J_{k+1} = \text{identity}$ and $\theta_{k+1}h_k(C_{k+1}) \supset B_{k+1}$, and let $h_{k+1} = h'_{k+1}\theta_{k+1}h_k$. Then $h_{k+1}$ moves no point of $\boldsymbol{R}^4$ farther than $2\sum_{i=1}^{k+1} \varepsilon_i$. Thus the sequence $h_1, h_2, \ldots$ converges to a map $h$ of $\boldsymbol{R}^4$ onto itself. By construction $h|J_1 = \lim h_i|J_1 = h''_1\colon J_1 \to \psi([0, 1])$ is a homeomorphism and $h|\boldsymbol{R}^4 - B_1 = \text{identity}$. Let $p$ be a point of $B_1 - J_1$. There exists $i$ such that $p \in B_1 - C_i$. Thus $h_{i+1}(p) \notin B_{i+1}$ and so $h(p) = h_{i+2}(p)$. This implies that $h|B_1 - J_1$ is a homeomorphism of $B_1 - J_1$ onto $B_1 - \psi([0, 1])$, and hence $h$ is a homeomorphism. The homeomorphism $H = h^{-1}$ takes $\psi([0, 1])$ onto an arc $(J_1)$ that is known to be locally flat. Hence $\psi([0, 1])$ is locally flat.

4. **Proof of the main theorem.** In the following, $K^0$ denotes the set of vertices of the complex $K$.

THEOREM 4.1. *Let $K$ be a finite 1-complex topologically embedded in the interior of the PL 4-manifold $M^4$. Suppose that $K-K^0$ has mapping cylinder neighborhoods in $M^4$ at each point. Then $K$ is tame in $M^4$.*

**Proof.** For each $x \in K-K^0$, let $V_x$ be an open arc in $K-K^0$ containing $x$ such that there exist an open 3-manifold $U_x$, a proper map $\phi_x$ of $U_x$ onto $V_x$, and an embedding $\psi_x$ of $Z_{\phi_x}$ into $M$ such that $\psi_x(v)=v$ for all $v \in V_x$. Applying (1.2), we see that $\phi_x^{-1}(t)$ has property uv$^1$ $(\boldsymbol{Z})$ for each $t \in V_x$. Applying (2.1), we see that there is a locally finite subset $F_x$ of $V_x$ such that if $J$ is an open interval in $V_x - F_x$ then $\phi_x^{-1}(J) \approx S^2 \times \boldsymbol{R}$. By shrinking the interval $V_x$ somewhat, we may assume that $F_x$ is actually finite.

Now, let $\{V_{x_1}, V_{x_2}, \ldots\}$ be a locally finite subcollection of $\{V_x\}$ which covers $K-K^0$, and let $F=K^0 \cup \bigcup_{i=1}^{\infty} F_{x_i}$. $F$ is a compact countable subset of $K$. The crucial property that $F$ enjoys is the following: For each point $x \in K-F$, $V_x$, $U_x$, $\phi_x$, $\psi_x$ may be chosen so that $\phi_x^{-1}(J) \approx S^2 \times \boldsymbol{R}$ for any open interval $J \subset V_x$. By the special case (3.1), $\psi_x(V_x)$ is locally flat in $M^4$. Hence, $K$ is locally tame in $M^4$ at each point of $K-F$. It follows from [5] and [8] that $K$ is tame in $M^4$.

COROLLARY 4.2. *If $S$ is a 1-sphere in $S^4$ which has mapping cylinder neighborhoods in $S^4$ at every point, then $S$ is flat.*

## REFERENCES


1. S. Armentrout and T. M. Price, *Decompositions into compact sets with UV properties*, Trans. Amer. Math. Soc. **141** (1969), 433–442. MR **39** #6307.

2. M. Brown, *A proof of the generalized Schoenflies theorem*, Bull. Amer. Math. Soc. **66** (1960), 74–76. MR **22** #8470b.

3. J. L. Bryant, *On embeddings of compacta in Euclidean space*, Proc. Amer. Math. Soc. **23** (1969), 46–51. MR **39** #6286.

4. J. L. Bryant and D. W. Sumners, *On embeddings of 1-dimensional compacta in a hyperplane in $E^4$*, Pacific J. Math. **33** (1970), 555–557.

5. J. C. Cantrell, *n-frames in euclidean k-space*, Proc. Amer. Math. Soc. **15** (1964), 574–578. MR **29** #1627.

6. A. V. Černavskiĭ, *Topological embeddings of manifolds*, Dokl. Akad. Nauk SSSR **187** (1969), 1247–1250=Soviet Math. Dokl. **10** (1969), 1037–1041.

7. L. C. Glaser, "On double suspensions of arbitrary nonsimply connected homology *n*-spheres," *Topology of manifolds*, J. C. Cantrell and C. H. Edwards, Jr. (editors), Markham, Chicago, 1970.

8. H. Gluck, *Embeddings in the trivial range*, Ann. of Math. (2) **81** (1965), 195–210. MR **30** #3456.

9. L. S. Husch and T. M. Price, *Finding a boundary for a 3-manifold*, Ann. of Math. (2) **91** (1970), 223–235.

10. R. C. Kirby, *On the set of non-locally flat points of a submanifold of codimension one*, Ann. of Math. (2) **88** (1968), 281–290. MR **38** #5193.

11. H. Kneser, *Geschlossene Flächen in dreidimensionalen Mannigfaltigkeiten*, Jber. Deutsch. Math.-Verein **38** (1929), 248–260.

12. R. C. Lacher, *Cell-like mappings*. I, Pacific J. Math. **30** (1969), 717–731. MR **40** #4941.

13. ———, *Cell-like mappings*. II, Pacific J. Math. **35** (1970), 649–660.

14. R. C. Lacher and A. H. Wright, "Mapping cylinders and 4-manifolds," *Topology of manifolds*, J. C. Cantrell and C. H. Edwards, Jr. (editors), Markham, Chicago, 1970.

15. D. R. McMillan, Jr., *Acyclicity in three-manifolds*, Bull. Amer. Math. Soc. **76** (1970), 942–964.

16. ———, UV *properties and related topics*, Lecture Notes, Florida State University, Tallahassee, Fla., 1970.

17. V. Nicholson, *Mapping cylinder neighborhoods*, Trans. Amer. Math. Soc. **143** (1969) 259–268. MR **40** #2038.

18. A. H. Wright, *Mappings from* 3-*manifolds onto* 3-*manifolds* (to appear).

Department of Mathematics, Florida State University, Tallahassee, Florida 32306