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SOME REMARKS ON QUASI-ANALYTIC VECTORS[1]

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Abstract. Recently a number of authors have developed conditions of a generalized quasi-analytic nature which imply essential selfadjointness for semibounded symmetric operators in Hilbert space. We give a unified derivation of these results by reducing them to the basic theorems of Nelson and Nussbaum. In addition we present an extension of Nussbaum's quasi-analytic vector theorem to the setting of semigroups in Banach spaces.

1. **Introduction.** Let A be a linear operator in a Banach space with domain $\mathscr{D}(A)$. A vector x will be called a C^∞ vector for A provided $x \in \bigcap_{n=1}^{\infty} \mathscr{D}(A^n) = \mathscr{D}^\infty(A)$. We shall distinguish the following subsets of $\mathscr{D}^\infty(A)$:

(1) *analytic vectors*: $x \in \mathscr{D}_{\mathrm{a}}(A)$ if $\sum_{n=0}^{\infty} (t^n/n!)\|A^n x\|$ converges for some $t>0$;

(2) *quasi-analytic vectors*: $x \in \mathscr{D}_{\mathrm{qa}}(A)$ if $\sum_{n=0}^{\infty} \|A^n x\|^{-1/n} = \infty$;

(3) *semi-analytic vectors*: $x \in \mathscr{D}_{\mathrm{sa}}(A)$ if $\sum_{n=0}^{\infty} (t^n/(2n)!)\|A^n x\|$ converges for some $t>0$;

(4) *Stieltjes vectors*: $x \in \mathscr{D}_{\mathrm{s}}(A)$ if $\sum_{n=0}^{\infty} \|A^n x\|^{-1/2n} = \infty$.

The sets $\mathscr{D}_{\mathrm{a}}$ and $\mathscr{D}_{\mathrm{sa}}$ are linear subspaces of $\mathscr{D}^\infty(A)$, but this is not necessarily true for $\mathscr{D}_{\mathrm{qa}}$ and $\mathscr{D}_{\mathrm{s}}$. The following inclusion relations are immediate:

$$\begin{array}{ccc} \mathscr{D}_{\mathrm{a}} & \subset & \mathscr{D}_{\mathrm{qa}} \\ \cap & & \cap \\ \mathscr{D}_{\mathrm{sa}} & \subset & \mathscr{D}_{\mathrm{s}} \end{array}$$

Analytic vectors were invented by Nelson; his paper [6] contains (among many things) the following fundamental fact.

THEOREM A. *Let A be a symmetric operator on Hilbert space. If A has a dense set of analytic vectors then the closure of A is selfadjoint.*

A significant generalization was made by Nussbaum [7], who introduced the notion of quasi-analytic vector and proved the analogue of Nelson's theorem:

THEOREM QA. *Let A be a symmetric operator on Hilbert space. If A has a total set of quasi-analytic vectors then the closure of A is selfadjoint.*

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More recently Nussbaum [8] proved the following result, which was discovered independently by Masson and McClary [5], to whom the term "Stieltjes vector" is due. In [5] this result is applied to quantum field theory.

THEOREM S. *Let A be a symmetric and semibounded operator on Hilbert space. If A has a total set of Stieltjes vectors then its closure is selfadjoint.*

A simplified proof of Theorem S has been given by Simon [11], who also proved the following result, which is to Theorem A as Theorem S is to Theorem QA.

THEOREM SA. *Let A be symmetric and semibounded. If A has a dense set of semi-analytic vectors then A has selfadjoint closure.*

Our new results are these: first we shall show via a simple trick that Theorems SA and S can be deduced in a unified way from their respective analogues, Theorems A and QA. We remark that in [8] Nussbaum deduced QA from S, but felt that it would be difficult to deduce S from QA. Secondly, Hasegawa [2] has devised a very simple proof of Theorem QA, avoiding the moment-problem methods of Nussbaum by a direct appeal to Carleman's theorem on quasi-analytic functions. Moreover Hasegawa has generalized the theorem to the context of contraction semigroups on Hilbert space. We shall extend his results to general strongly continuous semigroups on Banach spaces.

I thank Barry Simon for some useful correspondence.

2. **The unified derivation.** Our key idea is very simple. Let A be a semibounded operator on the Hilbert space $\boldsymbol{H}$. In addition to A consider the operator

$$iB = i\begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix}$$

on a suitable space. Then semianalytic (or Stieltjes) vectors for A yield analytic (respectively, quasi-analytic) vectors for B. We can then apply Theorem A (or QA) to B to deduce that B, and so ultimately A, has selfadjoint closure.

The rest of this section consists of the technical details. We begin with a preliminary result which shows that $\mathscr{D}_{\mathrm{qa}}(A)$ and $\mathscr{D}_{\mathrm{s}}(A)$ are stable under suitable operations.

LEMMA 2.1. *Let A be a symmetric operator on a Hilbert space $\boldsymbol{H}$. Let x be a Stieltjes (or quasi-analytic) vector for A. Let p be any polynomial. Then $p(A)x$ is Stieltjes (respectively, quasi-analytic).*

Proof. The quasi-analytic case is contained in Theorem 4 of [7]. The Stieltjes case is quite similar, but for completeness we shall write out the argument.

Obviously $p(A)x$ is a C^∞ vector. We can assume it is nonzero. Also

$$\begin{aligned}\|A^n p(A)x\|^2 &= \langle A^n p(A)x, A^n p(A)x\rangle = \langle A^{2n}x, p(A)^* p(A)x\rangle \\ &\leqq \|A^{2n}x\| \cdot \|p(A)^* p(A)x\|.\end{aligned}$$

Hence to show that $\sum_{n=1}^{\infty} \|A^n p(A)x\|^{-1/2n}$ diverges it is enough to show that $\sum_{n=1}^{\infty} \|A^{2n}x\|^{-1/4n}$ diverges.

We can obviously assume $\|x\|=1$. But then [7, p. 183] $\|A^n x\|^{-1/2n}$ is a decreasing function of n. Thus

$$2\sum_{n=1}^{\infty} \|A^{2n}x\|^{-1/4n} \geqq \sum_{n=1}^{\infty} \{\|A^{2n}x\|^{-1/4n}+\|A^{2n+1}x\|^{-1/(4n+2)}\}$$
$$= \sum_{k=2}^{\infty} \|A^k x\|^{-1/2k}$$

which diverges since x is a Stieltjes vector. □

Now let A satisfy the hypothesis of Theorem SA or Theorem S. Since A is semibounded we can assume it is bounded below; and since adding a constant to A does not affect the semi-analytic or Stieltjes property (see [8] and [11]) we can assume that $A \geqq I$.

Let $\mathscr{D}_0 \subset \mathscr{D}(A)$ be the set of semi-analytic vectors or, alternatively, the linear span of the Stieltjes vectors. Then $\mathscr{D}_0$ is, by assumption of the hypothesis of either Theorem SA or Theorem S, a dense linear subspace. Moreover, $\mathscr{D}_0$ is stable under A by Lemma 1. Let A_0 be the restriction of A to $\mathscr{D}_0$. We shall show that A_0 has selfadjoint closure.

Since A_0 is bounded below it has a Friedrichs extension A_1, which is selfadjoint and $\geqq I$. Moreover, by construction the domain of $(A_1)^{1/2}$ is the completion of $\mathscr{D}_0$ with respect to the inner product $[x, y]=\langle A_0x, y\rangle$. In other words, $\mathscr{D}_0$ is not merely dense in $\boldsymbol{H}$ but also dense in $\mathscr{D}((A_1)^{1/2})$ with respect to the appropriate norm, a fact which will be useful later in Lemmas 2.4 and 2.5.

LEMMA 2.2. *Let $A_1 \geqq I$ be any selfadjoint operator on $\boldsymbol{H}$. Form the space $\boldsymbol{K}=\mathscr{D}((A_1)^{1/2}) \oplus \boldsymbol{H}$, which is a Hilbert space with inner product*

$$\left\langle \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u' \\ v' \end{pmatrix} \right\rangle = \langle A_1 u, u'\rangle + \langle v, v'\rangle.$$

Let B_1 be given on $\boldsymbol{K}$ by the operator matrix

$$B_1 = \begin{pmatrix} 0 & I \\ -A_1 & 0 \end{pmatrix};$$

its domain is $\mathscr{D}(A_1) \oplus \mathscr{D}((A_1)^{1/2})$. Then B_1 is skew-adjoint.

Proof. An easy computation shows that B_1 is skew-symmetric. Indeed if $\binom{u}{v}$ and $\binom{u'}{v'} \in \mathscr{D}(B_1)$ we have

$$\left\langle B_1\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u' \\ v' \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} v \\ -A_1 u \end{pmatrix}, \begin{pmatrix} u' \\ v' \end{pmatrix} \right\rangle = \langle A_1 v, u'\rangle - \langle A_1 u, v'\rangle$$
$$= \langle v, A_1 u'\rangle - \langle u, A_1 v'\rangle = -\left\langle \begin{pmatrix} u \\ v \end{pmatrix}, B_1\begin{pmatrix} u' \\ v' \end{pmatrix} \right\rangle.$$

Next we show that $I+B_1$ is onto. Given $\binom{x}{y} \in \mathscr{D}((A_1)^{1/2}) \oplus \boldsymbol{H}$ we seek $\binom{u}{v} \in \mathscr{D}(A_1) \oplus \mathscr{D}((A_1)^{1/2})$ with

$$u+v = x, \qquad -A_1u+v = y.$$

Subtracting, we want $(I+A_1)u=x-y$.

This determines $u \in \mathscr{D}(A_1)$ since $I+A_1$ is one-one and onto. Then $v=x-u$ belongs to $\mathscr{D}((A_1)^{1/2})+\mathscr{D}(A_1)=\mathscr{D}((A_1)^{1/2})$. So $\binom{u}{v} \in \mathscr{D}(B_1)$ and $(I+B_1)\binom{u}{v}=\binom{x}{y}$.

A similar argument shows that $I-B_1$ is onto. □

Thus e^{tB_1} is a one-parameter unitary group on $\boldsymbol{K}$. It is given in operator matrix form by the formula

$$e^{tB_1} = \begin{pmatrix} \cos t(A_1)^{1/2} & (\sin t(A_1)^{1/2})/(A_1)^{1/2} \\ -(A_1)^{1/2} \sin t(A_1)^{1/2} & \cos t(A_1)^{1/2} \end{pmatrix}.$$

(For a more general theorem along these lines see Goldstein [1].)

Return to our specific Friedrichs extension operator A_1. Define an operator B_0 on $\boldsymbol{K}$ by

$$B_0 = \begin{pmatrix} 0 & I \\ -A_0 & 0 \end{pmatrix}$$

with domain $\mathscr{D}(B_0)=\mathscr{D}(A_0) \oplus \mathscr{D}((A_1)^{1/2})$. Then B_0 is a restriction of B_1.

LEMMA 2.3. *The closure of B_0 is*

$$\bar{B}_0 = \begin{pmatrix} 0 & I \\ -\bar{A}_0 & 0 \end{pmatrix}$$

with domain $\mathscr{D}(\bar{B}_0)=\mathscr{D}(\bar{A}_0) \oplus \mathscr{D}((A_1)^{1/2})$.

Proof. Denote $\left(\begin{smallmatrix} 0 & I \\ -\bar{A}_0 & 0 \end{smallmatrix}\right)$ by C. We claim first that C is closed. Indeed if

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} \to \begin{pmatrix} u \\ v \end{pmatrix} \quad \text{and} \quad C\begin{pmatrix} u_n \\ v_n \end{pmatrix} \to \begin{pmatrix} w \\ z \end{pmatrix},$$

we have

$$\begin{pmatrix} v_n \\ -\bar{A}_0 u_n \end{pmatrix} \to \begin{pmatrix} w \\ z \end{pmatrix} \quad \text{in } \boldsymbol{K}.$$

Hence $v=w$. Also $u_n \to u$ and $\bar{A}_0 u_n \to -z$. Since $\bar{A}_0$ is closed we must have $u \in \mathscr{D}(\bar{A}_0)$ and $\bar{A}_0 u=-z$. Thus $\binom{u}{v} \in \mathscr{D}(C)$ and $C\binom{u}{v}=\binom{w}{z}$.

Since C is closed it extends $\bar{B}_0$. For the opposite inclusion, suppose $\binom{u}{v} \in \mathscr{D}(C)$. Thus $u \in \mathscr{D}(\bar{A}_0)$, $v \in \mathscr{D}((A_1)^{1/2})$. So there is a sequence $u_n \in \mathscr{D}(A_0)$ with $u_n \to u$ and $A_0 u_n \to \bar{A}_0 u$. Since $\bar{A}_0 \subset A_1$ it follows that $(A_1)^{1/2} u_n \to (A_1)^{1/2} u$ as well. In other words

$$\begin{pmatrix} u_n \\ v \end{pmatrix} \to \begin{pmatrix} u \\ v \end{pmatrix} \quad \text{in the topology of } \boldsymbol{K}.$$

Moreover,

$$B_0\begin{pmatrix} u_n \\ v \end{pmatrix} = \begin{pmatrix} v \\ -A_0 u_n \end{pmatrix} \to \begin{pmatrix} v \\ -\bar{A}_0 u \end{pmatrix} = C\begin{pmatrix} u \\ v \end{pmatrix}. \quad \square$$

Now we are ready for the heart of our argument. By virtue of Lemma 2.3, if we can show that $\bar{B}_0 = B_1$ it will follow that $\bar{A}_0 = A_1$, and thus that $\bar{A} = A_1$, a selfadjoint operator.

We first treat the semi-analytic case.

LEMMA 2.4. *If x and y are semi-analytic for A_0 then $\binom{x}{y}$ is analytic for B_0.*

Proof. A straightforward computation. Note that

$$B_0^{2k} = (-1)^k \begin{pmatrix} A_0^k & 0 \\ 0 & A_0^k \end{pmatrix},$$

$$B_0^{2k+1} = (-1)^k \begin{pmatrix} 0 & A_0^k \\ -A_0^{k+1} & 0 \end{pmatrix},$$

with the obvious domains in $\boldsymbol{K}$.

Clearly, $\binom{x}{y}$ is a C^∞ vector for B_0. Also

$$B_0^{2k}\begin{pmatrix} x \\ y \end{pmatrix} = (-1)^k \begin{pmatrix} A_0^k x \\ A_0^k y \end{pmatrix}$$

so

$$\left\| B_0^{2k}\begin{pmatrix} x \\ y \end{pmatrix} \right\|^2 = (A_1 A_0^k x, A_0^k x) + (A_0^k y, A_0^k y)$$
$$\leqq \|A_0^{k+1} x\|^2 + \|A_0^k y\|^2,$$

whence

$$\left\| B_0^{2k}\begin{pmatrix} x \\ y \end{pmatrix} \right\| \leqq \|A_0^{k+1} x\| + \|A_0^k y\|.$$

Similarly,

$$\left\| B_0^{2k+1}\begin{pmatrix} x \\ y \end{pmatrix} \right\| \leqq \|A_0^{k+1} y\| + \|A_0^{k+1} x\|.$$

Hence

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \left\| B_0^n \begin{pmatrix} x \\ y \end{pmatrix} \right\| \leqq \sum_{n=0}^{\infty} \frac{t^n}{n!} \{ \|A_0^{[n/2]+1} x\| + \|A_0^{[n/2]+1} y\| \}$$

which converges for small t by semi-analyticity of x and y. $\square$

Thus if A satisfies the hypothesis (SA) the set of vectors $\{\binom{x}{y} : x, y \in \mathscr{D}_{sa}(A)\}$, namely $\mathscr{D}_0 \oplus \mathscr{D}_0$, is a set of analytic vectors for B_0. Moreover, $\mathscr{D}_0 \oplus \mathscr{D}_0$ is dense in $\boldsymbol{K}$ because $\mathscr{D}_0$ is dense in $\boldsymbol{H}$ by assumption, and dense in $\mathscr{D}((A_1)^{1/2})$ by the Friedrichs construction. So by Theorem A the operator iB_0 has selfadjoint closure.

Thus $\bar{B}_0 = B_1$ by Lemma 2.2 and therefore $\bar{A}_0 = A_1$ by Lemma 2.3. This completes the argument in the semi-analytic case.

Finally, we finish the Stieltjes case. The following result is the analogue of Lemma 2.4.

LEMMA 2.5. *Let x be a Stieltjes vector for A_0. Then $\binom{x}{0}$ and $\binom{0}{x}$ are both quasi-analytic vectors for B_0.*

Proof. As in Lemma 2.4 we have $\|B_0^{2k}\binom{x}{0}\| \leqq \|A_0^{k+1}x\|$. So $\|B_0^{2k}\binom{x}{0}\|^{-1/2k} \geqq \|A_0^{k+1}x\|^{-1/2k}$.

Now by Lemma 2.1, A_0x is Stieltjes because x is, so

$$\sum_{k=1}^{\infty} \|A_0^{k+1}x\|^{-1/2k} = \sum_{k=1}^{\infty} \|A_0^k A_0 x\|^{-1/2k} = \infty.$$

Hence

$$\sum_{n=1}^{\infty} \left\| B_0^n \begin{pmatrix} x \\ 0 \end{pmatrix} \right\|^{-1/n} \geqq \sum_{n=1}^{\infty} \left\| B_0^{2k} \begin{pmatrix} x \\ 0 \end{pmatrix} \right\|^{-1/2k} = \infty.$$

Similarly $\|B_0^{2k}\binom{0}{x}\| = \|A_0^k x\|$ so $\binom{0}{x}$ is also quasi-analytic for B_0. □

Now the set of vectors x which are Stieltjes for A_0, namely $\mathscr{D}_s(A)$, spans $\mathscr{D}_0$. So $\mathscr{D}_s$ is total in $\mathscr{D}((A_1)^{1/2})$. Therefore the set of all vectors of the form $\binom{x}{0}$ or $\binom{0}{y}$, $x, y \in \mathscr{D}_s$, is total in $\boldsymbol{K}$. At this point we can apply Theorem QA. This finishes the Stieltjes case.

REMARK. One might at first think that the above "doubling" argument could be used to prove a "hemi-semi-analytic vector" theorem by substituting Theorem SA for Theorem A in the proof. However, this is not the case; for although we would indeed end up with a dense set of semi-analytic vectors for B_0 we could go no further, since the passage from A_0 to B_0 destroys the semiboundedness. In fact, according to Simon (personal communication), such a "theorem" (with $(4n)!$ replacing $(2n)!$) is false.

3. Quasi-analytic vectors and semigroup generators.

The main result is this:

THEOREM 3.1. *Let A be a closed operator on a Banach space X. Assume that A has an extension $\tilde{A}$ which generates a (C_0) semigroup. Assume also that A has a total set of quasi-analytic vectors. Then $A = \tilde{A}$.*

REMARK. In case X is a Hilbert space and A is dissipative this is the theorem of Hasegawa [2]. In this case it is known that the extension $\tilde{A}$ automatically exists; cf. Phillips [10].

For the proof of the theorem we need the following simple fact.

LEMMA 3.2. *Let $\{M_n\}^\infty$ be a nonnegative sequence. Suppose that $\sum_{n=1}^{\infty} M_n^{-1/n} = \infty$. Assume that $0 \leqq K_n \leqq aM_n + b^n$, $a, b \geqq 0$.*

Then $\sum_{n=1}^{\infty} K_n^{-1/n} = \infty$.

Proof. It is obviously enough to assume $a=1$. Thus we suppose $K_n \leqq M_n + b^n$. Let $E=\{n : M_n \leqq b^n\}$. Perhaps E is finite. Then for all large n we have $b^n \leqq M_n$ so $K_n \leqq 2M_n$, which obviously implies that $\sum K_n^{-1/n}$ diverges. On the other hand, if E is infinite we have $K_n \leqq 2b^n$ for infinitely many n, so that $K_n^{-1/n} \geqq 2^{-1/n} b^{-1} \geqq (2b)^{-1}$ for infinitely many n; and again $\sum K_n^{-1/n}$ diverges. □

Proof of Theorem 3.1. Let $U_t = e^{t\tilde{A}}$. Since U_t is a (C_0) semigroup there are constants $M, \beta < \infty$ with $\|U_t\| \leqq Me^{\beta t}$. (For all facts about semigroups see Hille-Phillips [3].) Now if $\lambda > \beta$ the operator $\lambda - \tilde{A}$ is surjective and bounded below. But $\lambda - \tilde{A}$ is an extension of $\lambda - A$. Thus $\lambda - A$ is also bounded below, and it has a closed range because A is closed. It will follow that $\lambda - A = \lambda - \tilde{A}$, i.e. that $A = \tilde{A}$, if we can show that $\lambda - A$ has a dense range.

Suppose that $\phi \in X^*$ annihilates the range of $\lambda - A$. Thus for all $x \in \mathscr{D}(A)$,

$$\langle \phi, Ax \rangle = \lambda \langle \phi, x \rangle. \tag{1}$$

Assume that x is quasi-analytic for A. Then in particular x is a C^∞ vector, which means that the function $f(t) = \langle \phi, U_t x \rangle$ is C^∞ for $t \geqq 0$. Moreover, $f^{(n)}(t) = \langle \phi, U_t A^n x \rangle$. It follows from (1) that

$$f^{(n)}(0^+) = \langle \phi, A^n x \rangle = \lambda^n \langle \phi, x \rangle. \tag{2}$$

Define a function g by

$$\begin{aligned} g(t) &= f(t) - e^{\lambda t}\langle \phi, x \rangle \quad &&\text{for } t \geqq 0, \\ g(t) &= 0 &&\text{for } t \leqq 0. \end{aligned}$$

From (2) we see that g is C^∞. Moreover, for $t \geqq 0$,

$$g^{(n)}(t) = \langle \phi, U_t A^n x \rangle - \lambda^n e^{\lambda t} \langle \phi, x \rangle$$

so that, because $\|U_t\|$ is bounded on any bounded interval $[0, T]$,

$$K_n = \sup_{0 \leqq t \leqq T} |g^{(n)}(t)| \leqq c[\|A^n x\| + \lambda^n]$$

where c is independent of n. By Lemma 3.1, $\sum K_n^{-1/n} = +\infty$ because x is quasi-analytic. Thus g belongs to a quasi-analytic class on every finite interval.

Since $g(t)$ vanishes for $t \leqq 0$ it follows from Carleman's theorem (cf. Paley-Wiener [9]) that g is identically zero. Therefore

$$\langle \phi, U_t x \rangle = e^{\lambda t} \langle \phi, x \rangle \tag{3}$$

for every $x \in \mathscr{D}_{qa}(A)$. Since these vectors form a total set, (3) must hold for all $x \in X$. But then

$$U_t^* \phi = e^{\lambda t} \phi. \tag{4}$$

Suppose $\phi \neq 0$. Then (4) implies that $\|U_t\| = \|U_t^*\| \geqq e^{\lambda t}$. But this contradicts the estimate $\|U_t\| \leqq Me^{\beta t}$ since $\lambda > \beta$. So ϕ must be 0. Thus $\lambda - A$ has a dense range. □

REMARKS. (1) Lumer and Phillips [4, Theorem 3.2] have proven an analytic vector theorem for dissipative operators on a Banach space. Moreover, their theorem does not require our rather unsatisfactory hypothesis that an *a priori* extension to a semigroup generator exists. They avoid this essentially by directly constructing our U_t using power series.

(2) On the other hand [4] also gives an example of a dissipative operator A on a Banach space X (not a Hilbert space, of course) which does not extend to the generator of a contraction semigroup on X. It does not seem to be known if this phenomenon occurs in all non-Hilbert spaces. However, it is interesting to note that the space X in their example can be embedded in a larger space Y such that A does extend to a generator in Y. The question of when such extensions are possible is an interesting one which we hope to discuss in a future publication. But we make the point here that the hypothesis of Theorem 3.1 can be weakened in the following way: we do not need to assume that A extends to a generator $\tilde{A}$ on X; we can allow $\tilde{A}$ to act on a larger space containing X.

To be precise, assume that there is a Banach space Y with X embedded as a closed subspace in Y (the embedding need not be isometric). Assume that there is an operator B on Y with $A \subset B$, and that B generates a (C_0) semigroup U_t on Y. Suppose that A has a total set of quasi-analytic vectors. We shall show that U_t leaves X invariant. This means that the generator of U_t on X is a suitable extension $\tilde{A}$ of A, so that the hypothesis of Theorem 3.1 is satisfied.

Here is the proof that U_t leaves X invariant. Let x be quasi-analytic for A. It is enough to show that $U_t x \in X$ for all t. Note that x is a C^∞ vector for B and that

$$(d^n/dt^n)U_t x = U_t B^n x = U_t A^n x.$$

Suppose $\psi \in Y^*$ annihilates X. Define a function h by

$$h(t) = \langle \psi, U_t x \rangle, \quad t \geqq 0, \qquad h(t) = 0, \quad t \leqq 0.$$

Then h is quasi-analytic and vanishes for $t \leqq 0$, so h is identically zero. Thus $\langle \psi, U_t x \rangle = 0$. Since this is true for all $\psi \in X^\perp$ it follows that $U_t x \in X$.

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