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ABSTRACT EVOLUTION EQUATIONS AND THE MIXED PROBLEM FOR SYMMETRIC HYPERBOLIC SYSTEMS

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Abstract. In this paper we show that Kato's theory of linear evolution equations may be applied to the mixed problem for first order symmetric hyperbolic systems of partial differential equations.

1. **Introduction.** This paper is concerned with the mixed problem for the following symmetric hyperbolic system of partial differential equations:

$$(1.1)\qquad \begin{aligned} \frac{\partial u}{\partial t}+\sum_{j=1}^{m} a_j(x,t)\frac{\partial u}{\partial x_j}+b(x,t)u &= f(x,t), && x\in\Omega,\quad 0\leqq t\leqq T;\\ u(x,0) &= \phi(x), && x\in\Omega;\\ u(x,t) &\in P(x,t), && x\in\Gamma,\quad 0\leqq t\leqq T. \end{aligned}$$

The unknown $u=(u_1,\ldots,u_N)$ is a real vector-valued function, the coefficients, a_j and b, are real $N\times N$ matrix-valued functions, and the a_j are symmetric. We assume a_j and b are of class C^2 and C^1 on $\bar{\Omega}\times[0,T]$, respectively. Ω is a bounded open subset of R^m with boundary Γ of class C^3.

The results are restricted to the regular case where the *boundary matrix*

$$a_n(x,t)=\sum n_j(x)a_j(x,t),\qquad x\in\Gamma,\quad 0\leqq t\leqq T,$$

is nonsingular on $\Gamma\times[0,T]$. Here $n=(n_1,\ldots,n_m)$ is the exterior unit normal to Ω. Limits of summation will be omitted when they are clear from the context.

The *boundary subspace* $P(x,t)$ is a linear subspace of R^N which varies in a C^3 manner with $(x,t)\in\Gamma\times[0,T]$, and it is *maximal nonnegative* for each x, t. This means

$$(a_n(x,t)u,u)\geqq 0,\qquad u\in P(x,t),$$

and $P(x,t)$ is not contained in any other subspace of R^N having this property. Here (u,v) denotes the usual inner product of $u,v\in R^N$.

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Symmetric hyperbolic systems have been studied by Friedrichs ([4], [5]), Lax and Phillips [10], Cordes and Moyer [2], and others. Here we treat equation (1.1) using Kato's results [9] on linear evolution equations of the form

$$du/dt+A(t)u = f(t), \qquad 0 \leqq t \leqq T,$$

where the linear operators $-A(t)$ are the infinitesimal generators of C_0-semigroups in a Banach space. Kato established sufficient conditions for the existence of the *evolution operator* $\{U(t, s) : 0 \leqq s \leqq t \leqq T\}$ for the family $\{A(t)\}$, and he applied these results to the equation (1.1) in the case where $\Omega = R^m$ is the whole space. In this paper we extend Kato's method to the case where Ω is a bounded domain.

Before proceeding further, we make some remarks concerning notation. We shall reserve the terminology *vector-valued* function for a function $u=(u_1, \ldots, u_N)$ which has N real-valued components. X and $L^2(\Omega)$ will both be used to denote the Hilbert space of square-integrable vector-valued functions on Ω. $H^k(\Omega)$, $k=0, 1, \ldots$, is the Sobolev space of those $u \in X$ whose partial derivatives of order up to k also lie in X. $H^{-k}(\Omega)$ is the dual space of $H_0^k(\Omega)$, where $H_0^k(\Omega)$ is the closure in $H^k(\Omega)$ of the set of C^∞ vector-valued functions with compact support in Ω. The spaces $L^2(R^m)$, $H^k(R^m)$, $H^{-k}(R^m)$ are the corresponding spaces on R^m. $L^2(\Gamma)$ is the space of vector-valued functions on Γ which are square integrable with respect to the natural surface measure on Γ. The Sobolev space $H^k(\Gamma)$, $0 \leqq k \leqq 3$, consists of those vector-valued functions on Γ which coincide on coordinate neighborhoods with functions in $H^k(R^{m-1})$, and $H^{-k}(\Gamma)$ is the dual space to $H^k(\Gamma)$. Unless otherwise stated, Sobolev spaces H^k are of integral order k, so $0 \leqq k \leqq 3$, for example, means $k=0, 1, 2, 3$. We use $\| \ \|$ and $(\ , \ )$ to denote the norm and inner product in X, $L^2(R^m)$, or $L^2(\Gamma)$, and $\| \ \|_k$ to denote the norm in $H^k(\Omega)$, $H^k(R^m)$, or $H^k(\Gamma)$. The underlying space, Ω, R^m, or Γ, should be clear from the context.

If $u \in H^k(\Omega)$, $1 \leqq k \leqq 3$, then $u_0 \in H^{k-1}(\Gamma)$ denotes the trace of u on Γ. If $P(x)$, $x \in \Gamma$, is a linear subspace of R^N which varies continuously with x, then $H_P^1(\Omega)$ denotes the closed subspace of $H^1(\Omega)$ consisting of those u which satisfy the boundary conditions $u_0(x) \in P(x)$ for (almost all) $x \in \Gamma$. For basic properties of Sobolev spaces, see Hörmander [6], Lions and Magenes [11], Morrey [12], and Seeley [13].

If $u \in R^N$, then $|u|$ denotes the usual Euclidean norm of u. If a is an $N \times N$ matrix, then $|a| = \sup\{|au| : |u|=1\}$, and ${}^t a$ is the transpose of a.

If X_1, X_2 are Banach spaces, then $B(X_1, X_2)$ denotes the space of bounded operators from X_1 to X_2, and $B(X_1) \equiv B(X_1, X_1)$.

Let $A_0(t)$ be the operator defined by

$$A_0(t)u = \sum a_j(x, t)D_j u + b(x, t)u,$$

with domain, $D(A_0(t))$, equal to $H_{P_t}^1(\Omega)$. Here $P_t(x) \equiv P(x, t)$ and $D_j \equiv \partial/\partial x_j$. Let $A(t)$ denote the closure of $A_0(t)$ regarded as an unbounded operator in X. Friedrichs [5] and Lax and Phillips [10] have shown that $A(t)+\beta_t$ is m-accretive if

$$\beta_t = \sup\{|b'(x, t)| : x \in \Omega\}, \tag{1.2}$$

where $b' = \frac{1}{2}(b + {}^t b - \sum \partial a_j/\partial x_j)$. (Recall that an operator A in a real Hilbert space X is *accretive* if $(Au, u) \geqq 0$ for all $u \in D(A)$. A is *m-accretive* if $A + \lambda$ has range X for all $\lambda > 0$. This implies $-A$ generates a C_0-semigroup of contractions in X.) Thus $-A(t)$ generates a C_0-semigroup in X.

The following are the main results.

THEOREM 1. *There exists an isomorphism $S(t)$ from $H^1_{P_t}(\Omega)$ onto X such that*

$$S(t)A(t)S(t)^{-1} = A(t) + B(t),$$

where $B(t)$ is a bounded operator on X.

REMARK. According to Proposition 2.4 of [9], Theorem 1 implies that the subspace $H^1_{P_t}(\Omega)$ is *admissible* with respect to $A(t)$. This means (see [9, Definition 2.1]) the semigroup generated by $-A(t)$ leaves $H^1_{P_t}(\Omega)$ invariant and forms a C_0-semigroup in this space.

THEOREM 2. *If $P(x, t) = P(x)$ does not vary with t, then $S(t)$ in Theorem* 1 *may be chosen so that it is continuously differentiable on* $[0, T]$ *to* $B(H^1_P(\Omega), X)$ *and* $B(t)$ *is continuous on* $[0, T]$ *to* $B(X)$.

REMARK. Suppose that $P(x, t) = P(x)$ does not vary with t so that Theorems 1 and 2 are true. Then Theorems 4.1 and 6.1 of [9] may be applied to the family $\{A(t)\}$, taking $H^1_P(\Omega)$ for the subspace Y in those two theorems. Note that the stability condition (i) of Theorem 4.1 is true with $M = 1$ and $\beta = \sup_t \{\beta_t\}$, since the operators $A(t) + \beta_t$ are m-accretive. The condition (iii) of Theorem 4.1 is also easily seen to be true.

We then have the following result for equation (1.1).

THEOREM 3. *Suppose $\phi \in H^1_{P_0}(\Omega)$ and the map $t \to f(\cdot, t)$ is continuous on* $[0, T]$ *to $H^1(\Omega)$ so that $f(\cdot, t)$ belongs to $H^1_{P_t}(\Omega)$ for $0 \leqq t \leqq T$. Then* (1.1) *has a unique solution $u(x, t)$ such that the map $t \to u(\cdot, t)$ is continuously differentiable on* $[0, T]$ *to X and $u(\cdot, t)$ belongs to $H^1_{P_t}(\Omega)$ for $0 \leqq t \leqq T$.*

REMARK. If $P(x, t) = P(x)$ is independent of t, then the conclusions of Theorem 3 follow directly from Theorem 7.1 of [9]. We shall show that the general case where $P(x, t)$ varies with t may be reduced to the case $P(x, t) = P(x, 0)$ by an orthogonal transformation of the dependent variables.

The remainder of the paper is devoted to proving the above results. In §2 we construct the operator $S(t)$. §3 contains inequalities involving commutators. These are used in §4 to show that $S(t)$ is an isomorphism from $H^1_{P_t}(\Omega)$ onto X and to establish a regularity result for $S(t)$. Theorems 1, 2, and 3 are proved in §5.

The author wishes to thank Professor T. Kato, who suggested this problem, for his assistance and guidance in this research.

2. **Construction of the operator** S. We first consider Theorem 1. Here the variable t is only a parameter, and we shall omit it in the discussion and simply write $a_j(x)$, $S, \ldots,$ for $a_j(x, t)$, $S(t)$, etc.

By the Stone-Weierstrass Theorem, the set of all real-valued functions of class C^3 on Γ is dense in the Banach space of real-valued, continuous functions on Γ. It follows that, given any $\varepsilon > 0$, there exists a C^3 vector field ν such that $|-n(y)-\nu(y)| \leqq \varepsilon$ for all $y \in \Gamma$, where n is the exterior unit normal to Γ. We choose such a ν so that this inequality is satisfied for some $\varepsilon < 1$. Then $\nu(y)$ points into the interior of Ω for each $y \in \Gamma$, i.e. $(\nu(y), n(y)) < 0$.

Using this ν, we introduce new coordinates near Γ as follows. Let $\omega = \Gamma \times [0, \sigma]$, where σ is chosen small enough so that certain conditions stated below are satisfied. Consider the mapping $\omega \to R^m$ defined by

$$(y, s) \to y + s\nu(y), \qquad (y, s) \in \omega. \tag{2.1}$$

Using the fact that $\nu(y)$ is nowhere tangential to Γ, it follows that the derivative of (2.1) is nonsingular for $s=0$. Using the inverse function theorem, one may then show that (2.1) is a diffeomorphism if σ is chosen sufficiently small. Denoting the range of (2.1) by Ω' (note that $\Omega' \subset \bar{\Omega}$), the inverse $\Omega' \to \omega$ has the form

$$x \to (y(x), s(x)), \qquad x \in \Omega'. \tag{2.2}$$

Thus, $y(x) \in \Gamma$ and $s(x) \in [0, \sigma]$ may be thought of as new coordinates for $x \in \Omega'$.

The matrix

$$c(x) = \sum a_j(x)(\partial s/\partial x_j), \qquad x \in \Omega', \tag{2.3}$$

has the property that it is a strictly negative scalar multiple of the boundary matrix $a_n(x)$ for $x \in \Gamma$. This is because the vector $(\partial s/\partial x_1, \ldots, \partial s/\partial x_m)$ is an interior normal to Ω, since $\Gamma = \{x \in \Omega' : s(x) = 0\}$. Since a_n is nonsingular, σ may be chosen so that $c(x)$ is nonsingular for $x \in \Omega'$.

The spaces $L^2(\Omega')$ and $H^k(\Omega')$ are defined in the same way as $L^2(\Omega)$ and $H^k(\Omega)$. $L^2(\omega)$ denotes the space of vector-valued functions on $\omega = \Gamma \times [0, \sigma]$ which are square integrable with respect to the product measure on ω. We consider ω as a compact, C^3 manifold-with-boundary, and we shall use the Sobolev spaces $H^k(\omega)$, $-3 \leqq k \leqq 3$. For the definition and basic properties of these spaces, see Hörmander [6].

Let the operators U_0, U: $L^2(\Omega') \to L^2(\omega)$ be defined by

$$\begin{aligned} U_0 u(y, s) &= u(y + s\nu(y)), \qquad (y, s) \in \omega, \\ Uu &= U_0 h u, \end{aligned} \tag{2.4}$$

for $u \in L^2(\Omega')$. Here $h(x) = |j(x)|^{-1/2}$, where $j(x)$ is the Jacobian of the mapping (2.2). Since the map (2.1) is of class C^3, U_0 is an isomorphism from $H^k(\Omega')$ onto $H^k(\omega)$ for $0 \leqq k \leqq 3$. However h is only of class C^2, so U is an isomorphism between $H^k(\Omega')$ (resp. $H_0^k(\Omega')$) and $H^k(\omega)$ (resp. $H_0^k(\omega)$) only for $0 \leqq k \leqq 2$. Using the change of variables formula for integrals, one sees that U is unitary from $L^2(\Omega')$ to $L^2(\omega)$. By duality, U extends to an isomorphism between $H^k(\Omega')$ and $H^k(\omega)$ for $k = -1$, -2.

Let $\phi^4+\psi^4=1$ be a C^3 partition of unity for R^m with the following properties:

(i) $\phi=1$ and $\psi=0$ in a neighborhood of $\Omega\sim\Omega'$.

(ii) The support of ϕ is compact in Ω, so $\phi=0$ and $\psi=1$ near Γ.

(iii) For $x\in\Omega'$, $\phi(x)$ and $\psi(x)$ depend only on $s(x)$. The reason for this last assumption will be discussed in a moment.

In the definition of S we shall use certain matrix-valued functions defined on Γ which locally transform the boundary subspace $P(y)$ into a subspace which does not vary with y. To construct these functions, we shall use the following lemma.

LEMMA 1. *Given $y_0\in\Gamma$, there exists a neighborhood $\mathscr{U}$ of y_0 (with respect to Γ) and an orthogonal matrix-valued function $r\in C^2(\Gamma)$ such that, for $y\in\mathscr{U}$, $r(y)$ maps $P(y)$ onto $P=\{u\in R^N : u_1=\cdots=u_p=0\}$, where p is the common codimension of $P(y)$ for y belonging to the connected component of Γ which contains y_0.*

Proof. Clearly there exists $\mathscr{U}$ and r of class C^2 on $\mathscr{U}$ with the property that $r(y)P(y)=P$ for $y\in\mathscr{U}$. The problem is to extend r to all of Γ. This can be done by modifying r near $\partial\mathscr{U}$ so that it is equal to $r(y_0)$ there. For example, by shrinking the neighborhood $\mathscr{U}$, if necessary, and introducing local coordinates, we may assume that we are working in R^{m-1}, $y_0=0$, and $\mathscr{U}$ is the ball about y_0 of radius 1. We choose a C^∞ real-valued function ρ on [0, 1] with the property that $\rho(t)=1$ for t near 0 and $\rho(t)=0$ for t near 1. Then $r'(y)\equiv r(\rho(|y|)y)$ has the desired properties. □

Using this lemma, we can find an open covering $\mathscr{U}_1,\ldots,\mathscr{U}_K$ of Γ together with orthogonal matrix-valued functions $r_1,\ldots,r_K\in C^2(\Gamma)$ such that, for $k=1,\ldots,K$, and $y\in\mathscr{U}_k$, $r_k(y)$ maps $P(y)$ onto

(2.5) $$P_k = \{u\in R^N : u_1 = \cdots = u_{p_k} = 0\}.$$

(p_k is the same for those $\mathscr{U}_k$ lying in the same component of Γ.)

We choose a partition of unity for Γ

(2.6) $$\sum_{k=1}^{K} \zeta_k^2 = 1$$

such that ζ_k has support in $\mathscr{U}_k$ for each k.

Let Δ_Γ denote the Laplace-Beltrami operator on Γ. This is a negative self-adjoint operator in $L^2(\Gamma)$ if we choose $D(\Delta_\Gamma)=H^2(\Gamma)$. Let $\Lambda_\Gamma=(1-\Delta_\Gamma)^{1/2}$. Then Λ_Γ is an isomorphism from $H^k(\Gamma)$ onto $H^{k-1}(\Gamma)$ for $1\leqq k\leqq 3$. By duality Λ_Γ extends to an isomorphism from $H^k(\Gamma)$ onto $H^{k-1}(\Gamma)$ for $-2\leqq k\leqq 0$.

We shall frequently use the natural correspondence whereby a function $u(y, s)$ on $\omega=\Gamma\times[0,\sigma]$ is regarded as a function $u(\cdot, s)$ on $[0,\sigma]$ whose values are functions on Γ. Under this correspondence $L^2(\omega)$ is naturally isomorphic to $L^2([0,\sigma]; L^2(\Gamma))$, the space of square-integrable functions on $[0,\sigma]$ with values in $L^2(\Gamma)$. We also have

(2.7) $$H^k(\omega) \cong H^0([0,\sigma]; H^k(\Gamma)) \cap\cdots\cap H^k([0,\sigma]; H^0(\Gamma)),$$

for $k=1, 2, 3$. Using this correspondence Λ_Γ may be regarded as a bounded operator from $H^k(\omega)$ to $H^{k-1}(\omega)$ for $1\leqq k\leqq 3$. Λ_Γ maps $H_0^k(\omega)$ into $H_0^{k-1}(\omega)$, so, by duality, Λ_Γ extends to a bounded operator from $H^k(\omega)$ to $H^{k-1}(\omega)$ for $-2\leqq k \leqq 0$.

Let $M=U^{-1}\Lambda_\Gamma U$. Then M is a bounded operator from $H^k(\Omega')$ to $H^{k-1}(\Omega')$ for $-1\leqq k\leqq 2$. Since U is unitary, M is symmetric and bounded below by 1 if it is considered as an operator in $L^2(\Omega')$ by restricting its domain to $H^1(\Omega')$. If we regard ϕ and ψ as functions on Ω', then it follows from the assumption (iii) above that $M\phi u=\phi M u$ for $u\in H^{-1}(\Omega')$ and similarly for ψ.

Let $\Lambda=(1-\Delta)^{1/2}$ where Δ is the Laplacian in the whole space R^m. Λ is an isomorphism from $H^k(R^m)$ onto $H^{k-1}(R^m)$ for all k.

The expressions $u\to\phi\Lambda\phi u$ and $u\to\psi\zeta_k r_k^{-1}Mr_k\zeta_k\psi u$ define bounded operators from $H^k(\Omega)$ to $H^{k-1}(\Omega)$ for $-2\leqq k\leqq 3$ in the first case and for $-1\leqq k\leqq 2$ in the second. We are regarding multiplication by ϕ as a bounded operator from $H^k(\Omega)$ to $H^k(R^m)$ and also from $H^k(R^m)$ to $H^k(\Omega)$ since it has compact support in Ω. In the second case ψ vanishes in a neighborhood of $\Omega\sim\Omega'$ so that it is a bounded operator from $H^k(\Omega)$ to $H^k(\Omega')$ and also from $H^k(\Omega')$ to $H^k(\Omega)$.

The operator S is now defined by

$$Su = (A_0+\beta)u+\phi\Lambda\phi u+\sum \psi\zeta_k r_k^{-1}Mr_k\zeta_k\psi u, \tag{2.8}$$

for $u\in D(S)\equiv H_P^1(\Omega)$. Here $\beta=\beta_t$ has the value given by (1.2). One sees that S is a bounded operator from $H_P^1(\Omega)$ to X. We shall show in §§4 and 5 that S fulfills the requirements of Theorem 1.

3. **Inequalities involving commutators.** This section contains results which will be used later to prove Theorems 1 and 2. We begin with the following proposition which is due to T. Kato (unpublished).

PROPOSITION 1. *Let A be a strictly positive selfadjoint operator in a Hilbert space H, and let $B\in B(H)$ be such that B maps $D(A)$ into itself with $ABA^{-1}\in B(H)$. If*

$$\|[A, B]u\| \leqq C\|A^{1/2}u\|, \qquad \|A^{-1/2}[A, B]u\| \leqq C\|u\|, \tag{3.1}$$

for $u\in D(A)$, with a constant C, then

$$\|[A^{1/2}, B]u\| \leqq (C/2)\|u\|, \qquad u\in D(A^{1/2}), \tag{3.2}$$

$$\|[A^{1/4}, B]u\| \leqq (C/2)\pi^{-1/2}\|A^{-1/4}u\|, \qquad u\in D(A^{1/4}). \tag{3.3}$$

Here $[A, B]=AB-BA$ denotes the commutator.

Proof. Using interpolation (see Kato [7]), one sees that B maps $D(A^\alpha)$ into itself with $A^\alpha BA^{-\alpha}\in B(H)$ for $0\leqq\alpha\leqq 1$. Thus $[A^{1/2}, B]u$ and $[A^{1/4}, B]u$ are well defined for $u\in D(A^{1/2})$ and $u\in D(A^{1/4})$ respectively.

We claim that (3.1) implies

$$\|A^{-1/4}[A, B]u\| \leqq C\|A^{1/4}u\|, \qquad u\in D(A). \tag{3.4}$$

In order to see this, let $Tu = A^{-1/2}[A, B]u$, $u \in D(A)$. By (3.1), T and $A^{1/2}TA^{-1/2}$ extend to bounded operators on H with norm bounded by C. Using interpolation, we obtain $\|A^{1/4}TA^{-1/4}\| \leqq C$, which implies (3.4).

We now consider $[A^{1/2}, B]$. Since $A^{1/2}BA^{-1/2}$ belongs to $B(H)$ and $D(A)$ is a core of $A^{1/2}$, it suffices to prove (3.2) only for $u \in D(A)$. For such u, $A^{1/2}u$ is given by

$$A^{1/2}u = \pi^{-1}\int_0^\infty \lambda^{-1/2}R(\lambda)Au\, d\lambda,$$

where $R(\lambda) = (\lambda + A)^{-1}$ is the resolvent of $-A$. For fractional powers, see Kato [8] and Yosida [14]. Since $[R(\lambda)A, B] = \lambda[B, R(\lambda)] = \lambda R(\lambda)[A, B]R(\lambda)$, it follows that

$$[A^{1/2}, B]u = \pi^{-1}\int \lambda^{1/2}R(\lambda)[A, B]R(\lambda)u\, d\lambda.$$

Here and in the following, integrals are from 0 to ∞. If $v \in D(A)$, then

$$([A^{1/2}, B]u, v) = \pi^{-1}\int \lambda^{1/2}(A^{-1/4}[A, B]R(\lambda)u, A^{1/4}R(\lambda)v)\, d\lambda.$$

Using (3.4), it follows that

$$\begin{aligned} |([A^{1/2}, B]u, v)| &\leqq \pi^{-1}C\int \lambda^{1/2}\|A^{1/4}R(\lambda)u\|\ \|A^{1/4}R(\lambda)v\|\, d\lambda \\ &\leqq \pi^{-1}C\left\{\int \lambda^{1/2}(R(\lambda)^2A^{1/2}u, u)\, d\lambda\right\}^{1/2} \\ &\quad \times\left\{\int \lambda^{1/2}(R(\lambda)^2A^{1/2}v, v)\, d\lambda\right\}^{1/2}. \end{aligned}$$

Using the spectral theorem, one has $\int \lambda^{1/2}R(\lambda)^2\, d\lambda = (\pi/2)A^{-1/2}$. It follows that $|([A^{1/2}, B]u, v)| \leqq (C/2)\|u\|\ \|v\|$, from which (3.2) follows.

The inequality (3.3) is proved in a similar manner. Starting from the formula

$$A^{1/4}u = 2^{-1/2}\pi^{-1}\int_0^\infty \lambda^{-3/4}R(\lambda)Au\, d\lambda,$$

one proceeds as before to obtain

$$([A^{1/4}, B]u, v) = 2^{-1/2}\pi^{-1}\int \lambda^{1/4}(R(\lambda)u, [B, A]R(\lambda)v)\, d\lambda.$$

Using the first half of (3.1), one obtains

$$\begin{aligned} |([A^{1/4}, B]u, v)| &\leqq 2^{-1/2}\pi^{-1}C\left\{\int \lambda^{1/2}(R(\lambda)^2u, u)\, d\lambda\right\}^{1/2}\left\{\int (R(\lambda)^2Av, v)\, d\lambda\right\}^{1/2} \\ &= (C/2)\pi^{-1/2}\|A^{-1/4}u\|\ \|v\|, \end{aligned}$$

from which (3.3) follows. □

COROLLARY. *Let A be as in Proposition* 1. *Let T be an unbounded operator in H with $D(T)=D(A^{1/2})$ and $TA^{-1/2}\in B(H)$ and such that T maps $D(A^{3/2})$ into $D(A)$ with $ATA^{-3/2}\in B(H)$. If*

$$\|[A, T]v\| \leqq C\|Av\|, \qquad \|A^{-1/2}[A, T]v\| \leqq C\|A^{1/2}v\|, \tag{3.5}$$

$v\in D(A^{3/2})$, with a constant C, then

$$\|[A^{1/2}, T]v\| \leqq (C/2)\|A^{1/2}v\|, \qquad v\in D(A),$$
$$\|[A^{1/4}, T]v\| \leqq (C/2)\pi^{-1/2}\|A^{1/4}v\|, \qquad v\in D(A^{3/4}).$$

Proof. This follows by applying Proposition 1 to $B=TA^{-1/2}$. □

The next theorem is a consequence of results from the theory of singular integrals proved by Calderón [1], and it will be used frequently later.

THEOREM (CALDERÓN). *Let Δ be the Laplacian on R^m and $\Lambda=(1-\Delta)^{1/2}$. If $a\in C^1(R^m)$, then*

$$\|[\Lambda, a]u\| \leqq C\|a\|_{C^1}\|u\|, \qquad u\in H^1(R^m),$$

with a constant C independent of a and u. If $a\in C^2(R^m)$, then

$$\|[\Lambda, a]u\|_1 \leqq C'\|a\|_{C^2}\|u\|_1, \qquad u\in H^2(R^m).$$

Here $C^k(R^m)$, $k=1, 2$, is the class of real $N\times N$ matrix-valued functions on R^m which together with their first k derivatives are continuous and bounded on R^m. $\| \ \|_{C^k}$ is the usual supremum norm in this space. Also, $\| \ \|$ and $\| \ \|_1$ are the norms in $L^2(R^m)$ and $H^1(R^m)$, respectively.

The following result is essentially due to Seeley [13], but we give another proof here.

LEMMA 2. *Let Δ_Γ be the Laplace-Beltrami operator on Γ and $\Lambda_\Gamma=(1-\Delta_\Gamma)^{1/2}$. If $a\in C^2(\Gamma)$, then*

$$\|[\Lambda_\Gamma, a]u\|_k \leqq C\|a\|_{C^2}\|u\|_k, \qquad u\in H^{k+1}(\Gamma), \quad k = 0, 1, \tag{3.6}$$

with a constant C independent of a and u. Here $C^2(\Gamma)$ is the Banach space of C^2 matrix-valued functions on Γ, and $\| \ \|_k$ denotes the norm in $H^k(\Gamma)$.

Proof. For $k=0$ this can be established using Proposition 1 with $H=L^2(\Gamma)$, $A=1-\Delta_\Gamma$, and B being the operator of multiplication by a. The hypothesis (3.1) reduces to showing

$$\|[\Delta_\Gamma, a]u\|_{k-1} \leqq \text{const.}\ \|a\|_{C^2}\|u\|_k, \qquad u\in H^2(\Gamma), \quad k = 0, 1. \tag{3.7}$$

We have

$$[\Delta_\Gamma, a]u = 2(\text{grad}\,(a), \text{grad}\,(u))+u\Delta_\Gamma a,$$

and the inequality (3.7) follows from this. Therefore (3.6) is true for $k=0$.

The case $k=1$ may be proved from the case $k=0$ using the relation $\Lambda_\Gamma[\Lambda_\Gamma, a]u = [a, \Delta_\Gamma]u - [\Lambda_\Gamma, a]\Lambda_\Gamma u$. □

LEMMA 3. *Let Δ_Γ, Λ_Γ be as in Lemma 2. If $a \in C^2(\Gamma)$, then*

$$\|[\Lambda_\Gamma^{1/2}, a]u\| \leqq C\|a\|_{C^2}\|\Lambda_\Gamma^{-1/2}u\|, \qquad u \in H^1(\Gamma),$$

with a constant C independent of a and u. Here $\|\ \|$ denotes the norm of $L^2(\Gamma)$.

Proof. This is proved using Proposition 1 in the same way that (3.6) was established in the case $k=0$. □

The following lemma is again due to Seeley [13], but before stating it we make some notational comments. If V is a vector field on Γ and u is a function on Γ, then $Vu=(V, \operatorname{grad}(u))$ is the directional derivative of u in the direction V. The set of vector fields of class C^k on Γ ($k=0, 1, 2$) is a Banach space, and a norm for this space may be defined as follows. Let $\{U_i\}$ be a finite covering of Γ such that each U_i is the domain of a coordinate chart which maps U_i onto the unit ball $B \subset R^{m-1}$. With respect to the local coordinates for U_i the vector field V can be represented as $(V_{i,1}, \ldots, V_{i,m-1})$, where the $V_{i,j}$ are C^k real-valued functions on B. We define

$$\|V\|_{C^k} = \sum_{i,j} \|V_{i,j}\|_{C^k(B)}.$$

LEMMA 4. *Let Δ_Γ, Λ_Γ be as in Lemma 2. If V is a C^2 vector field on Γ, then*

$$\|[\Lambda_\Gamma, V]u\|_{k-1} \leqq C\|V\|_{C^2}\|u\|_k, \qquad u \in H^{k+1}(\Gamma), \quad k = 1, 2, \tag{3.8}$$

where the constant C is independent of V and u.

Proof. For $k=1$ the inequality (3.8) can be established using the Corollary to Proposition 1 where one takes $H=L^2(\Gamma)$, $A=1-\Delta_\Gamma$, and $Tu=Vu$. The hypothesis (3.5) reduces to showing

$$\|[\Delta_\Gamma, V]u\|_{k-2} \leqq \text{const.}\ \|V\|_{C^2}\|u\|_k, \qquad u \in H^3(\Gamma), \quad k = 1, 2. \tag{3.9}$$

This inequality can be proved by showing that it holds in the domain of any coordinate chart. When restricted to such a domain we may assume we are working in R^{m-1}. The operators V and Δ_Γ become first and second order differential operators respectively, and the inequality (3.9) is shown to be true. Thus (3.8) holds for $k=1$.

The inequality (3.8) for $k=2$ follows from the case $k=1$ together with (3.9) and the relation $\Lambda_\Gamma[\Lambda_\Gamma, V]u=[V, \Delta_\Gamma]u-[\Lambda_\Gamma, V]\Lambda_\Gamma u$. □

Now we obtain formulas for the transformation of the differential operators $D_j=\partial/\partial x_j$ under the mappings U, U_0 defined by (2.4). Using these, we consider commutators involving the operator M.

Let $(y, s) \to y+s\nu(y)$, $x \to (y(x), s(x))$, $c(x)$, and $h(x)$ be as in §2. Define $h_j=h\,\partial(h^{-1})/\partial x_j$, $y_j=\partial y/\partial x_j$, $s_j=\partial s/\partial x_j$. Put $\alpha_j(y, s)=a_j(y+s\nu(y))$ for $(y, s) \in \omega$, and let γ, δ_j, σ_j, V_j be defined in the same way from c, h_j, s_j, y_j, respectively. (Note that V_j is a vector field on ω.) Let $\delta=\sum \alpha_j\delta_j$ and

$$\mathscr{L}v = \gamma\, \partial v/\partial s+\sum \alpha_j V_j v+\delta v, \qquad v\in H^1(\omega). \tag{3.10}$$

Then we have

LEMMA 5.

$$U_0D_jU_0^{-1}v = V_jv+\sigma_j(\partial v/\partial s), \qquad UD_jU^{-1}v = V_jv+\sigma_j(\partial v/\partial s)+\delta_jv,$$
$$U\sum a_jD_jU^{-1}v = \mathscr{L}v, \qquad \|V_jv\| \leqq C\|\Lambda_\Gamma v\|,$$

$v\in H^1(\omega)$. Here $\|\ \|$ denotes the norm in $L^2(\omega)$.

Proof. In order to show the first relation, note that $U_0^{-1}v(x)=v(y(x), s(x))$. Using the chain rule, we have $D_jU_0^{-1}v=V_jv+(\partial v/\partial s)(\partial s/\partial x_j)$, where V_jv and $\partial v/\partial s$ are evaluated at $(y, s)=(y(x), s(x))$. The formula for $U_0D_jU_0^{-1}$ then follows. Since $U=U_0h$, one may obtain the formula for UD_jU^{-1} from the formula for $U_0D_jU_0^{-1}$. Using this and (2.3), one arrives at the third formula.

Note that the vector field V_j is tangential to Γ, i.e. $V_j(y, s)$ is a tangent vector to Γ for each $(y, s)\in\omega$. Therefore $(V_jv)(s)=V_j(s)v(s)$ for $s\in[0, \sigma]$. Here $V_j(s)$ denotes the vector field on Γ defined by $V_j(s)(y)=V_j(y, s)$, $y\in\Gamma$. It follows that $\|V_jv(s)\|_\Gamma^2\leqq C\|\Lambda_\Gamma v(s)\|_\Gamma^2$, where C is a constant independent of s and $\|\ \|_\Gamma$ is the norm in $L^2(\Gamma)$. Integrating from 0 to σ gives the last inequality in the lemma. □

LEMMA 6. *If $a\in C^2(\Omega')$, then*

$$\|[M, a]u\|_k \leqq C\|a\|_{C^2}\|u\|_k, \qquad u\in H^{k+1}(\Omega'), \quad k = 0, 1,$$

with a constant C independent of a and u. Here $\|\ \|_k$ is the norm in $H^k(\Omega')$.

Proof. Under the mapping U this inequality corresponds to the inequality $\|[\Lambda_\Gamma, \alpha]v\|_k\leqq C\|\alpha\|_{C^2}\|v\|_k$, where $\alpha(y, s)=a(y+sv(y))$, $v=Uu$, and the norms are now with respect to ω. This inequality follows from Lemma 2. □

LEMMA 7.

$$\|[M, D_j]u\| \leqq C\|u\|_1, \qquad u\in H^2(\Omega').$$

Here $\|\ \|$ is the norm in $L^2(\Omega')$.

Proof. Using Lemma 5 and transforming to ω by the mapping U, one sees that it suffices to prove

$$\|[\Lambda_\Gamma, V_j+\sigma_j(\partial/\partial s)+\delta_j]v\| \leqq \text{const. } \|v\|_1,$$

where $v=Uu$. This inequality is easily obtained from Lemmas 2 and 4. □

LEMMA 8. *If ρ is a C^3 function with support in the interior of Ω', then*

$$\|[\rho M\rho, \Lambda]u\| \leqq C\|u\|_1, \qquad \|[\rho M\rho, \Lambda^{1/2}]u\| \leqq C\|\Lambda^{1/2}u\|,$$

$u \in H^2(R^m)$. Here $\| \;\|$ and $\| \;\|_1$ denote the norms in $L^2(R^m)$ and $H^1(R^m)$. We are regarding $u \to \rho M \rho u$ as a bounded operator from $H^k(R^m)$ to $H^{k-1}(R^m)$ for $k=1$, $\frac{3}{2}$, 2.

Proof. Let $M' = U_0^{-1}\Lambda_\Gamma U_0 = hMh^{-1}$. M' has the advantage that it maps $H^3(\Omega')$ into $H^2(\Omega')$, while M does not. Note that $(M-M')u = h[h^{-1}, M]u$, $u \in H^1(\Omega')$. It follows from Lemma 6 that $M-M'$ extends to a bounded operator on $H^k(\Omega')$, $k=0, 1$. Thus $\rho M \rho - \rho M' \rho$ extends to a bounded operator on $H^k(R^m)$, $k=0, 1$. Using interpolation, one sees that this is also true for $k=\frac{1}{2}$. Consequently it suffices to prove the inequalities in the lemma with M replaced by M'.

We can use the Corollary to Proposition 1 with $H = L^2(R^m)$, $A = 1-\Delta$, and $Tu = \rho M' \rho u$. The hypothesis reduces to showing

$$\|[\Delta, \rho M'\rho]u\|_{k-2} \leqq C\|u\|_k, \qquad u \in H^3(R^m), \quad k = 1, 2. \tag{3.11}$$

One has

$$\begin{aligned} [\Delta, \rho M'\rho]u &= \sum (D_j[D_j, \rho M'\rho]u + [D_j, \rho M'\rho]D_j u), \\ [D_j, \rho M'\rho]u &= (\partial\rho/\partial x_j)M'\rho u + \rho[D_j, M']\rho u + \rho M'(\partial\rho/\partial x_j)u. \end{aligned} \tag{3.12}$$

Using Lemma 5, we have

$$[D_j, M']v = U_0^{-1}[V_j, \Lambda_\Gamma]U_0 v + U_0^{-1}[\sigma_j, \Lambda_\Gamma](\partial/\partial s)U_0 v,$$

$v \in H^2(\Omega')$. It follows from Lemmas 2 and 4 that

$$\|[D_j, M']v\|_{k-1} \leqq C\|v\|_k, \qquad v \in H^{k+1}(\Omega'), \quad k = 1, 2.$$

Hence

$$\|[D_j, \rho M'\rho]u\|_{k-1} \leqq C\|u\|_k, \qquad u \in H^{k+1}(R^m), \quad k = 1, 2.$$

Using duality and the fact that $(M'u, v) = (u, h^{-2}M'h^2 v)$, $u, v \in H^1(\Omega')$, one can show that this inequality holds for $k=0$. The inequality (3.11) can now be proved by combining this with (3.12). □

4. **Properties of the operator S.**

PROPOSITION 2. *S is an isomorphism from $H^{\frac{1}{2}}_P(\Omega)$ onto X.*

In the proof of this proposition we shall use the operator R defined by

$$Ru = (A_0 + \beta)u + \phi\Lambda\phi u + \psi M \psi u, \qquad u \in D(R) \equiv D(S).$$

LEMMA 9. *$S-1$ and $R-1$ are accretive when considered as operators in X, and $S-R$ extends to a bounded operator on X. It follows that Proposition 2 is true if and only if R is an isomorphism from $H^{\frac{1}{2}}_P(\Omega)$ onto X.*

Proof. If we expand (Su, u) using the formula (2.8), then $((A_0+\beta)u, u)$ is non-negative because $A_0+\beta$ is accretive, and $(\phi\Lambda\phi u, u)$ is bounded below by $\|\phi u\|^2$

because Λ is selfadjoint and bounded below by 1 when considered as an operator in $L^2(R^m)$. We have

$$\sum (Mr_k\zeta_k\psi u, r_k\zeta_k\psi u) \geqq \sum (r_k\zeta_k\psi u, r_k\zeta_k\psi u) = \|\psi u\|^2.$$

The inequality on the left is a consequence of the fact that M is symmetric and bounded below by 1 when considered as an operator in $L^2(\Omega')$, and the equality on the right follows from (2.6) and the fact that the r_k are orthogonal. Therefore

$$(Su, u) \geqq \|\phi u\|^2+\|\psi u\|^2 \geqq \|u\|^2, \qquad u \in D(S).$$

Here we have used the fact that $\phi^2+\psi^2 \geqq \phi^4+\psi^4=1$. The proof that $R-1$ is accretive is the same except the r_k and ζ_k do not appear.

One has

$$(4.1) \qquad (S-R)u = \sum \psi\zeta_k r_k^{-1}[M, r_k\zeta_k]\psi u, \qquad u \in D(S).$$

By Lemma 6, $S-R$ extends to a bounded operator on X.

Since $S-1$ and $R-1$ are accretive, it follows that S has range equal to X precisely when $S-1$ is m-accretive and similarly for R. In general, if two accretive operators differ by a bounded operator, then one is m-accretive if the other is. Thus, Proposition 2 is equivalent to the range of R being X. □

We prove that R has range equal to X by showing (1) R is closed when regarded as an operator in X, and (2) its adjoint R^* is one-to-one. The first assertion is a consequence of the following lemma.

LEMMA 10. *The operator R satisfies*

$$(4.2) \qquad \|u\|_1 \leqq C\|Ru\|, \qquad u \in D(R),$$

with a constant C.

Proof. Since $R-1$ is accretive, one has $\|u\| \leqq \|Ru\|$, $u \in D(R)$. Therefore, it suffices to show

$$(4.3) \qquad \|u\|_1^2 \leqq C(\|Ru\|^2+\|u\|^2), \qquad u \in D(R).$$

Here and in the following, C denotes a constant.

Since $\phi^4+\psi^4=1$, we have

$$(4.4) \qquad \|u\|_1^2 \leqq C(\|\phi^2 u\|_1^2+\|\psi^2 u\|_1^2), \qquad u \in H^1(\Omega).$$

Using Calderón's theorem, it is not hard to show

$$(4.5) \qquad \|\phi^2 u\|_1^2 \leqq C(\|\phi\Lambda\phi u\|^2+\|u\|^2), \qquad u \in H^1(\Omega).$$

For the term $\psi^2 u$, we shall establish the following estimate:

$$(4.6) \qquad \|\psi^2 u\|_1^2 \leqq C(\|\mathscr{A}u\|^2+\|\psi M\psi u\|^2+\|u\|^2), \qquad u \in H^1(\Omega),$$

where $\mathscr{A}$ denotes the formal differential operator $\sum a_j D_j$. In order to show (4.6), we first note that

$$\|w\|_1 \leqq C(\|\partial w/\partial s\| + \|\Lambda_\Gamma w\| + \|w\|), \qquad w \in H^1(\omega),$$

where $\| \;\|$ and $\| \;\|_1$ are the norms in $L^2(\omega)$ and $H^1(\omega)$. This inequality follows from (2.7) and the fact that Λ_Γ is an isomorphism from $H^1(\Gamma)$ onto $L^2(\Gamma)$. Now let $\mathscr{L}$ be the operator defined by (3.10). Since γ is nonsingular, one has

$$\|\partial w/\partial s\| \leqq C\left(\|\mathscr{L}w\| + \sum \|V_j w\| + \|w\|\right), \qquad w \in H^1(\omega).$$

If one combines this with the preceding inequality and uses the fact $\|V_j w\| \leqq C\|\Lambda_\Gamma w\|$, proved in Lemma 5, one obtains

$$\|w\|_1 \leqq C(\|\mathscr{L}w\| + \|\Lambda_\Gamma w\| + \|w\|), \qquad w \in H^1(\omega).$$

Using the formula $U\mathscr{A} = \mathscr{L}U$, proved in Lemma 5, it follows that

$$\|v\|_1 \leqq C(\|\mathscr{A}v\| + \|Mv\| + \|v\|), \qquad v \in H^1(\Omega'),$$

where the norms are with respect to Ω'. Putting $v = \psi^2 u$ and using the fact that M commutes with multiplication by ψ, one obtains (4.6).

Combination of (4.4), (4.5), and (4.6) leads to

$$\|u\|_1^2 \leqq C(\|\mathscr{A}u\|^2 + \|\phi\Lambda\phi u\|^2 + \|\psi M\psi u\|^2 + \|u\|^2), \tag{4.7}$$

$u \in H^1(\Omega)$. Keeping this in mind, we expand $\|Ru\|^2$. Note that in proving (4.3), we may ignore terms in R which act as bounded operators on X. Thus we may assume without loss of generality that $b = 0$, $\beta = 0$. Then

$$\|Ru\|^2 = \|\mathscr{A}u\|^2 + \|\phi\Lambda\phi u\|^2 + \|\psi M\psi u\|^2 + 2(\mathscr{A}u, Ku) + 2(\phi\Lambda\phi u, \psi M\psi u), \tag{4.8}$$

where $Ku = \phi\Lambda\phi u + \psi M\psi u$. In a moment we shall prove

$$-(\phi\Lambda\phi u, \psi M\psi u) \leqq C\|u\|_1\|u\|, \tag{4.9a}$$

$$-(\mathscr{A}u, Ku) \leqq C\|u\|_1\|u\|, \tag{4.9b}$$

$u \in D(R)$. Combining (4.7), (4.8), and (4.9ab), one arrives at

$$\|u\|_1^2 \leqq C(\|Ru\|^2 + \|u\|_1\|u\|).$$

Since $2\|u\|_1\|u\| \leqq C^{-1}\|u\|_1^2 + C\|u\|^2$, one obtains (4.3).

We now show (4.9a). Let ρ be a C^3, real-valued function on R^m such that (1) $\rho(x) = \psi(x)$ for x belonging to the support of ϕ, and (2) the support of ρ is contained in the interior of Ω'. Then $\phi\psi = \phi\rho$. It follows that $\phi\psi M\psi u = \rho M\rho\phi u$ since ϕ and M commute. Therefore, it suffices to show $-(\Lambda w, \rho M\rho w) \leqq C\|w\|_1\|w\|$, where $w = \phi u \in H^1(R^m)$ and the inner product and norms are with respect to R^m. We may assume $w \in H^2(R^m)$ since this set is dense in $H^1(R^m)$. We have

$$-(\Lambda w, \rho M\rho w) = -(\rho\Lambda^{1/2}w, M\rho\Lambda^{1/2}w) - ([\rho M\rho, \Lambda^{1/2}]\Lambda^{1/2}w, w).$$

The first term on the right is nonpositive since $M \geqq 1$. Using Lemma 8, one sees that the second term is bounded in absolute value by $C\|w\|_1\|w\|$.

Finally we show (4.9b). We may assume $u \in H^2(\Omega) \cap H_P^1(\Omega)$ since this set is dense in $H_P^1(\Omega)$. Since K is symmetric, one has $(\mathscr{A}u, Ku) = (\mathscr{A}Ku, u) + ([K, \mathscr{A}]u, u)$. If one integrates the first term by parts, one obtains $(\mathscr{A}Ku, u) = -(Ku, (\mathscr{A}+e)u) + ((Ku)_0, a_n v)_\Gamma$, where $e = \sum \partial a_j/\partial x_j$. Here $v = u_0$ and $(Ku)_0$ are the traces of u and Ku on Γ, and $(\ ,\ )_\Gamma$ denotes the inner product in $L^2(\Gamma)$. Since $\phi = 0$ on Γ, we have $(Ku)_0 = (\psi M \psi u)_0$. Using the fact that $Mu = h^{-1}U_0^{-1}\Lambda_\Gamma U_0 hu$, one can show $(Ku)_0 = h_0^{-1}\Lambda_\Gamma h_0 v$, where h_0 is the restriction of h to Γ. It follows that

$$2(\mathscr{A}u, Ku) = ([K, \mathscr{A}]u, u) - (Ku, eu) + (h_0^{-1}\Lambda_\Gamma h_0 v, a_n v)_\Gamma.$$

Using Calderón's theorem and Lemmas 6 and 7, one can show

$$\|[K, \mathscr{A}]w\| \leqq C\|w\|_1 \sum \|a_j\|_{C^2}, \qquad w \in H^2(\Omega). \tag{4.10}$$

Therefore, in order to finish the proof of (4.9b), it suffices to show

$$-(h_0^{-1}\Lambda_\Gamma h_0 v, a_n v)_\Gamma \leqq C\|u\|_1\|u\|. \tag{4.11}$$

Let $\pi(y)$, $y \in \Gamma$, be the orthogonal projection of R^N onto $P(y)$. Then $\pi(y)v(y) = v(y)$ since $u \in D(R)$. Thus

$$(h_0^{-1}\Lambda_\Gamma h_0 v, a_n v)_\Gamma = (\Lambda_\Gamma v, \alpha v)_\Gamma + (h_0^{-1}[\Lambda_\Gamma, h_0\pi]v, a_n v)_\Gamma,$$

where $\alpha = \pi a_n \pi$. Note that $\alpha(y) \geqq 0$ for $y \in \Gamma$. By Lemma 2, the second term on the right is bounded in absolute value by $C\|v\|_\Gamma^2$, where $\|\ \|_\Gamma$ denotes the norm in $L^2(\Gamma)$. For the first term on the right we have

$$(\Lambda_\Gamma v, \alpha v)_\Gamma = (\Lambda_\Gamma^{1/2}v, \alpha\Lambda_\Gamma^{1/2}v)_\Gamma + ([\alpha, \Lambda_\Gamma^{1/2}]\Lambda_\Gamma^{1/2}v, v)_\Gamma.$$

The first term on the right is nonnegative since $\alpha \geqq 0$, and the second term is bounded by $C\|v\|_\Gamma^2$, by Lemma 3. This proves $-(h_0^{-1}\Lambda_\Gamma h_0 v, a_n v)_\Gamma \leqq C\|v\|_\Gamma^2$. In order to complete the proof of (4.11), it suffices to show $\|v\|_\Gamma^2 \leqq C\|u\|_1\|u\|$. Using a partition of unity and change of variables, this inequality can be reduced to the case where $\Omega = \{x \in R^m : x_m > 0\}$ and $\Gamma = \{x \in R^m : x_m = 0\}$. Using integration by parts, we have $(v, v)_\Gamma = -2(D_m u, u)$, and the desired inequality follows from this. □

In order to finish the proof that the range of R is X, it remains to show that R^* is one-to-one, where R^* is the adjoint of R regarded as an operator in X.

The formal adjoint of A_0 is given by

$$B_0 v = -\sum D_j a_j v + {}^t b v, \qquad v \in D(B_0) \equiv H_Q^1(\Omega),$$

where $Q(x) = (a_n(x)P(x))^\perp$, $x \in \Gamma$, is the boundary subspace formally adjoint to P. The formal adjoint of R is then defined by

$$Tv = (B_0 + \beta)v + \phi\Lambda\phi v + \psi M \psi v, \qquad v \in D(T) \equiv H_Q^1(\Omega).$$

R and T are formally adjoint to each other, i.e. $(Ru, v)=(u, Tv)$, $u \in D(R)$, $v \in D(T)$. It follows that T is closable when regarded as an operator in X and $\tilde{T} \subseteq R^*$, where $\tilde{T}$ is the closure of T.

The operator $B_0+\beta$ is accretive; this is proved by Friedrichs [5]. Using the argument in Lemma 9, one sees that $T-1$ is accretive. Hence $\|u\| \leqq \|\tilde{T}u\|$, $u \in D(\tilde{T})$, and $\tilde{T}$ is one-to-one. In order to show R^* is one-to-one (and to complete the proof of Proposition 2), it suffices to prove the following lemma.

LEMMA 11. $\tilde{T}=R^*$.

Proof. In the terminology of Friedrichs [3], $\tilde{T}$ is the strong extension of T and R^* is the weak extension of T, so the proposition asserts the equivalence of the weak and strong extensions of T. Friedrichs ([3], [4], [5]) and Lax and Phillips [10] have shown the identity of weak and strong extensions of first order partial differential operators, and this proof is an extension of their methods to the case at hand.

Let $\xi^2+\eta^2=1$ be a C^3 partition of unity for R^m with the following properties:

(i) $\xi=1$ and $\eta=0$ in a neighborhood of the support of ϕ, so that $\xi\phi=\phi$, $\eta\phi=0$, and $\eta\psi=\eta$.

(ii) The support of ξ is relatively compact in Ω; in particular, $\xi=0$ and $\eta=1$ near Γ.

(iii) For $x \in \Omega'$, $\xi(x)$ and $\eta(x)$ depend only on $s(x)$. This implies that $M\xi u=\xi Mu$, $u \in H^{-1}(\Omega')$, and similarly for η.

Since the inclusion $\tilde{T} \subseteq R^*$ has already been shown, it remains to prove the opposite inclusion. Let $v \in D(R^*)$ with $R^*v=f$. We shall show $\xi^2 v$, $\eta^2 v \in D(\tilde{T})$.

We first show $\xi^2 v \in D(\tilde{T})$. Let $J_\varepsilon=(1+\varepsilon\Lambda)^{-1}$, $\varepsilon>0$. For each k, J_ε maps $H^k(R^m)$ into itself with norm uniformly bounded in ε. If $u \in H^k(R^m)$, then $J_\varepsilon u \to u$ in $H^k(R^m)$ as $\varepsilon \to 0$. Furthermore J_ε maps $H^k(R^m)$ into $H^{k+1}(R^m)$.

Let $H_\varepsilon u=\xi J_\varepsilon \xi u$, where we regard multiplication by ξ as an operator from $H^k(\Omega)$ to $H^k(R^m)$ and also from $H^k(R^m)$ to $H^k(\Omega)$, $|k| \leqq 3$. Then, for $|k| \leqq 3$, H_ε maps $H^k(\Omega)$ into itself with norm uniformly bounded in ε. If $u \in H^k(\Omega)$, then $H_\varepsilon u \to \xi^2 u$ in $H^k(\Omega)$. (From now on we omit "$\varepsilon \to 0$" when it is clear from the context.) For $-3 \leqq k \leqq 2$, H_ε maps $H^k(\Omega)$ into $H^{k+1}(\Omega)$.

Let $v_\varepsilon=H_\varepsilon v$. Then $v_\varepsilon \in D(T)$ since $\xi=0$ near Γ. Also $v_\varepsilon \to \xi^2 v$ in X. In order to show $\xi^2 v \in D(\tilde{T})$, it remains to show Tv_ε converges in X.

Let $T_1\colon X \to H^{-1}(\Omega)$ be defined by

$$T_1 u = -\sum D_j a_j u+({}^t b+\beta)u+\phi\Lambda\phi u+\psi M\psi u, \qquad u \in X.$$

Note that T_1 is an extension of T. It is not hard to see that $T_1 v=f$. Then we have $Tv_\varepsilon=H_\varepsilon f+[T_1, H_\varepsilon]v$. Since $H_\varepsilon f \to \xi^2 f$ in X, it remains to show $[T_1, H_\varepsilon]v$ converges in X. In fact we shall show

$$[T_1, H_\varepsilon]u \text{ converges in } X \quad \text{for all } u \in X. \tag{4.12}$$

The first step is to show

$$\|[T_1, H_\varepsilon]u\| \leqq C\|u\|, \qquad u \in X, \quad \varepsilon > 0, \tag{4.13}$$

with a constant C independent of ε.

Let χ be a C^∞ real-valued function with compact support in Ω with the property that $\chi(x)=1$ for x belonging to the support of ξ. Then $\chi\xi=\xi$ and

$$[T_1, H_\varepsilon]u = [T_1, \xi]\chi J_\varepsilon \xi u + \xi[\chi T_1 \chi, J_\varepsilon]\xi u + \xi J_\varepsilon \chi[T_1, \xi]u. \tag{4.14}$$

In the first and third terms on the right we have

$$\|[T_1, \xi]w\| \leqq C\|w\|, \qquad w \in H^1(\Omega). \tag{4.15}$$

This follows from Calderón's theorem and Lemma 6. In the second term we have

$$[\chi T_1 \chi, J_\varepsilon]w = \varepsilon J_\varepsilon[\Lambda, \chi T_1 \chi]J_\varepsilon w = J_\varepsilon[\Lambda, \chi T_1 \chi]\Lambda^{-1}(1-J_\varepsilon)w,$$

$w \in L^2(R^m)$. It follows from Calderón's theorem and Lemma 8 that $\|[\Lambda, \chi T_1 \chi]v\| \leqq C\|v\|_1$, $v \in H^2(R^m)$. Therefore

$$\|[\chi T_1 \chi, J_\varepsilon]w\| \leqq C\|w\|, \qquad w \in H^1(R^m), \quad \varepsilon > 0.$$

Since $H^1(R^m)$ is dense in $L^2(R^m)$, this holds for $u \in L^2(R^m)$. Combining this inequality and (4.15) with (4.14) proves (4.13).

Having established the inequality (4.13), we can prove (4.12) using the following well-known principle:

(4.16) Let $A_\varepsilon: X_1 \to X_2$, $\varepsilon>0$, be a family of continuous linear operators between Banach spaces X_1 and X_2. Suppose the A_ε are uniformly bounded in norm with respect to ε and $A_\varepsilon u$ converges in X_2 as $\varepsilon \to 0$ for all u belonging to a dense subset of X_1. Then $A_\varepsilon u$ converges in X_2 for all u in X_1.

In the particular case of (4.12) one sees that $[T_1, H_\varepsilon]u$ converges in X for all $u \in H^1(\Omega)$. Since $H^1(\Omega)$ is dense in X, the assertion (4.12) follows. This completes the proof that $\xi^2 v \in D(\tilde{T})$.

In order to show $\eta^2 v \in D(\tilde{T})$, we transform from Ω to ω. However, we first make the following observation. Suppose C is a bounded operator on X. Then Lemma 11 is true if and only if $(R+C)^*$ coincides with the closure of $T+C^*$. Thus we may assume without loss of generality that $b=0$, $\beta=0$ in R and T.

Let

$$Eu = \mathscr{L}u + \Lambda_\Gamma u, \tag{4.17}$$

where the domain of E consists of those $u \in H^1(\omega)$ which satisfy the boundary conditions $u(y, 0) \in P(y)$, $u(y, \sigma)=0$ for a.a. $y \in \Gamma$. If $u \in D(E)$, then $\eta U^{-1}u \in D(R)$ and

$$R\eta U^{-1}u = \eta U^{-1}Eu + \sum a_j(\partial\eta/\partial x_j)U^{-1}u. \tag{4.18}$$

Here we have used Lemma 5 and the fact that $\eta\phi=0$. It follows that if E^* is the adjoint of E regarded as an operator in $L^2(\omega)$, then $U\eta v \in D(E^*)$ with

$$E^*U\eta v = f' \equiv U\left(\eta f-\sum (\partial\eta/\partial x_j)a_j v\right).$$

The formal adjoint of E is given by $Fw=\mathscr{M}w+\Lambda_\Gamma w$, where

$$\mathscr{M}w = -(\partial/\partial s)\gamma w-\sum V_j\alpha_j w+\delta_1 w,$$

with $\delta_1=\delta-\sum \operatorname{div}(V_j)\alpha_j$. Here we are using the fact that if V is a vector field on Γ, then V and $-V-\operatorname{div}(V)$ are formally adjoint as operators in $L^2(\Gamma)$. The domain of F consists of those $w \in H^1(\omega)$ which satisfy the boundary conditions $w(y, 0) \in Q(y)$ for a.a. $y \in \Gamma$. We note that $(Eu, w)=(u, Fw)$, $u \in D(E)$, $w \in D(F)$. Thus $\tilde{F}\subseteq E^*$. In a moment we shall show $E^*=\tilde{F}$. Assuming this, it follows that there exists a sequence $\{w_n\}\subset D(\tilde{F})$ with $w_n \to U\eta v$ and $Fw_n \to f'$ in $L^2(\omega)$. Then $\eta U^{-1}w_n \in D(T)$, $\eta U^{-1}w_n \to \eta^2 v$ in X. Using the fact that $U(-\sum D_j a_j)U^{-1}=\mathscr{M}$, one can show

$$T\eta U^{-1}w_n = \eta U^{-1}Fw_n-\sum (\partial\eta/\partial x_j)a_j U^{-1}w_n.$$

Thus $T\eta U^{-1}w_n$ converges in X. We conclude $\eta^2 v \in D(\tilde{T})$.

It remains to show $E^*=\tilde{F}$. We first reduce the problem to the case where the boundary subspace $P(y)$ is independent of y. Let r_k, ζ_k be as in §2. Let $v \in D(E^*)$. In order to show $v \in D(\tilde{F})$, it suffices to show $\zeta_k^2 v \in D(\tilde{F})$, $k=1, \dots, K$. Fix k and let $\zeta=\zeta_k$, $r=r_k$ and

$$Lu = \gamma r^{-1}\,\partial u/\partial s+\sum \alpha_j r^{-1}V_j u+\delta r^{-1}u+\Lambda_\Gamma r^{-1}u,$$

with the domain of L consisting of those $u \in H^1(\omega)$ which satisfy the boundary conditions

$$u(y, 0) \in P_k, \qquad u(y, \sigma) = 0, \quad \text{for a.a. } y \in \Gamma. \tag{4.19}$$

If $u \in D(L)$, then $\zeta r^{-1}u \in D(E)$ and

$$E\zeta r^{-1}u = \zeta Lu+\sum \alpha_j V_j(\zeta r^{-1})u+[\Lambda_\Gamma, \zeta]r^{-1}u.$$

By Lemma 2, $[\Lambda_\Gamma, \zeta]$ extends to a bounded operator on $L^2(\omega)$. It follows that $\zeta v \in D(L^*)$ with

$$L^*\zeta v = \zeta r E^* v-\sum V_j(\zeta r)\alpha_j v-r[\Lambda_\Gamma, \zeta]v.$$

The formal adjoint of L is

$$Mw = -(\partial/\partial s)r\gamma w-\sum V_j r\alpha_j w+r\delta_1 w+r\Lambda_\Gamma w,$$

where the domain of M consists of those $w \in H^1(\omega)$ which satisfy the adjoint boundary conditions $w(y, 0) \in Q_k(y)$, where

$$Q_k(y) = (\gamma(y, 0)r(y)^{-1}P_k)^\perp.$$

We have $(Lu, w)=(u, Mw)$, $u \in D(L)$, $w \in D(M)$, so $\tilde{M} \subseteq L^*$. In a moment we shall show $L^*=\tilde{M}$. Assuming this, it follows that there exists a sequence $\{w_n\} \subset D(M)$ with $w_n \to \zeta v$, $Mw_n \to L^*\zeta v$ in $L^2(\omega)$. For y belonging to the support of ζ, we have $r^{-1}(y)P_k=P(y)$. For such y, one has $Q_k(y)=Q(y)$. Thus $\{\zeta w_n\} \subset D(F)$. Also $\zeta w_n \to \zeta^2 v$ and

$$F\zeta w_n = \zeta r^{-1}Mw_n - \sum V_j(\zeta r^{-1})r\alpha_j w_n + [\Lambda_\Gamma, \zeta]w_n,$$

which converges in $L^2(\omega)$ since $[\Lambda_\Gamma, \zeta]$ extends to a bounded operator on $L^2(\omega)$. It follows that $\zeta^2 v \in D(\tilde{F})$.

We now show $L^*=\tilde{M}$. By multiplying L by $r\gamma^{-1}$, we may assume L has the form $Lu=\partial u/\partial s+Gu$, where $Gu=\sum a_j V_j u+bu+c_1\Lambda_\Gamma c_2 u$, where $a_j=r\gamma^{-1}\alpha_j r^{-1}$, $b=r\gamma^{-1}\delta r^{-1}$, $c_1=r\gamma^{-1}$, and $c_2=r^{-1}$. Then $Mv=-\partial v/\partial s+Hv$, where

$$Hv = -\sum V_j\, {}^t a_j v+\left({}^t b-\sum \operatorname{div}(V_j)\, {}^t a_j\right)v+{}^t c_2 \Lambda_\Gamma\, {}^t c_1 v,$$

and the adjoint boundary conditions become $v(y, 0) \in P_k^\perp$.

To show $L^* \subseteq \tilde{M}$, we use the mollifier $K_\varepsilon=(1+\varepsilon\Lambda_\Gamma)^{-1}$, $\varepsilon>0$. For $-1 \leqq k \leqq 1$, K_ε maps $H^k(\Gamma)$ into itself with norm uniformly bounded in ε. For $u \in H^k(\Gamma)$, $K_\varepsilon u \to u$ in $H^k(\Gamma)$. Furthermore K_ε maps $H^k(\Gamma)$ into $H^{k+1}(\Gamma)$ for $k=-1, 0$.

Using the natural correspondence (2.7), we shall regard K_ε as mapping $H^k(\omega)$ into itself for $-1 \leqq k \leqq 1$. Then, for $u \in H^k(\omega)$, $K_\varepsilon u \to u$ in $H^k(\omega)$.

Given $v \in D(L^*)$, let $v_\varepsilon=K_\varepsilon v$. Then $v_\varepsilon \to v$ in $L^2(\omega)$. Since K_ε maps $H^0(\Gamma)$ into $H^1(\Gamma)$, it follows that the first order derivatives of v_ε along directions tangential to Γ lie in $L^2(\omega)$. In order to show v_ε belongs to $H^1(\omega)$, it suffices to show $\partial v_\varepsilon/\partial s$ belongs to $L^2(\omega)$.

If $L^*v=f$, then $-\partial v/\partial s+Hv=f$ when we regard $\partial v/\partial s$ and Hv as elements of $H^{-1}(\omega)$. Then $\partial v_\varepsilon/\partial s=K_\varepsilon(\partial/\partial s)v=-K_\varepsilon f+K_\varepsilon Hv$. One has $K_\varepsilon Hv \in L^2(\omega)$, since K_ε maps $H^{-1}(\Gamma)$ and $L^2(\Gamma)$ and the operator H only involves differentiation in the y component of a function $v(y, s)$ on ω. Thus $v_\varepsilon \in H^1(\omega)$.

We claim v_ε satisfies the boundary conditions $v_\varepsilon(y, 0) \in P_k^\perp$. Suppose $u \in D(L)$. Let $u(0)$, $(K_\varepsilon u)(0)$, denote the traces of u and $K_\varepsilon u$ on $\Gamma \times \{0\}$. Then $(K_\varepsilon u)(0) =K_\varepsilon(u(0))$. It follows that $K_\varepsilon u$ satisfies the boundary conditions (4.19) and hence belongs to $D(L)$. We have

$$(LK_\varepsilon u, v) = (K_\varepsilon u, f) = (u, K_\varepsilon f), \tag{4.20}$$

since $L^*v=f$. On the other hand, we have $(LK_\varepsilon u, v)=(\partial u/\partial s, v_\varepsilon)+(GK_\varepsilon u, v)$, since K_ε and $\partial/\partial s$ commute. Integrating by parts, one obtains

$$(\partial u/\partial s, v_\varepsilon) = -(u, \partial v_\varepsilon/\partial s)-(u(0), v_\varepsilon(0))_\Gamma.$$

We use here the fact that $u(y, \sigma)=0$ for a.a. $y \in \Gamma$. Thus

$$(LK_\varepsilon u, v) = -(u, \partial v_\varepsilon/\partial s)-(u(0), v_\varepsilon(0))_\Gamma+(v, K_\varepsilon Hv) = (u, K_\varepsilon f)-(u(0), v_\varepsilon(0))_\Gamma.$$

Combining this with (4.20) gives $(u(0), v_\varepsilon(0))_\Gamma = 0$, $u \in D(L)$. Since the values of $u(0)$ can be chosen arbitrarily subject to the restriction (4.19), it follows that v_ε satisfies the boundary conditions $v_\varepsilon(y, 0) \in P_k^\perp$. Therefore v_ε belongs to $D(M)$.

It remains to show $Mv_\varepsilon \to f$ in $L^2(\omega)$. We have $Mv_\varepsilon = K_\varepsilon f + [H, K_\varepsilon]v$. Since $f \in L^2(\omega)$, $K_\varepsilon f \to f$ in $L^2(\omega)$. We claim $[H, K_\varepsilon]v \to 0$ in $L^2(\omega)$. In fact, we shall show

$$(4.21) \qquad [H, K_\varepsilon]w \to 0 \quad \text{in } L^2(\omega) \quad \text{for all } w \in L^2(\omega).$$

The proof of this is very similar to (4.12). The crucial step is to establish the inequality

$$(4.22) \qquad \|[H, K_\varepsilon]w\| \leqq C\|w\|, \qquad w \in L^2(\omega), \quad \varepsilon > 0,$$

and then we can apply the principle (4.16). We have

$$[H, K_\varepsilon]w = K_\varepsilon[\Lambda_\Gamma, H]\Lambda_\Gamma^{-1}(1-K_\varepsilon)w.$$

Using Lemmas 2 and 4, one sees that $\|[\Lambda_\Gamma, H]\Lambda_\Gamma^{-1}v\| \leqq C\|v\|$, $v \in H^1(\omega)$. Using this inequality, one easily obtains (4.22).

Since $[H, K_\varepsilon]w \to 0$ in $L^2(\omega)$ for $w \in H^1(\omega)$, and $H^1(\omega)$ is dense in $L^2(\omega)$, one obtains (4.21). Thus $Mv_\varepsilon \to f$ and $v \in D(\tilde{M})$. □

This concludes the proof of Proposition 2. We now prove a regularity result for S.

PROPOSITION 3. *If $v \in H^1(\Omega)$ then $S^{-1}v \in H^2(\Omega)$.*

Proof. Let $u = S^{-1}v$. Then $u \in H_P^1(\Omega)$, $Su \in H^1(\Omega)$, and we must show $u \in H^2(\Omega)$. $(S-R)u$ is given by (4.1) and, by Lemma 6, $[M, r_k\zeta_k]$ is a bounded operator on $H^1(\Omega')$. Therefore $(S-R)u$ belongs to $H^1(\Omega)$. Since $Su \in H^1(\Omega)$, it follows that $Ru \in H^1(\Omega)$.

Let $\xi^2 + \eta^2 = 1$ be the partition of unity introduced in the proof of Lemma 11. We must show $\xi^2 u$, $\eta^2 u \in H^2(\Omega)$.

Let $\{J_\varepsilon : \varepsilon > 0\}$ be the operators introduced in the proof of Lemma 11 and $G_\varepsilon u = \varepsilon^{-1}\xi(1-J_\varepsilon)\xi u$. Note that $\varepsilon^{-1}(1-J_\varepsilon) = \Lambda J_\varepsilon$. We claim that in order to show $\xi^2 u \in H^2(\Omega)$, it suffices to show $G_\varepsilon u$ converges in $H^1(\Omega)$. Suppose the latter is true. Then

$$G_\varepsilon u - \varepsilon^{-1}(1-J_\varepsilon)\xi^2 u = \varepsilon^{-1}[J_\varepsilon, \xi]\xi u = J_\varepsilon[\xi, \Lambda]J_\varepsilon \xi u.$$

By Calderón's theorem, $[\xi, \Lambda]$ is a bounded operator on $H^1(R^m)$, so the right side converges in $H^1(R^m)$. Therefore $J_\varepsilon \Lambda \xi^2 u$ converges in $H^1(R^m)$. So $\Lambda \xi^2 u \in H^1(R^m)$ which implies $\xi^2 u \in H^2(\Omega)$. This proves the claim.

It remains to show $G_\varepsilon u$ converges in $H^1(\Omega)$. Note that $G_\varepsilon u \in D(R)$ since $\xi = 0$ near Γ. By (4.2), we are reduced to proving $RG_\varepsilon u$ converges in X. We have

$$RG_\varepsilon u = G_\varepsilon Ru + [R, G_\varepsilon]u.$$

Since $Ru \in H^1(\Omega)$, $G_\varepsilon Ru \to \xi\Lambda\xi Ru$ in X.

It remains to show that $[R, G_\varepsilon]u$ converges in X. In fact we shall show that $[R, G_\varepsilon]w$ converges in X for all $w \in H^1(\Omega)$. (Note that in proving this we do not use the fact that w satisfies the boundary conditions $w_0(y) \in P(y)$, $y \in \Gamma$, so we shall assume that $P(y) = R^N$ and $D(R) = H^1(\Omega)$.) The proof is similar to the proof of (4.12). We establish the inequality

$$\|[R, G_\varepsilon]w\| \leqq C\|w\|_1, \qquad w \in H^1(\Omega), \quad \varepsilon > 0, \tag{4.23}$$

and then apply the principle (4.16). Let χ be the function introduced in connection with the proof of (4.13). Then

$$[R, G_\varepsilon]w = [R, \xi]\chi\Lambda J_\varepsilon \xi w + \xi J_\varepsilon[\chi R\chi, \Lambda]J_\varepsilon \xi w + \xi \Lambda J_\varepsilon \chi[R, \xi]w. \tag{4.24}$$

Application of Calderón's theorem and Lemma 6 shows that

$$\|[R, \xi]w\|_k \leqq C\|w\|_k, \qquad w \in H^{k+1}(\Omega), \quad k = 0, 1.$$

Using Calderón's theorem and Lemma 8, one obtains

$$\|[\chi R\chi, \Lambda]v\| \leqq C\|v\|_1, \qquad v \in H^2(R^m).$$

Combination of these last two inequalities with (4.24) proves (4.23).

Note that if $w \in H^2(\Omega)$, then $G_\varepsilon w \to \xi\Lambda\xi w$ in $H^1(\Omega)$. Thus $[R, G_\varepsilon]w \to [R, \xi\Lambda\xi]w$ in X for $w \in H^2(\Omega)$. Since $H^2(\Omega)$ is dense in $H^1(\Omega)$, it follows that $[R, G_\varepsilon]w$ converges in X for all $w \in H^1(\Omega)$. Thus we have proved $\xi^2 u \in H^2(\Omega)$.

We now show $\eta^2 u \in H^2(\Omega)$. It suffices to show $U\eta^2 u \in H^2(\omega)$ which in turn reduces to showing $\Lambda_\Gamma U\eta^2 u$ and $(\partial/\partial s)U\eta^2 u$ belong to $H^1(\omega)$. To show this, we shall use the operators $\{K_\varepsilon : \varepsilon > 0\}$ introduced in the proof of Lemma 11. In order to show $\Lambda_\Gamma U\eta^2 u \in H^1(\omega)$, it suffices to show $K_\varepsilon \Lambda_\Gamma U\eta^2 u = \varepsilon^{-1}(1-K_\varepsilon)U\eta^2 u$ converges in $H^1(\omega)$. Since η commutes with $U^{-1}K_\varepsilon U$, it suffices to show $\varepsilon^{-1}\eta U^{-1}(1-K_\varepsilon)w$ converges in $H^1(\Omega)$, where $w = U\eta u$.

Let $L_\varepsilon = \sum \zeta_k r_k^{-1} K_\varepsilon r_k \zeta_k$. Using (2.6), we have

$$\varepsilon^{-1}(K_\varepsilon - L_\varepsilon)w = \varepsilon^{-1} \sum \zeta_k r_k^{-1}[r_k \zeta_k, K_\varepsilon]w. \tag{4.25}$$

Note that $\varepsilon^{-1}[r_k\zeta_k, K_\varepsilon]w = K_\varepsilon[\Lambda_\Gamma, r_k\zeta_k]K_\varepsilon w$. This converges in $H^1(\omega)$ as $\varepsilon \to 0$ by Lemma 2. Thus it suffices to show

$$\varepsilon^{-1}\eta U^{-1}(1-L_\varepsilon)w \quad \text{converges in } H^1(\Omega). \tag{4.26}$$

We claim that $\varepsilon^{-1}\eta U^{-1}(1-L_\varepsilon)w$ belongs to $D(R)$. Since $u \in D(R)$, it follows that $w(y, 0) \in P(y)$ for a.a. $y \in \Gamma$. Therefore $r_k(y)\zeta_k(y)w(y, 0) \in P_k$ for a.a. $y \in \Gamma$, which implies $(K_\varepsilon r_k \zeta_k w)(y, 0) \in P_k$ for a.a. $y \in \Gamma$. From this one concludes $(L_\varepsilon w)(y, 0) \in P(y)$ for a.a. $y \in \Gamma$. The claim then follows.

Using (4.2), one sees that the assertion (4.26) reduces to showing $\varepsilon^{-1}R\eta U^{-1}(1-L_\varepsilon)w$ converges in X. We may assume without loss of generality that $b=0$, $\beta=0$ in R. Let E be the operator given by (4.17). Using (4.18), one sees that it suffices to show $\varepsilon^{-1}E(1-L_\varepsilon)w$ converges in $L^2(\omega)$. For this we do not use the fact that w satisfies the boundary conditions $w(y,0)\in P(y)$, $y\in\Gamma$, so we shall assume that $P(y)=R^N$ and $D(E)=H^1(\omega)$. Again using (4.25), one sees that it suffices to show $\varepsilon^{-1}E(1-K_\varepsilon)w$ converges in $L^2(\omega)$. One has

$$\varepsilon^{-1}E(1-K_\varepsilon)w = \varepsilon^{-1}(1-K_\varepsilon)Ew+\varepsilon^{-1}[K_\varepsilon, E]w.$$

It is not hard to show that $Ru\in H^1(\Omega)$ implies $Ew\in H^1(\omega)$. Therefore the first term on the right converges to $\Lambda_\Gamma Ew$ in $L^2(\omega)$.

It remains to show $\varepsilon^{-1}[K_\varepsilon, E]w$ converges in $L^2(\omega)$. We shall show that $\varepsilon^{-1}[K_\varepsilon, E]v$ converges in X for all $v\in H^1(\omega)$. We first establish the inequality

$$\|\varepsilon^{-1}[K_\varepsilon, E]v\| \leqq C\|v\|_1, \qquad v\in H^1(\omega), \quad \varepsilon>0, \tag{4.27}$$

and then apply (4.16). We have

$$\varepsilon^{-1}[K_\varepsilon, E]v = K_\varepsilon[E, \Lambda_\Gamma]K_\varepsilon v. \tag{4.28}$$

Using Lemmas 2 and 5, one sees that $\|[E, \Lambda_\Gamma]u\|\leqq C\|u\|_1$, $u\in H^2(\omega)$. Combining this with (4.28), one obtains (4.27). Since $\varepsilon^{-1}[K_\varepsilon, E]v=\varepsilon^{-1}[(I-K_\varepsilon), E]v$ converges to $[\Lambda_\Gamma, E]v$ in $L^2(\omega)$ for $v\in H^2(\omega)$ and $H^2(\omega)$ is dense in $H^1(\omega)$, it follows that $\varepsilon^{-1}[K_\varepsilon, E]v$ converges in X for all $v\in H^1(\omega)$. This concludes the proof that $\Lambda_\Gamma U\eta^2u\in H^1(\omega)$.

It remains to show $(\partial/\partial s)U\eta^2u\in H^1(\omega)$. It was noted above that $Ew=EU\eta u \in H^1(\omega)$. It follows that $EU\eta^2u\in H^1(\omega)$. Using the fact that $\Lambda_\Gamma U\eta^2u\in H^1(\omega)$ and the fact that the $V_j\Lambda_\Gamma^{-1}$ are bounded operators in $H^1(\omega)$, it follows that $\gamma(\partial/\partial s)U\eta^2u\in H^1(\omega)$. Since γ is nonsingular, this implies $(\partial/\partial s)U\eta^2u\in H^1(\omega)$. □

5. **Proof of Theorems 1–3.** In order to show that S defined by (2.8) satisfies the requirements of Theorem 1, it remains to prove the following proposition.

PROPOSITION 4. *$SAS^{-1}=A+B$, where $B\in B(X)$.*

Proof. We first show that $u\in H^1_P(\Omega)$ implies $AS^{-1}u\in H^1_P(\Omega)$. Let $v=S^{-1}u$. Then $Sv=u\in H^1_P(\Omega)$ and, by Proposition 3, $v\in H^1_P(\Omega)\cap H^2(\Omega)$. We must show $Av\in H^1_P(\Omega)$.

Note that $\phi\Lambda\phi v\in H^1_P(\Omega)$ since $\phi=0$ near Γ. Next we claim

$$\psi\zeta_k r_k^{-1}Mr_k\zeta_k\psi v\in H^1_P(\Omega), \qquad k=1,\ldots,K. \tag{5.1}$$

To prove this, note that $v\in H^1_P(\Omega)\cap H^2(\Omega)$ implies $r_k\zeta_k\psi v\in H^2(\Omega')$ and $(r_k\zeta_k\psi v)(y)\in P_k$ for a.a. $y\in\Gamma$ since $r_k(y)$ maps $P(y)$ onto P_k for y belonging to the support of ζ_k. Therefore $w=Ur_k\zeta_k\psi v\in H^2(\omega)$ and $w(y,0)\in P_k$ for a.a. $y\in\Gamma$. It follows that $(\Lambda_\Gamma w)(y,0)\in P_k$ for a.a. $y\in\Gamma$ and $(Mr_k\zeta_k\psi v)(y)\in P_k$ for a.a. $y\in\Gamma$. The assertion (5.1) follows from this. We conclude $Av\in H^1_P(\Omega)$.

It follows that $H^1_P(\Omega)\subset D(SAS^{-1})$, and for $u\in H^1_P(\Omega)$ we have

$$SAS^{-1}u-Au = [S, A]S^{-1}u = [Z, A]S^{-1}u,$$

with

$$Zv = \phi\Lambda\phi v+\sum \psi\zeta_k r_k^{-1}Mr_k\zeta_k\psi v, \qquad v\in H^1(\Omega).$$

Note that Z is an extension of $S-A-\beta$. We claim

$$\|[Z, A]v\| \leqq C\|v\|_1\left(\sum\|a_j\|_{C^2}+\|b\|_{C^1}\right), \qquad v\in H^1_P(\Omega)\cap H^2(\Omega). \tag{5.2}$$

If $Kv=\phi\Lambda\phi v+\psi M\psi v$ is as in Lemma 10, then $(Z-K)v=(S-R)v$ is given by (4.1) with u replaced by v. We have already noted that $S-R$ is a bounded operator on both X and $H^1(\Omega)$. Therefore (5.2) follows from (4.10) since $Av=\mathscr{A}v+bv$.

It follows from (5.2) that $B=[Z, A]S^{-1}$ extends to a bounded operator on X. Thus we have $SAS^{-1}u=Au+Bu$ for $u\in H^1_P(\Omega)$. Since $H^1_P(\Omega)$ is a core of A, it follows that this holds for $u\in D(A)$. So SAS^{-1} is an extension of $A+B$. It follows that $S(A+\lambda)S^{-1}\supset A+B+\lambda$ for all λ. If λ is large, the right side has range X and the left side is one-to-one. Therefore we have equality: $SAS^{-1}=A+B$. □

Theorem 1 is a direct consequence of Propositions 2 and 4.

Proof of Theorem 2. Let

$$S(t)u = (A_0(t)+\beta)u+\phi\Lambda\phi u+\sum \psi\zeta_k r_k^{-1}Mr_k\zeta_k\psi u,$$

for $u\in D(S(t))\equiv H^1_P(\Omega)$. Here $\beta=\sup_t\{\beta_t\}$, where β_t has the value (1.2). The matrix

$$c(x, t) = \sum a_j(x, t)\,\partial s/\partial x_j, \qquad x\in\Omega', \quad 0\leqq t\leqq T,$$

defined by (2.3), is now a function of t, as well as x. Therefore the value of σ in (2.1) should be chosen so that $c(x, t)$ is nonsingular for $x\in\Omega'$, $0\leqq t\leqq T$.

Note that the only part of $S(t)$ which varies with t is $A_0(t)$. Thus $S(t)$ is continuously differentiable on $[0, T]$ to $B(H^1_P(\Omega), X)$, since this is true for $A_0(t)$.

According to Propositions 2 and 4, $S(t)$ is an isomorphism from $H^1_P(\Omega)$ onto X, and $S(t)A(t)S(t)^{-1}=A(t)+B(t)$, where $B(t)\in B(X)$. It follows from the inequality (5.2) that the map $t\to B(t)=[Z, A(t)]S(t)^{-1}$ is continuous on $[0, T]$ to $B(X)$. □

Proof of Theorem 3. We noted in the remark after Theorem 3 that the conclusions of Theorem 3 are true if $P(x, t)=P(x)$ is independent of t. We turn now to the general case where $P(x, t)$ varies with t.

Suppose $r(x, t)$ is an orthogonal matrix-valued function of class C^2 on $\bar\Omega\times[0, T]$ with the property that

(5.3) $\qquad r(y, t)$ maps $P(y, 0)$ onto $P(y, t)$ for $y\in\Gamma$, $0\leqq t\leqq T$.

If one makes the change of variables

$$v(x, t) = r(x, t)^{-1}u(x, t), \qquad x\in\bar\Omega, \quad 0\leqq t\leqq T,$$

then equation (1.1) for u corresponds to the following equation for v.

$$\begin{aligned}
&\partial v/\partial t+\sum r^{-1}a_j r\,\partial v/\partial x_j+b'v = r^{-1}f,\\
&v(x, 0) = r(x, 0)^{-1}\phi(x),\\
&v(x, t)\in P(x, 0), \qquad x\in\Gamma, \quad 0\leqq t\leqq T,
\end{aligned}$$

where $b' = r^{-1}(\partial r/\partial t + \sum a_j\, \partial r/\partial x_j + br)$. In this equation the boundary subspace $P(x, 0)$ does not vary with t. We also note that the assumption $f(\cdot, t) \in H^1_{P_t}(\Omega)$ implies $r(\cdot, t)^{-1} f(\cdot, t) \in H^1_{P_0}(\Omega)$. Therefore, as noted above, the conclusions of Theorem 3 hold for this equation. It follows that $u = rv$ is the desired solution to equation (1.1).

To complete the argument, one must show that there is a function $r(x, t)$ with the above properties. The condition (5.3) can be restated as

$$r(y, t)e(y, 0)r(y, t)^{-1} = e(y, t), \qquad y \in \Gamma, \quad 0 \leqq t \leqq T, \tag{5.4}$$

where $e(y, t)$ is the orthogonal projection of R^N onto $P(y, t)$. Such an r is called a transformation function for $e(y, t)$, and it may be constructed using Kato's method (see [8, p. 99]). We take for r the solution of the differential equation

$$(\partial/\partial t)r(x, t) = q(x, t)r(x, t), \qquad 0 \leqq t \leqq T,$$

$r(x, 0) = 1$, where $q = [\partial e/\partial t, e]$. Since q is antisymmetric, this r is an orthogonal matrix-valued function of class C^2 on $\Gamma \times [0, T]$ which satisfies (5.4) with $r(y, 0) = 1$. We must extend r to $\bar{\Omega} \times [0, T]$. Let $r'(y, s, t) = r(y, t\rho(s))$, where ρ is a function mapping $[0, \sigma]$ into $[0, 1]$ with $\rho(0) = 1$ and $\rho(s) = 0$ for $s \geqq \sigma/2$. r' is defined on $\Gamma \times [0, \sigma] \times [0, T]$ with $r'(y, 0, t) = r(y, t)$ and $r'(y, s, t) = 1$ for $s \geqq \sigma/2$. Letting $y(x)$, $s(x)$ be the coordinates in Ω' introduced in (2.2), one may change variables to obtain $r(x, t) = r'(y(x), s(x), t)$. This r is defined on $\Omega' \times [0, T]$ with $r(x, t) = 1$ if $s(x) \geqq \sigma/2$. r may then be extended to $\bar{\Omega} \times [0, T]$ by setting $r(x, t) = 1$ for $x \in \bar{\Omega} \sim \Omega'$. □

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